On Near Mean Graphs

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Abstract: Let $G = (V, E)$ be a graph with $p$ vertices and $q$ edges and let $f : V(G) \to \{0, 1, 2, \ldots, q - 1, q + 1\}$ be an injection. The graph $G$ is said to have a near mean labeling if for each edge, there exist an induced injective map $f^* : E(G) \to \{1, 2, \ldots, q\}$ defined by

$$f^*(uv) = \begin{cases} \frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even,} \\ \frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd.} \end{cases}$$

We extend this notion to Smarandachely near $m$-mean labeling (as in [9]) if for each edge $e = uv$ and an integer $m \geq 2$, the induced Smarandachely $m$-labeling $f^*$ is defined by

$$f^*(e) = \left\lceil \frac{f(u) + f(v)}{m} \right\rceil.$$

A graph that admits a Smarandachely near mean $m$-labeling is called Smarandachely near $m$-mean graph. The graph that admits a near mean labeling is called a near mean graph (NMG). In this paper, we proved that the graphs $P_n, C_n, K_{2,n}$ are near mean graphs and $K_n (n > 4)$ and $K_{1,n} (n > 4)$ are not near mean graphs.

Key Words: Labeling, near mean labeling, near mean graph, Smarandachely near $m$-labeling, Smarandachely near $m$-mean graph.

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§1. Introduction

By a graph, we mean a finite simple and undirected graph. The vertex set and edge set of a graph $G$ denoted are by $V(G)$ and $E(G)$ respectively. Let $f : V(G) \to \{0, 1, 2, \ldots, q - 1, q + 1\}$ be an injection. The graph $G$ is said to have a near mean labeling if for each edge, there exist an induced injective map $f^* : E(G) \to \{1, 2, \ldots, q\}$ defined by

$$f^*(uv) = \begin{cases} \frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even,} \\ \frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd.} \end{cases}$$

We extend this notion to Smarandachely near $m$-mean labeling (as in [9]) if for each edge $e = uv$
and an integer \( m \geq 2 \), the induced Smarandachely \( m \)-labeling \( f^* \) is defined by
\[
f^*(e) = \left\lceil \frac{f(u) + f(v)}{m} \right\rceil.
\]
A graph that admits a Smarandachely near mean \( m \)-labeling is called Smarandachely near mean graph. A path \( P_n \) is a graph of length \( n - 1 \cdot K_n \) and \( C_n \) are complete graph and cycle with \( n \) vertices respectively. Terms and notations not used here are as in [2].

\section{Preli\(\text{m}\)naries}

The mean labeling was introduced in [3]. Let \( G \) be a \((p,q)\) graph. In [4], we proved that the graphs Book \( B_n \), Ladder \( L_n \), Grid \( P_n \times P_n \), Prism \( P_m \times C_3 \) and \( L_n \odot K_1 \) are near mean graphs. In [5], we proved that Join of graphs, \( K_2 + mK_1, K_1^3 + 2K_2, S_m + K_1P_n + 2K_1 \) and double fan are near mean graphs. In [6], we proved Family of trees, Bi-star, Sub-division Bi-star \( P_m \ominus 2K_1, P_m \ominus 3K_1, P_m \ominus K_{1,4} \) and \( P_m \ominus K_{1,3} \) are near mean graphs. In [7], special class of graphs triangular snake, quadrilateral snake, \( C_4 + n \), \( S_m, 3 \), \( S_m, 4 \), and parachutes are proved as near mean graphs. In [8], we proved the graphs armed and double armed crown of \( C_3 \) and \( C_4 \) are near mean graphs. In this paper we proved that the graphs \( P_n, C_n, K_2 \), and \( K_1(n > 4) \) and \( K_1,n(n > 4) \) are not near mean graphs.

\section{Near Mean Graphs}

\textbf{Theorem 3.1} The path \( P_n \) is a near mean graph.

\textit{Proof} Let \( P_n \) be a path of \( n \) vertices with \( V(P_n) = \{u_1, u_2, \ldots, u_n\} \) and \( E(P_n) = \{(u_iu_{i+1})/i = 1, 2, \ldots, n-1\} \). Define \( f : V(P_n) \to \{0, 1, 2, \ldots, n-1, n+1\} \) by
\[
f(u_i) = i - 1, 1 \leq i \leq n \] 
\[
f(u_n) = n + 1.
\]
Clearly, \( f \) is injective. It can be verified that the induced edge labeling given by \( f^*(u_iu_{i+1}) = i(1 \leq i \leq n) \) are distinct. Hence, \( P_n \) is a near mean graph.

\textbf{Example 3.2} A near mean labeling of \( P_4 \) is shown in Figure 1.

\begin{figure} [h]
\centering
\begin{tikzpicture}
    \node (1) at (0,0) [circle, fill=black, inner sep=2pt]{};
    \node (2) at (1,0) [circle, fill=black, inner sep=2pt]{};
    \node (3) at (2,0) [circle, fill=black, inner sep=2pt]{};
    \node (4) at (3,0) [circle, fill=black, inner sep=2pt]{};
    \node (5) at (4,0) [circle, fill=black, inner sep=2pt]{};
    \draw (1) -- (2);
    \draw (2) -- (3);
    \draw (3) -- (4);
    \draw (4) -- (5);
    \node at (0,0.2) {u_1};
    \node at (1,0.2) {u_2};
    \node at (2,0.2) {u_3};
    \node at (3,0.2) {u_4};
    \node at (0,-0.2) {0};
    \node at (1,-0.2) {1};
    \node at (2,-0.2) {2};
    \node at (3,-0.2) {3};
    \node at (4,-0.2) {4};
\end{tikzpicture}
\caption{\( P_4 \)}
\end{figure}

\textbf{Theorem 3.3} \( K_n, (n > 4) \) is not a near mean graph.

\textit{Proof} Let \( f : V(G) \to \{0, 1, 2, \ldots, q-1, q+1\} \). To get the edge label 1 we must have either 0 and 1 as vertex labels or 0 and 2 as vertex labels.
In either case 0 must be label of some vertex. In the same way to get edge label \( q \), we must have either \( q - 1 \) and \( q + 1 \) as vertex labels or \( q - 2 \) and \( q + 1 \) as vertex labels. Let \( u \) be a vertex whose vertex label 0.

**Case i.** To get the edge label \( q \). Assign vertex labels \( q - 1 \) and \( q + 1 \) to the vertices \( w \) and \( x \) respectively.

**Subcase a.** Let \( v \) be a vertex whose vertex label be 2, then the edges \( vw \) and \( ux \) get the same label.

**Subcase b.** Let \( v \) be a vertex whose vertex label be 1.

Then the edges \( vw \) and \( ux \) get the same label when \( q \) is odd. Similarly, when \( q \) is even, the edges \( uw \) and \( vw \) get the same label as well the edges \( ux \) and \( vx \) get the same label.

**Case ii.** To get the edge label \( q \) assign the vertex label \( q - 2 \) and \( q + 1 \) to the vertices \( w \) and \( x \) respectively.

**Subcase a.** Let \( v \) be the vertex whose vertex label be 1.

As \( n > 4 \), to get edge label 2, there should be a vertex whose vertex label is either 3 or 4. Let it be \( z \) (say). When vertex label of \( z \) is 3, the edges \( ux \) and \( wz \) have the same label also the edges \( uz \) and \( vz \) get the same edge label. When the vertex label of \( z \) is 4, the edges \( vx \) and \( wz \) have the same label.

**Subcase b.** Let \( v \) be a vertex whose vertex label 2.

As \( n > 4 \), to get edge label 2, there should be a vertex, say \( z \) whose vertex label is either 3 or 4. When vertex label of \( z \) is 3, the edges \( ux \) and \( wz \) get the same label. Suppose the vertex label of \( z \) is 4.

If \( q \) is even then the edges \( ux \) and \( wz \) have the same label. If \( q \) is odd then the edges \( vw \) and \( ux \) have the same label. Hence \( K_n(n \geq 5) \) is not a near mean graph. \( \Box \)

**Remark 3.4** \( K_2, K_3 \) and \( K_4 \) are near mean graphs.

**Theorem 3.5** A cycle \( C_n \) is a near mean graph for any integer \( n \geq 1 \).
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Proof Let $V(C_n) = \{u_1, u_2, u_3, \ldots, u_n, u_1\}$ and $E(C_n) = \{(u_i u_{i+1}) : 1 \leq i \leq n-1\} \cup \{(u_1 u_n)\}$.

Case i Let $n$ be even, say $n = 2m$.

Define $f : V(C_n) \to \{0, 1, 2, \ldots, 2m, 2m + 2\}$ by

$f(u_i) = i - 1, 1 \leq i \leq m.\\
\quad f(u_{m+j}) = m + j, 1 \leq j < m.\\
\quad f(u_n) = 2m + 1.$

Clearly $f$ is injective. The set of edge labels of $C_n$ is $\{1, 2, \ldots, q\}$. □

Case ii. Let $n$ be odd, say $n = 2m + 1$.

Define $f : V(C_n) \to \{0, 1, 2, \ldots, 2m - 1, 2m + 1\}$ by

$f(u_i) = i - 1, 1 \leq i \leq m.\\
\quad f(u_{m+j}) = m + j, 1 \leq j \leq m.\\
\quad f(u_{2m+1}) = 2m + 2.$

Clearly $f$ is injective. The set of edge labels of $C_n$ is $\{1, 2, \ldots, q\}$. □

Example 3.6 A near mean labeling of $C_6$ and $C_7$ is shown in Figure 3.

Figure 3: $C_6, C_7$

Theorem 3.7 $K_{1,n}(n > 4)$ is not a near mean graph.

Proof Let $V(K_{1,n}) = \{u, v_i : 1 \leq i \leq n\}$ and $E(K_{1,n}) = \{(uv_i) : 1 \leq i \leq n\}$. To get the edge label 1, either 0 and 1 (or) 0 and 2 are assigned to $u$ and $v_i$ for some $i$. In either case 0 must be label of some vertex.

Suppose if $f(u) = 0$, then we can not find an edge label $q$. Suppose if $f(v_1) = 0$, then either $f(u) = 1$ or $f(u) = 2$.

Case i. Let $f(u) = 1$.

To get edge label $q$, we need the following possibilities either $q - 1$ and $q + 1$ or $q - 2$ and $q + 1$. If $f(u) = 1$, it is possible only when $q$ is either 2 or 3. But $q > 4$, so it is not possible to get edge value $q$. 
Case ii. Let \( f(u) = 2 \).

As in Case i, if \( f(u) = 2 \) and if one of the edge value is \( q \), then the value of \( q \) is either 3 or 4. From both the cases it is not possible to get the edge value \( q \), when \( q > 4 \).

Hence, \( K_{1,n}(n > 5) \) is not a near mean graph. \( \square \)

Remark 3.8 \( K_{1,n}, n \leq 4 \) is a near mean graph. For example, one such a near mean labeling is shown in Figure 4.

![Figure 4: \( K_{1,n}, n \leq 4 \)](image)

Theorem 3.9 \( K_{2,n} \) admits near mean graph.

Proof Let \((V_1, V_2)\) be the bipartition of \( V(K_{2,n}) \) with \( V_1 = \{u_1u_2\} \) and \( V_2 = \{v_1, v_2, \ldots, v_n\} \).

Define an injective map \( f : V(K_{2,n}) \to \{0, 1, 2, \ldots, 2n - 1, 2n + 1\} \) by

\[
\begin{align*}
f(u_1) &= 1 \\
f(u_2) &= 2n + 1 \\
f(v_i) &= 2(i - 1), 1 \leq i \leq n.
\end{align*}
\]

Then, it can be verified \( f^{*}(u_1v_i) = i, 1 \leq i \leq n, f^{*}(u_2v_i) = n + i, 1 \leq i \leq n \) and the edge values are distinct. Hence, \( K_{2,n} \) is a near mean graph. \( \square \)
Example 3.10  A near mean labeling of $K_{2,4}$ is shown in Figure 5.

![Graph of $K_{2,4}$](image)

Figure 5: $K_{2,4}$

References


