A Note on 1-Edge Balance Index Set

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Abstract: Let $G$ be a graph with vertex set $V$ and edge set $E$, and $Z_2 = \{0, 1\}$. Let $f$ be a labeling from $E$ to $Z_2$, so that the labels of the edges are 0 or 1. The edges labelled 1 are called 1-edges and edges labelled 0 are called 0-edges. The edge labeling $f$ induces a vertex labeling $f^* : V \to Z_2$ defined by

$$f^*(v) = \begin{cases} 1 & \text{if the number of 1-edges incident on } v \text{ is odd,} \\ 0 & \text{if the number of 1-edges incident on } v \text{ is even.} \end{cases}$$

For $i \in Z_2$ let $e_f(i) = e(i) = |\{ e \in E : f(e) = i \}|$ and $v_f(i) = v(i) = |\{ v \in V : f^*(v) = i \}|$. A labeling $f$ is said to be edge-friendly if $|e_f(0) - e_f(1)| \leq 1$. The 1-edge balance index set (OEBI) of a graph $G$ is defined by $\{|v_f(0) - v_f(1)| : \text{the edge labeling } f \text{ is edge-friendly}\}$. The main purpose of this paper is to completely determine the 1-edge balance index set of wheel and Mycielskian graph of a path.

Key Words: Mycielskian graph, edge labeling, edge-friendly, 1-edge balance index set, Smarandachely induced vertex labeling, Smarandachely edge-friendly graph.

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§1. Introduction

A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. Varieties of graph labeling have been investigated by many authors [2], [3] [5] and they serve as useful models for broad range of applications.

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$ and $Z_2 = \{0, 1\}$. Let $f$ be a labeling from $E(G)$ to $Z_2$, so that the labels of the edges are 0 or 1. The edges labelled 1 are called 1-edges and edges labelled 0 are called 0-edges. The edge labeling $f$ induces a vertex labeling $f^* : V(G) \to Z_2$, defined by

$$f^*(v) = \begin{cases} 1 & \text{if the number of 1-edges incident on } v \text{ is odd,} \\ 0 & \text{if the number of 1-edges incident on } v \text{ is even.} \end{cases}$$

For $i \in Z_2$, let $e_f(i) = e(i) = |\{ e \in E(G) : f(e) = i \}|$ and $v_f(i) = v(i) = |\{ v \in V(G) : f^*(v) = i \}|$. Generally, let $f : E(G) \to Z_p$ be a labeling from $E(G)$ to $Z_p$ for an integer

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A Smarandachely induced vertex labeling on $G$ is defined by $f^v = (l_1, l_2, \cdots, l_p)$ with $n_k(v) \equiv l_k(\text{mod} p)$, where $n_k(v)$ is the number of $k$-edges, i.e., edges labeled with an integer $k$ incident on $v$. Let

$$e_k(G) = \frac{1}{2} \sum_{e \in E(G)} n_k(v)$$

for an integer $1 \leq k \leq p$. Then a Smarandachely edge-friendly graph is defined as follows.

**Definition 1.1** A graph $G$ is said to be Smarandachely edge-friendly if $|c_k(G) - c_{k+1}(G)| \leq 1$ for integers $1 \leq k \leq p$. Particularly, if $p = 2$, such a Smarandachely edge-friendly graph is abbreviated to an edge-friendly graph.

**Definition 1.2** The 1-edge balance index set of a graph $G$, denoted by $OEBI(G)$, is defined as $|v_f(1) - v_f(0)|$: if $f$ is edge-friendly.

For convenience, a vertex is called 0-vertex if its induced vertex label is 0 and 1-vertex, if its induced vertex label is 1.

In the mid 20th century there was a question regarding the construction of triangle-free $k$-chromatic graphs, where $k \leq 3$. In this search Mycielski [4] developed an interesting graph transformation known as the Mycielskian which is defined as follows:

**Definition 1.3** For a graph $G = (V, E)$, the Mycielskian of $G$ is the graph $\mu(G)$ with vertex set consisting of the disjoint union $V \cup V^\prime \cup \{v_0\}$, where $V^\prime = \{x^\prime : x \in V\}$ and edge set $E \cup \{x^\prime y : xy \in E\} \cup \{x^\prime v_0 : x^\prime \in V^\prime\}$.

![Figure 1 Mycielskian graph of the path $P_n$](image)

Recently Chandrashekar Adiga et al. [1] have introduced and studied the 1-edge balance index set of several classes of graphs. In Section 2, we completely determine the 1-edge balance index set of the Mycielskian graph of path $P_n$. In Section 3, we establish that $OEBI(W_n) = \{0, 4, 8, \ldots, n\}$ if $n \equiv 0(\text{mod } 4)$, $OEBI(W_n) = \{2, 6, 10, \ldots, n\}$ if $n \equiv 2(\text{mod } 4)$ and $OEBI(W_n) = \{1, 2, 5, \ldots, n\}$ if $n$ is odd.
§2. The 1-Edge Balance Index Set of $\mu(P_n)$

In this section we consider the Mycielskian graph of the path $P_n$ ($n \geq 2$), which consists of $2n+1$ vertices and $4n-3$ edges. To determine the $OEBI(\mu(P_n))$ we need the following theorem, whose proof is similar to the proof of the Theorem 1 in [6].

**Theorem 2.1** If the number of vertices in a graph $G$ is even(odd) then the 1-edge balance index set contains only even(odd) numbers.

Now we divide the problem of finding $OEBI(\mu(P_n))$ into two cases, viz,

\[ n \equiv 0(\mod 2) \quad \text{and} \quad n \equiv 1(\mod 2), \]

Denoted by $\max\{OEBI(\mu(P_n))\}$ the largest number in the 1-edge balance index set of $\mu(P_n)$. Then we get the following result.

**Theorem 2.2** If $n \equiv 0(\mod 2)$ i.e, $n = 2k(k \in N)$, then $OEBI(\mu(P_n)) = \{1, 3, 5, \ldots, 2n+1\}$.

**Proof** Let $f$ be an edge-friendly labeling on $\mu(P_n)$. Since the graph has $2n+1 = 4k+1$ vertices, $4n-3 = 8k-3$ edges, we have two possibilities: i) $e(0) = 4k - 1$, $e(1) = 4k - 2$ ii) $e(0) = 4k - 2$, $e(1) = 4k - 1$. Now we consider the first case namely $e(0) = 4k - 1$ and $e(1) = 4k - 2$. Denote the vertices of $\mu(P_n)$ as in the Figure 1. Now we label the edges $u_{2q-1}v_{2q}$, $u_{2q+1}v_{2q}$ for $1 \leq q \leq k - 1$, $u_qu_{q+1}$ for $1 \leq q \leq 2k-3$, $u_{2k-2}v_{2k-1}$, $u_{2k}v_{2k-1}$ and $u_{2k-1}u_{2k}$ by 1 and label the remaining edges by 0. Then it is easy to observe that $v(0) = 4k + 1$ and there is no 1-vertex in the graph. Thus $|v(1) - v(0)| = 4k + 1 = max\{OEBI(\mu(P_n))\}$.  

Now we interchange the labels of the edges to get the remaining 1-edge balance index numbers. By interchanging the labels of edges $u_{2q}u_{2q+1}$ and $u_{2q}v_{2q+1}$ for $1 \leq q \leq k - 2$, we get, $|v(0) - v(1)| = 4k + 1 - 4q$. Further interchanging $u_{2k-1}u_{2k}$ and $u_{2k-1}v_{2k}$, we get $|v(0) - v(1)| = 5$.

In the next four steps we interchange two pairs of edges as follows to see that $1, 3, 7, 11 \in OEBI(\mu(P_n))$

\[ u_1v_2 \text{ and } v_1v_0, u_2v_3 \text{ and } v_2v_0, \]
\[ u_3v_4 \text{ and } v_3v_0, u_4v_5 \text{ and } v_4v_0, \]
\[ u_5v_6 \text{ and } v_5v_0, u_6v_7 \text{ and } v_6v_0. \]

Now we interchange $u_{2q\frac{2k+1}{2}+7}v_{2q\frac{2k-1}{2}+6}$ and $u_{2q+6}v_{2q+6}$, $v_{2q+7}v_{2q+7}$ and $v_{2q+8}v_0$ for $1 \leq q \leq k - 5$ to obtain $|v(0) - v(1)| = 4q + 11$. Finally by interchanging the labels of the edges $u_{2q\frac{2k-1}{2}+7}v_{2q\frac{2k-1}{2}+6}$ and $u_{2q-2}u_{2q-1}$ we get $|v(0) - v(1)| = 4k - 5$ and $u_{2q\frac{2k+1}{2}+7}v_{2q\frac{2k+1}{2}+6}$ and $u_{2k-1}v_0$ we get $|v(0) - v(1)| = 4k - 1$.

Proof of the second case follows similarly. Thus

\[ OEBI(\mu(P_n)) = \{1, 3, 5, \ldots, 2n + 1\}. \]

**Theorem 2.3** If $n \equiv 1(\mod 2)$ i.e, $n = 2k + 1(k \in N)$, then $OEBI(\mu(P_n)) = \{1, 3, 5, \ldots, 2n + 1\}$. 

\[ \square \]
Proof Let $f$ be an edge-friendly labeling on $\mu(P_n)$. Since the graph contains $2n+1 = 4k+3$ vertices, $4n - 3 = 8k + 1$ edges, we have two possibilities: i) $e(0) = 4k + 1$, $e(1) = 4k$ ii) $e(0) = 4k$, $e(1) = 4k + 1$. Now we consider the first case namely $e(0) = 4k + 1$ and $e(1) = 4k$. Denote the vertices of $\mu(P_n)$ as in the Figure 1. Now we label the edges $u_{2q-1}v_{2q}$, $u_{2q+1}v_{2q}$ for $1 \leq q \leq k$ and $u_qv_{q+1}$ for $1 \leq q \leq 2k$ by 1 and label the remaining edges by 0. Then it is easy to observe that $v(0) = 4k + 3$ and there is no 1-vertex in the graph. Thus $|v(1) - v(0)| = 4k + 3 = 2n + 1 = \max\{OEBI(\mu(P_n))\}$.

Now we interchange the labels of the edges to get the remaining 1-edge balance index numbers. By interchanging the labels of edges $u_{2q}u_{2q+1}$ and $u_{2q}v_{2q+1}$ for $1 \leq q \leq k$ we get $|v(0) - v(1)| = 4k + 3 - 4q$. Further interchanging $u_{2q}v_{2k+1}$ and $v_{2k+1}v_0$ we get $|v(0) - v(1)| = 1$.

In the next four steps we interchange two pairs of edges as follows to see that $5, 9, 13, 17 \in OEBI(\mu(P_n))$

$$
\begin{align*}
&u_1v_2 \text{ and } v_1v_0, u_2v_3 \text{ and } v_2v_0. \\
&u_3v_2 \text{ and } v_3v_0, u_3v_4 \text{ and } v_4v_0. \\
&u_4v_5 \text{ and } v_5v_0, u_5v_4 \text{ and } v_6v_0. \\
&u_5v_6 \text{ and } v_7v_0, u_6v_7 \text{ and } v_8v_0.
\end{align*}
$$

And finally by interchanging the labels of edges $u_{2\left\lfloor \frac{q-1}{2} \right\rfloor + 7}v_{2\left\lfloor \frac{q-1}{2} \right\rfloor + 6}$ and $v_{2q+7}v_0, u_{2q+6}v_{2q+7}$ and $v_{2q+8}v_0$ for $1 \leq q \leq k - 4$, we obtain $|v(0) - v(1)| = 4q + 17$.

Proof of the second case follows similarly. Thus

$$OEBI(\mu(P_n)) = \{1, 3, 5, \ldots, 2n + 1\}. \quad \square$$

§3. The 1-Edge Balance Index Set of Wheel

In this section we consider the wheel, denoted by $W_n$ which consists of $n$ vertices and $2n - 2$ edges. To determine the $OEBI(W_n)$ we consider four cases, namely,

$$n \equiv 0(\text{mod } 4), \quad n \equiv 1(\text{mod } 4),$$

$$n \equiv 2(\text{mod } 4), \quad n \equiv 3(\text{mod } 4).$$

Theorem 3.1 If $n \equiv 0(\text{mod } 4)$ i.e, $n = 4k(k \in N)$, then $OEBI(W_n) = \{0, 4, 8, \ldots, n\}$.

Proof Let $f$ be an edge-friendly labeling on $W_n$. Since the graph contains $n = 4k$ vertices, $2n - 2 = 8k - 2$ edges, we must have $e(0) = e(1) = 4k - 1$. Denote the vertices on the rim of the wheel by $v_0, v_1, v_2, \cdots, v_{4k-1}$ and denote the center by $v_0$. Now we label the edges $v_qv_{q+1}$ for $1 \leq q \leq 4k - 2$ and $v_{4k-1}v_1$ by 1 and label the remaining edges by 0. Then it is easy to observe that $v(0) = 4k$ and there is no 1-vertex in the graph. Thus $|v(1) - v(0)| = 4k = n = \max\{OEBI(W_n)\}$.

Now we interchange the labels of the edges to get the remaining 1-edge balance index numbers. By interchanging the labels of edges $v_{2q-1}v_{2q}$ and $v_{2q-1}v_0, v_{2q}v_{2q+1}$ and $v_{2q}v_0$ for $1 \leq q \leq k$ we get $|v(0) - v(1)| = 4k - 4q$. Thus $0, 4, 8, \cdots, n$ are elements of $OEBI(W_n)$.
Let \( a_i = \text{card}\{v \in V \mid \text{number of 1-edges incident on } v \text{ is equal to } i\} \), \( i = 1, 2, 3, \ldots, 4k-1 \). Then we have

\[
\sum_{i=1}^{4k-1} ia_i = a_1 + 2a_2 + 3a_3 + \ldots, +(4k-1)a_{4k-1} = 8k - 2
\]

implies that \( a_1 + 3a_3 + 5a_5 + \ldots, +(4k-1)a_{4k-1} \) is even, which is possible if and only if, \( a_1 + a_3 + a_5 + \ldots, +a_{4k-1} \) is even, that is, the number of 1-vertices is even and hence the number of 0-vertices is also even. Therefore, the numbers 2, 6, 10, \ldots, \( n - 2 \) are not elements of \( OEBI(W_n) \)

**Theorem 3.2** If \( n \equiv 1(\text{mod } 4) \) i.e., \( n = 4k + 1 (k \in N) \), then \( OEBI(W_n) = \{1, 3, 5, \ldots, n\} \).

**Proof** Let \( f \) be an edge-friendly labeling on \( W_n \). Since the graph contains \( n = 4k + 1 \) vertices, \( 2n - 2 = 8k \) edges, we must have \( e(0) = e(1) = 4k \). Denote the vertices on the rim of the wheel by \( v_0, v_1, v_2, \ldots, v_{4k} \) and denote the center by \( v_0 \). Now we label the edges \( v_qv_{q+1} \) for \( 1 \leq q \leq 4k - 1 \) and \( v_{4k}v_1 \) by 1 and label the remaining edges by 0. Then it is easy to observe that \( v(0) = 4k + 1 \) and there is no 1-vertex in the graph. Thus \( |v(1) - v(0)| = 4k + 1 = n = \max\{OEBI(W_n)\} \).

Now we interchange the labels of the edges to get the remaining 1-edge balance index numbers. By interchanging the labels of edges \( v_{2q-1}v_{2q} \) and \( v_{2q-1}v_0 \), \( v_{2q}v_{2q+1} \) and \( v_{2q}v_0 \) for \( 1 \leq q \leq 2k - 1 \), we get \( |v(0) - v(1)| = |4k + 1 - 4q| \) and by interchanging the labels of edges \( v_{4k-1}v_{4k} \) and \( v_{4k-1}v_0 \), \( v_{4k}v_1 \) and \( v_{4k}v_0 \), we get \( |v(0) - v(1)| = 4k - 1 \). Thus

\[
OEBI(W_n) = \{1, 3, 5, \ldots, n\}.
\]

Similarly one can prove the following results.

**Theorem 3.3** If \( n \equiv 2(\text{mod } 4) \) i.e., \( n = 4k + 2 (k \in N) \), then \( OEBI(W_n) = \{2, 6, 10, \ldots, n\} \).

**Theorem 3.4** If \( n \equiv 3(\text{mod } 4) \) i.e., \( n = 4k + 3 (k \in N) \), then \( OEBI(W_n) = \{1, 3, 5, \ldots, n\} \).

**References**


