A new additive function and the F. Smarandache function

Yanchun Guo

Department of Mathematics, Xianyang Normal University
Xianyang, Shaanxi, P.R.China

Abstract For any positive integer \( n \), we define the arithmetical function \( F(n) \) as \( F(1) = 0 \). If \( n > 1 \) and \( n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k} \) be the prime power factorization of \( n \), then \( F(n) = \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_k p_k \). Let \( S(n) \) be the Smarandache function. The main purpose of this paper is using the elementary method and the prime distribution theory to study the mean value properties of \((F(n) - S(n))^2\), and give a sharper asymptotic formula for it.

Keywords Additive function, Smarandache function, Mean square value, Elementary method, Asymptotic formula.

§1. Introduction and result

Let \( f(n) \) be an arithmetical function, we call \( f(n) \) as an additive function, if for any positive integers \( m, n \) with \( (m, n) = 1 \), we have \( f(mn) = f(m) + f(n) \). We call \( f(n) \) as a complete additive function, if for any positive integers \( m, n \) with \( (m, n) = 1 \), we have \( f(mn) = f(m) + f(n) \). In elementary number theory, there are many arithmetical functions satisfying the additive properties. For example, if \( n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k} \) denotes the prime power factorization of \( n \), then function \( \Omega(n) = \alpha_1 + \alpha_2 + \cdots + \alpha_k \) and logarithmic function \( f(n) = \ln n \) are two complete additive functions, \( \omega(n) = k \) is an additive function, but not a complete additive function. About the properties of the additive functions, one can find them in references [1], [2] and [5].

In this paper, we define a new additive function \( F(n) \) as follows: \( F(1) = 0 \); If \( n > 1 \) and \( n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k} \) denotes the prime power factorization of \( n \), then \( F(n) = \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_k p_k \). It is clear that this function is a complete additive function. In fact if \( m = p_1^{\beta_1}p_2^{\beta_2}\cdots p_k^{\beta_k} \) and \( n = p_1^{\beta_1}p_2^{\beta_2}\cdots p_k^{\beta_k} \), then we have \( mn = p_1^{\alpha_1+\beta_1}p_2^{\alpha_2+\beta_2}\cdots p_k^{\alpha_k+\beta_k} \). Therefore, \( F(mn) = (\alpha_1 + \beta_1)p_1 + (\alpha_2 + \beta_2)p_2 + \cdots + (\alpha_k + \beta_k)p_k = F(m) + F(n) \). So \( F(n) \) is a complete additive function. Now we let \( S(n) \) be the Smarandache function. That is, \( S(n) \) denotes the smallest positive integer \( m \) such that \( n \) divide \( m! \), or \( S(n) = \min\{m: n \mid m!\} \). About the properties of \( S(n) \), many authors had studied it, and obtained a series results, see references [7], [8] and [9]. The main purpose of this paper is using the elementary method and the prime distribution theory to study the mean value properties of \((F(n) - S(n))^2\), and give a sharper asymptotic formula for it. That is, we shall prove the following:

**Theorem.** Let \( N \) be any fixed positive integer. Then for any real number \( x > 1 \), we
have the asymptotic formula
\[
\sum_{n \leq x} (F(n) - S(n))^2 = \sum_{i=1}^{N} c_i \cdot \frac{x^2}{\ln^{i+1} x} + O\left(\frac{x^2}{\ln^{N+2} \sqrt{x}}\right),
\]
where \(c_i (i = 1, 2, \cdots, N)\) are computable constants, and \(c_1 = \frac{\pi^2}{6}\).

§2. Proof of the theorem

In this section, we use the elementary method and the prime distribution theory to complete the proof of the theorem. We use the idea in reference [4]. First we define four sets \(A, B, C, D\) as follows: \(A = \{n, n \in N, n\} has only one prime divisor \(p\) such that \(p \mid n \) and \(p^2 \nmid n, p > n^\frac{3}{2}\}; \(B = \{n, n \in N, n\} has only one prime divisor \(p\) such that \(p^2 \mid n \) and \(p > n^\frac{3}{2}\}; \(C = \{n, n \in N, n\} has two different prime divisors \(p_1\) and \(p_2\) such that \(p_1 p_2 \mid n, p_2 > p_1 > n^\frac{3}{2}\}; D = \{n, n \in N, any prime divisor \(p\) of \(n\) satisfying \(p \leq n^\frac{3}{2}\}, where \(N\) denotes the set of all positive integers. It is clear that from the definitions of \(A, B, C\) and \(D\) we have

\[
\sum_{n \leq x} (F(n) - S(n))^2 = \sum_{n \leq x, \ n \in A} (F(n) - S(n))^2 + \sum_{n \leq x, \ n \in B} (F(n) - S(n))^2 + \sum_{n \leq x, \ n \in C} (F(n) - S(n))^2 + \sum_{n \leq x, \ n \in D} (F(n) - S(n))^2 = W_1 + W_2 + W_3 + W_4. \tag{1}
\]

Now we estimate \(W_1, W_2, W_3\) and \(W_4\) in (1) respectively. Note that \(F(n)\) is a complete additive function, and if \(n \in A\) with \(n = pk\), then \(S(n) = S(p) = p\), and any prime divisor \(q\) of \(k\) satisfying \(q \leq n^\frac{3}{2}\), so \(F(k) \leq n^\frac{3}{2} \ln n\). From the Prime Theorem (See Chapter 3, Theorem 2 of [3]) we know that

\[
\pi(x) = \sum_{p \leq x} 1 = \sum_{i=1}^{k} c_i \cdot \frac{x}{\ln^{i} x} + O\left(\frac{x}{\ln^{i+1} x}\right), \tag{2}
\]

where \(c_i (i = 1, 2, \cdots, k)\) are computable constants, and \(c_1 = 1\). By these we have the estimate:

\[
W_1 = \sum_{n \leq x, \ n \in A} (F(n) - S(n))^2 = \sum_{pk \leq x, \ (pk) \in A} (F(pk) - p)^2 = \sum_{pk \leq x, \ (pk) \in A} F^2(k) \ll \sum_{k \leq \sqrt{\frac{x}{k}}} \sum_{k \leq \sqrt{\frac{x}{k}}} \sum_{k \leq \sqrt{\frac{x}{k}}} (pk)^2 \ln^2(pk) \leq (\ln x)^2 \sum_{k \leq \sqrt{\frac{x}{k}}} k^2 \sum_{k \leq \sqrt{\frac{x}{k}}} p^2 \ll (\ln x)^2 \sum_{k \leq \sqrt{\frac{x}{k}}} k^2 \left(\frac{x}{k}\right)^{\frac{3}{2}} \frac{1}{\ln^{\frac{3}{2}} k} \ll x^2 \ln^2 x. \tag{3}
\]
Finally, we estimate main term $W_2$. Note that $n < \frac{x}{\ln x}$, and $S(n) \leq \frac{n}{\ln x}$, so we have

$$W_2 = \sum_{n \leq x} (F(n) - S(n))^2 = \sum_{n \leq x} (F(p^2k) - 2p)^2$$

$$= \sum_{k \leq x} \sum_{k < p \leq \sqrt{\frac{x}{k}}} F^2(k) \ll \sum_{k \leq x} \sum_{k < p \leq \sqrt{\frac{x}{k}}} k^2$$

$$\ll \sum_{k \leq x} \frac{k^2 \cdot x^{\frac{1}{2}}}{k + \ln x} \ll \frac{x^{\frac{3}{2}}}{\ln x}. \quad (4)$$

If $n \in D$, then $F(n) \leq \frac{n}{\ln x}$ and $S(n) \leq \frac{n}{\ln x}$, so we have

$$W_4 = \sum_{n \leq x} (F(n) - S(n))^2 \ll \sum_{n \leq x} n^{\frac{3}{2}} \ln^2 x \ll x^{\frac{3}{2}} \ln^2 x. \quad (5)$$

Finally, we estimate main term $W_3$. Note that $n < \frac{x}{\ln x}$, and $S(n) \leq \frac{n}{\ln x}$, so we have

$$W_3 = \sum_{n \leq x} (F(n) - S(n))^2 = \sum_{n \leq x} (F(p_1p_2k) - p_2)^2 + O\left(x^{\frac{5}{2}} \ln^2 x\right)$$

$$= \sum_{k \leq x} \sum_{k < p_1 \leq \sqrt{\frac{x}{k}}} \sum_{p_2 > p_1 > k} p_1^2 + O\left(\sum_{k \leq x} \sum_{k < p_1 \leq \sqrt{\frac{x}{k}}} \sum_{p_1 < p_2 \leq \frac{x}{k}} k p_1\right) + O\left(x^{\frac{3}{2}} \ln^2 x\right)$$

$$= \sum_{k \leq x} \sum_{k < p_1 \leq \sqrt{\frac{x}{k}}} \sum_{p_2 < p_1} p_1^2 \left(\sum_{i=1}^N c_i \cdot \frac{x}{p_1 k \ln x} + O\left(\frac{x}{p_1 k \ln^{N+1} x}\right)\right) + O\left(x^{\frac{3}{2}} \ln^2 x\right)$$

$$- \sum_{k \leq x} \sum_{k < p_1 \leq \sqrt{\frac{x}{k}}} \sum_{p_2 < p_1} p_1^2 \sum_{i=1}^N c_i \cdot \frac{x}{p_1 k \ln x} + O\left(\sum_{k \leq x} \sum_{k < p_1 \leq \sqrt{\frac{x}{k}}} \sum_{p_1 < p_2 \leq \frac{x}{k}} k p_1\right). \quad (6)$$

Note that $\zeta(2) = \frac{\pi^2}{6}$, from the Abel’s identity (See Theorem 4.2 of [6]) and (2) we have

$$\sum_{k \leq x} \sum_{k < p_1 \leq \sqrt{\frac{x}{k}}} p_1^2 \sum_{p_2 < p_1} 1 = \sum_{k \leq x} \sum_{k < p_1 \leq \sqrt{\frac{x}{k}}} p_1^2 \sum_{i=1}^N c_i \cdot \frac{p_1}{\ln^i p_1} + O\left(\frac{p_1}{\ln^{N+1} p_1}\right)$$

$$= \sum_{i=1}^N \sum_{k \leq x} \sum_{k < p_1 \leq \sqrt{\frac{x}{k}}} c_i \cdot \frac{p_1^3}{\ln^i p_1} + O\left(\sum_{k \leq x} \sum_{k < p_1 \leq \sqrt{\frac{x}{k}}} \frac{p_1^3}{\ln^{N+1} p_1}\right)$$

$$= \sum_{i=1}^N \frac{d_i \cdot x^2}{\ln^{i+1} x} + O\left(\frac{2^N \cdot x^2}{\ln^{N+2} x}\right), \quad (7)$$
where $d_i (i = 1, 2, \cdots, N)$ are computable constants, and $d_1 = \frac{\pi^2}{6}$.

\[
\sum_{k \leq x^{\frac{1}{3}}} \sum_{k < p_1 \leq \sqrt{x}} \sum_{p_1 < p_2 \leq \sqrt{x}} k p_1 \ll \sum_{k \leq x^{\frac{1}{3}}} \sum_{p_1 \leq \sqrt{x}} p_1 \cdot \frac{x}{p_1 k \ln x} \ll \sum_{k \leq x^{\frac{1}{3}}} \frac{x^{\frac{3}{2}}}{\sqrt{k \ln^2 x}} \ll \frac{x^{\frac{3}{2}}}{\ln^2 x}.
\] (8)

\[
\sum_{k \leq x^{\frac{1}{3}}} \sum_{k < p_1 \leq \sqrt{x}} \frac{x \cdot p_1}{k \ln^{N+1} x} \ll \sum_{k \leq x^{\frac{1}{3}}} \frac{x^2}{k \ln^{N+2} x} \ll \frac{x^2}{\ln^{N+2} x}.
\] (9)

From the Abel’s identity and (2) we also have the estimate

\[
\sum_{k \leq x^{\frac{1}{3}}} \sum_{k < p_1 \leq \sqrt{x}} \frac{x^2}{k \ln^{N+1} x} = \sum_{k \leq x^{\frac{1}{3}}} \frac{1}{k} \sum_{k < p_1 \leq \sqrt{x}} \frac{x p_1}{\ln^{N+2} x}
\]

\[
= \sum_{i=1}^{N} b_i \cdot \frac{x^2}{\ln^{N+1} x} + O \left( \frac{x^2}{\ln^{N+2} x} \right),
\] (10)

where $b_i (i = 1, 2, \cdots, N)$ are computable constants, and $b_1 = \frac{\pi^2}{6}$.

Now combining (1), (3), (4), (5), (6), (7), (8), and (9) we may immediately deduce the asymptotic formula:

\[
\sum_{n \leq x} (F(n) - S(n))^2 = \sum_{i=1}^{N} a_i \cdot \frac{x^2}{\ln^{N+1} x} + O \left( \frac{x^2}{\ln^{N+2} \sqrt{x}} \right),
\]

where $a_i (i = 1, 2, \cdots, N)$ are computable constants, and $a_1 = b_1 - d_1 = \frac{\pi^2}{6}$.

This completes the proof of Theorem.

References