ON THE 82-TH SMARANDACHE’S PROBLEM

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Abstract  The main purpose of this paper is using the elementary method to study the asymptotic properties of the integer part of the $k$-th root positive integer, and give two interesting asymptotic formulae.

Keywords:  $k$-th root; Integer part; Asymptotic formula.

§ 1. Introduction And Results

For any positive integer $n$, let $s_k(n)$ denote the integer part of $k$-th root of $n$. For example, $s_k(1) = 1$, $s_k(2) = 1$, $s_k(3) = 1$, $s_k(4) = 1$, $s_k(2^k) = 2$, $s_k(2^k + 1) = 2$, $s_k(3^k) = 3$, $\ldots$. In problem 82 of [1], Professor F.Smarandache asked us to study the properties of the sequence $s_k(n)$. About this problem, some authors had studied it, and obtained some interesting results. For instance, the authors [5] used the elementary method to study the mean value properties of $\varphi(s_k(n))$, where Smarandache function $S(n)$ is defined as following:

$$S(n) = \min\{m : m \in N, n \mid m!\}.$$

In this paper, we use elementary method to study the asymptotic properties of this sequence in the following form: $\sum_{n \leq x} \frac{\varphi(s_k(n))}{s_k(n)}$ and $\sum_{n \leq x} \frac{1}{\varphi(s_k(n))}$, where $x \geq 1$ be a real number, $\varphi(n)$ be the Euler totient function, and give two interesting asymptotic formulae. That is, we shall prove the following:

Theorem 1. For any real number $x > 1$ and any fixed positive integer $k > 1$, we have the asymptotic formula

$$\sum_{n \leq x} \frac{\varphi(s_k(n))}{s_k(n)} = \frac{6}{\pi^2}x + O\left(x^{1 - \frac{1}{k} - \varepsilon}\right),$$

where $\varepsilon$ is any real number.
Theorem 2. For any real number \( x > 1 \) and any fixed positive integer \( k > 1 \), we have the asymptotic formula

\[
\sum_{n \leq x} \frac{1}{\varphi(s_k(n))} = \frac{k \zeta(2) \zeta(3)}{(k - 1) \zeta(6)} x^{1 - \frac{1}{k}} + A + O \left( x^{1 - \frac{2}{k} \log x} \right),
\]

where \( A = \gamma \sum_{n=1}^{\infty} \frac{\mu^2(n)}{n \varphi(n)} - \sum_{n=1}^{\infty} \frac{\mu^2(n) \log n}{n \varphi(n)} \).

§2. Proof of Theorems

In this section, we will complete the proof of Theorems. First we come to prove Theorem 1. For any real number \( x > 1 \), let \( M \) be a fixed positive integer with \( M^k \leq x \leq (M + 1)^k \), from the definition of \( s_k(n) \) we have

\[
\sum_{n \leq x} \frac{\varphi(s_k(n))}{s_k(n)} = \sum_{t=1}^{M} \sum_{(t-1)^k \leq n < t^k} \frac{\varphi(s_k(n))}{s_k(n)} + \sum_{M^k \leq n < x} \frac{\varphi(s_k(n))}{s_k(n)}
\]

\[
= \sum_{t=1}^{M-1} \sum_{t^k \leq n < (t+1)^k} \frac{\varphi(s_k(n))}{s_k(n)} + \sum_{M^k \leq n \leq x} \frac{\varphi(M)}{M}
\]

\[
= \sum_{t=1}^{M-1} \left[ (t + 1)^k - t^k \right] \frac{\varphi(t)}{t} + O \left( \sum_{M^k \leq n < (M + 1)^k} \frac{\varphi(M)}{M} \right)
\]

\[
= k \sum_{t=1}^{M} t^{k-1} \frac{\varphi(t)}{t} + O \left( M^{k-1} \right),
\]

where we have used the estimate \( \frac{\varphi(n)}{n} \ll n^{-\epsilon} \).

Note that (see reference [3])

\[
\sum_{n \leq x} \frac{\varphi(n)}{n} = \frac{6}{\pi^2} x + O \left( (\log x)^{\frac{3}{2}} (\log \log x)^{\frac{1}{2}} \right).
\]

Let \( B(y) = \sum_{t \leq y} \frac{\varphi(t)}{t} \), then by Abel’s identity (see Theorem 4.2 of [2]) and (2), we can easily deduce that

\[
\sum_{t=1}^{M} t^{k-1} \frac{\varphi(t)}{t} = M^{k-1} B(M) - B(1) - (k - 1) \int_{1}^{M} y^{k-2} B(y) dy
\]

\[
= M^{k-1} \left( \frac{6}{\pi^2} M + O \left( (\log M)^{\frac{3}{2}} (\log \log M)^{\frac{1}{2}} \right) \right)
\]

\[
- (k - 1) \int_{1}^{M} y^{k-2} \left( \frac{6}{\pi^2} y + O \left( (\log y)^{\frac{3}{2}} (\log \log y)^{\frac{1}{2}} \right) \right) dy
\]

\[
= \frac{6}{k \pi^2} M^k + O \left( (\log M)^{\frac{3}{2}} (\log \log M)^{\frac{1}{2}} \right).
\]
Applying (1) and (3) we can obtain the asymptotic formula

$$\sum_{n \leq x} \frac{\varphi(s_k(n))}{s_k(n)} = \frac{6}{\pi^2} M^k + O \left( M^{k-1-\varepsilon} \right).$$

(4)

On the other hand, note that the estimate

$$0 \leq x - M^k < (M + 1)^k - M^k \ll x^{\frac{k-1}{k}}$$

(5)

Now combining (4) and (5) we can immediately obtain the asymptotic formula

$$\sum_{n \leq x} \frac{\varphi(s_k(n))}{s_k(n)} = \frac{6}{\pi^2} x + O \left( x^{1-\frac{1}{k}+\varepsilon} \right).$$

This proves Theorem 1.

Similarly, note that (see reference [4])

$$\sum_{n \leq x} \frac{1}{\varphi(n)} = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \log x + A + O \left( \frac{\log x}{x} \right),$$

where

$$A = \gamma \sum_{n=1}^{\infty} \frac{\mu^2(n)}{n \varphi(n)} - \sum_{n=1}^{\infty} \frac{\mu^2(n) \log n}{n \varphi(n)}.$$

We can use the same method to obtain the result of Theorem 2.

References


