A note on $q$-nanologue of Sándor’s functions

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Abstract The additive analogues of Pseudo-Smarandache, Smarandache-simple functions and their duals have been recently studied by J. Sándor. In this note, we obtain $q$-analogues of Sándor’s theorems [6].

Keywords $q$-gamma function; Pseudo-Smarandache function; Smarandache-simple function; Asymptotic formula.

Dedicated to Sun-Yi Park on 90th birthday

§1. Introduction

The additive analogues of Smarandache functions $S$ and $S_*$ have been introduced by Sándor [5] as follows:

$$S(x) = \min \{m \in N : x \leq m!\}, \quad x \in (1, \infty),$$

and

$$S_*(x) = \max \{m \in N : m! \leq x\}, \quad x \in [1, \infty).$$

He has studied many important properties of $S_*$ relating to continuity, differentiability and Riemann integrability and also proved the following theorems:

Theorem 1.1.

$$S_* \sim \frac{\log x}{\log \log x} \quad (x \to \infty).$$

Theorem 1.2. The series

$$\sum_{n=1}^{\infty} \frac{1}{n(S_*(n))^\alpha},$$

is convergent for $\alpha > 1$ and divergent for $\alpha \leq 1$. 
In [1], Adiga and Kim have obtained generalizations of Theorems 1.1 and 1.2 by the use of Euler’s gamma function. Recently Adiga-Kim-Somashekara-Fathima [2] have established a q-analogues of these results on employing analogues of Pseudo-Smarandache, Smarandache-simple functions and their duals as follows:

\[
Z(x) = \min \left\{ m \in \mathbb{N} : x \leq \frac{m(m+1)}{2} \right\}, \quad x \in (0, \infty),
\]

\[
Z_*(x) = \max \left\{ m \in \mathbb{N} : \frac{m(m+1)}{2} \leq x \right\}, \quad x \in [1, \infty),
\]

\[
P(x) = \min \{ m \in \mathbb{N} : p^x \leq m! \}, \quad p > 1, \quad x \in (0, \infty),
\]

\[
P_*(x) = \max \{ m \in \mathbb{N} : m! \leq p^x \}, \quad p > 1, \quad x \in [1, \infty).
\]

He has also proved the following theorems:

**Theorem 1.3.**

\[
Z_* \sim \frac{1}{2} \sqrt{8x + 1} \quad (x \to \infty).
\]

**Theorem 1.4.** The series

\[
\sum_{n=1}^{\infty} \frac{1}{(Z_*(n))^\alpha},
\]

is convergent for \( \alpha > 2 \) and divergent for \( \alpha \leq 2 \). The series

\[
\sum_{n=1}^{\infty} \frac{1}{n(Z_*(n))^\alpha},
\]

is convergent for all \( \alpha > 0 \).

**Theorem 1.5.**

\[
\log P_*(x) \sim \log x \quad (x \to \infty),
\]

**Theorem 1.6.** The series

\[
\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\log \log n}{\log P_*(n)} \right)^\alpha
\]

is convergent for all \( \alpha > 1 \) and divergent for \( \alpha \leq 1 \).

The main purpose of this note is to obtain q-analogues of Sándor’s Theorems 1.3 and 1.5.

In what follows, we make use of the following notations and definitions. F. H. Jackson defined a q-analogues of the gamma function which extends the q-factorial

\[
(n!)_q = 1(1 + q)(1 + q + q^2) \cdots (1 + q + q^2 + \cdots + q^{n-1}), \quad \text{cf [3]},
\]

which becomes the ordinary factorial as \( q \to 1 \). He defined the q-analogue of the gamma function as

\[
\Gamma_q(x) = \frac{(q!q)_\infty}{(q^x!q)_\infty} (1 - q)^{1-x}, \quad 0 < q < 1,
\]
and
\[
\Gamma_q(x) = \frac{(q^{-1}; q^{-1})_\infty}{(q^{-x}; q^{-1})_\infty} (1 - q)^{1-x} q^x, \quad q > 1,
\]
where
\[
(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n).
\]

It is well known that \(\Gamma_q(x) \to \Gamma(x)\) as \(q \to 1\), where \(\Gamma(x)\) is the ordinary gamma function.

§2. Main Theorems

We now defined the q-analogues of \(Z\) and \(Z_*\) as follows:
\[
Z_q(x) = \min \left\{ \frac{1 - q^m}{1 - q} : x \leq \frac{\Gamma_q(m + 2)}{2\Gamma_q(m)} \right\}, \quad m \in N, \quad x \in (0, \infty),
\]
and
\[
Z^*_q(x) = \max \left\{ \frac{1 - q^m}{1 - q} : \frac{\Gamma_q(m + 2)}{2\Gamma_q(m)} \leq x \right\}, \quad m \in N, \quad x \in \left[ \frac{\Gamma_q(m + 2)}{2\Gamma_q(m)} , \infty \right),
\]
where \(0 < q < 1\). Clearly, \(Z_q(x) \to Z(x)\) and \(Z^*_q(x) \to Z_*(x)\) as \(q \to 1^{-}\). From the definitions of \(Z_q\) and \(Z^*_q\), it is clear that
\[
\begin{align*}
Z_q(x) &= \begin{cases} 
1, & \text{if } x \in \left(0, \frac{\Gamma_q(3)}{2\Gamma_q(1)}\right] \\
1 - \frac{q^m}{1 - q}, & \text{if } x \in \left(\frac{\Gamma_q(m + 1)}{2\Gamma_q(m)}, \frac{\Gamma_q(m + 2)}{2\Gamma_q(m)}\right], \quad m \geq 2,
\end{cases} \quad (1)
\end{align*}
\]
and
\[
\begin{align*}
Z^*_q(x) &= \frac{1 - q^m}{1 - q} \quad \text{if } x \in \left[ \frac{\Gamma_q(m + 2)}{2\Gamma_q(m)}, \frac{\Gamma_q(m + 3)}{2\Gamma_q(m + 1)} \right]. \quad (2)
\end{align*}
\]

Since
\[
\frac{1 - q^{m-1}}{1 - q} \leq \frac{1 - q^m}{1 - q} = \frac{1 - q^{m-1}}{1 - q} + q^{m-1} \leq \frac{1 - q^{m-1}}{1 - q} + 1,
\]
(1) and (2) imply that for \(x > \frac{\Gamma_q(3)}{2\Gamma_q(1)}\),
\[
Z^*_q \leq Z_q \leq Z^*_q + 1.
\]

Hence it suffices to study the function \(Z^*_q\). We now prove our main theorems.

**Theorem 2.1.** If \(0 < q < 1\), then
\[
\frac{\sqrt{1 + 8xq} - (1 + 2q)}{2q^2} < Z^*_q \leq \frac{\sqrt{1 + 8xq} - 1}{2q}, \quad x \geq \frac{\Gamma_q(3)}{2\Gamma_q(1)}.
\]

**Proof.** If
\[
\frac{\Gamma_q(k + 1)}{2\Gamma_q(k)} \leq x < \frac{\Gamma_q(k + 3)}{2\Gamma_q(k + 1)}, \quad (3)
\]
then
\[ Z_q^* = \frac{1 - q^k}{1 - q} \]
and
\[ (1 - q^k)(1 - q^{k+1}) - 2x(1 - q)^2 \leq 0 < (1 - q^{k+1})(1 - q^{k+2}) - 2x(1 - q)^2. \] (4)

Consider the functions \( f \) and \( g \) defined by
\[ f(y) = (1 - y)(1 - yq) - 2x(1 - q)^2 \]
and
\[ g(y) = (1 - yq)(1 - yq^2) - 2x(1 - q)^2. \]

Note that \( f \) is monotonically decreasing for \( y \leq \frac{1 + q}{2q} \) and \( g \) is strictly decreasing for \( y \leq \frac{1 + q}{2q^2} \).

Also \( f(y_1) = 0 = g(y_2) \) where
\[ y_1 = \frac{(1 + q) - (1 - q)\sqrt{1 + 8xq}}{2q}, \]
\[ y_2 = \frac{(q + q^3) - q(1 - q)\sqrt{1 + 8xq}}{2q^3}. \]

Since \( y_1 \leq \frac{1 + q}{2q} \), \( y_2 \leq \frac{1 + q}{2q^2} \) and \( q^k < \frac{1 + q}{2q} < \frac{1 + q}{2q^2} \), from (4), it follows that
\[ f(q^k) \leq f(y_1) = 0 = g(y_2) < g(q^k). \]

Thus \( y_1 < q^k < y_2 \) and hence
\[ \frac{1 - y_2}{1 - q} < \frac{1 - q^k}{1 - q} < \frac{1 - y_1}{1 - q}. \]
i.e.
\[ \frac{\sqrt{1 + 8xq} - (1 + 2q)}{2q^2} < Z_q^* \leq \frac{\sqrt{1 + 8xq} - 1}{2q}. \]

This completes the proof.

**Remark.** Letting \( q \to 1^- \) in the above theorem, we obtain Sándor’s Theorem 1.3.

We define the q-analogues of \( P \) and \( P^* \) as follows:
\[ P_q(x) = \min\{m \in N : p^x \leq \Gamma_q(m + 1)\}, \quad p > 1, \quad x \in (0, \infty), \]
and
\[ P_q^*(x) = \max\{m \in N : \Gamma_q(m + 1) \leq p^x\}, \quad p > 1, \quad x \in [1, \infty), \]
where \( 0 < q < 1 \). Clearly, \( P_q(x) \to P(x) \) and \( P_q^* \to P^*(x) \) as \( q \to 1^- \). From the definitions of \( P_q \) and \( P_q^* \), we have
\[ P_q^*(x) \leq P_q(x) \leq P_q^*(x) + 1. \]

Hence it is enough to study the function \( P_q^* \).

**Theorem 2.2.** If \( 0 < q < 1 \), then
\[ P_*(x) \sim \frac{x \log p}{\log \left(\frac{1}{1-q}\right)} \quad (x \to \infty). \]
**Proof.** If \( \Gamma_q(n+1) \leq p^x < \Gamma_q(n+2) \), then
\[
P_q^*(x) = n
\]
and
\[
\log \Gamma_q(n+1) \leq \log p^x < \log \Gamma_q(n+2).
\]

But by the \( q \)-analogue of Stirling’s formula established by Moak [4], we have
\[
\log \Gamma_q(n+1) \sim \left( n + \frac{1}{2} \right) \log \left( \frac{q^{n+1}}{q-1} \right) \sim n \log \left( \frac{1}{1-q} \right).
\]

Dividing (5) throughout by \( n \log \left( \frac{1}{1-q} \right) \), we obtain
\[
\frac{\log \Gamma_q(n+1)}{n \log \left( \frac{1}{1-q} \right)} \leq \frac{x \log p}{P_q^*(x)} \log \left( \frac{1}{1-q} \right) < \frac{\log \Gamma_q(n+2)}{n \log \left( \frac{1}{1-q} \right)}.
\]

Using (6) and (7), we deduce
\[
\lim_{x \to \infty} \frac{x \log p}{P_q^*(x) \log \left( \frac{1}{1-q} \right)} = 1.
\]

This completes the proof.

**References**


