ON $k$-FACTORIALS AND SMARANDACHEIALS

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Abstract
F. Smarandache defines a $k$-factorial as $n(n - k)(n - 2k)\cdots$, terminating when $n - xk$ is positive and $n - (x + 1)k$ is 0 or negative. Smarandacheials extend this definition into the negative numbers such that the factorial terminates when $|n - xk|$ is less than or equal to $n$ and $|n - (x + 1)k|$ is greater than $n$. This paper looks at some relations between these numbers.

Keywords: Smarandache function; Additive Analogue; Mean value formula.

$k$-factorials
We begin by looking at the $k$-factorial, represented by $k$ exclamation marks after the variable.

The $k$-factorial is merely the full factorial $n!$ with some of the terms omitted.

e.g. the factorial for $n = 8$;

$n! = 8! = 8.7.6.5.4.3.2.1$

If we look at the 2-factorial $n!!$, we have

$n!! = 8.6.4.2$

and we see that 7, 5, 3 and 1 are not present.

Similarly the 3-factorial $n!!!$ gives

$n!!! = 8.5.2$

and 7, 6, 4, 2 and 1 are not present.

In the first case we have 7!! omitted, so we may write $n!! = n! \mid (n - 1)!!$

For the 3-factorial, there are two sequences present, 7!!! and 6!!!!, so $n!!!! = n! \mid [(n - 1)!!!(n - 2)!!!]$

Using the notation $n!_k$ for a $k$-factorial, we can easily obtain the general formula

$$n!_k = \frac{n!}{\prod_{i=1}^{k-1}(n-k)_k}$$

A PARI/GP program to implement this is

```
{kfactorial(n, k)=local(result);
 result=n!;
 for (i = 1, k - 1,result/=kfactorial(n - i, k));
 result}
```
although this is highly ineffective and not recommended for use.

To access the $k$-factorial function use

```pascal
{kfactorial(n, k)=local(res);
 res=vector(n, i, if (i <= k, i, 0));
 for (i = k + 1, n, res[i]=i*res[i - k]);
 res[n];
}
```

This code stores 1 to $k$ in a vector in positions 1 to $k$. Then each progressive
term is calculated from the $k$-th previous entry and the current one. The above
code actually calculates $n!_k$ for all $n$ from 1 to $n$.

Even simpler, and the quickest yet is

```pascal
kf(n,k)=local(r,c);
 c=n
 if (c==0,c=k);
 r=c;
 while (c<n,c+=k;r*=c);
 r
```

The $r$ variable holds the result, and the $c$ variable is a counter. $c$ is set to $n$
mod $k$, and then incremented until it is $n$. $r$ is multiplied by $c$ at each stage.

We can also see that if $gcd(n, k) = k$, then $n!_k = k^{\frac{n}{k}}(\frac{n}{k})!$, so in this case we have

$$k^{\frac{n}{k}}(\frac{n}{k})! \prod_{i=1}^{k-1} (n - k)_k! = n!$$

If $gcd(n, k)! = k$, then the $k$-factorial seems more difficult to define. We
address this problem shortly.

**Smarandacheials**

In extending the $k$-factorial to the negative integers, we need to further de-
define the parameters involved.

If we let $n$ be the starting number, and $k$ be the decrease, then we also need
to define $m$ as $n \mod k$, and then $m' = k - m$.

If $m$ is greater than or equal to $m'$, we can see that $SM(n, k) = \pm [n!_k * (n - (m - m'))]_k$.

If $m$ is less than $m'$, then we have $SM(n, k) = \pm [n!_k * (n - (m - m') - k)]_k$.

The plus/minus sign is not known yet - this is developed later in this paper.

This result follows because $m$ represents the last integer from $n!_k$, and so
$k - m$ will be the first negative integer from $(n - x)!_k$, and so we determine $x$.

If $m$ is greater than or equal to $m'$, then the difference $n - (n - x)$ must be
the difference between $m$ and $m'$, so $x = m - m'$.

If $m < m'$, then we have a problem. The basic idea still works, however the
negative factorial will rise to a higher level than the original $n$, and this is not
allowed. So the adjustment from subtracting $k$ cuts the last negative integer out of the equation.

To combine these, define $m^*$ as the smallest positive value of $(m - m') \mod k$, and now we may write:

$$SM(n, k) = \pm [n!_k(n - m^*)].$$

For an example, consider $SM(13, 5)$. Then $m = 3$ and $m' = 2$, and $m^* = (3 - 2) \mod 5 = 1$, so we get:

$$SM(13, 5) = \pm 13!_5^*(13 - 1)!_5$$
$$= \pm 13!_5^*(12)!_5$$
$$= \pm 13.8.3\cdot12.7.2$$

However, for $SM(12, 5)$, $m^* = (2 - 3) \mod 5 = 4$

$$= \pm 12!_5^*(12 - 4)!_5$$
$$= \pm 12!_5^*8!_5$$
$$= \pm 12.7.2^8.3$$

The sign is then simply $(-1)^{\text{number of terms in second Smarandacheial}}$.

**Number of terms in $n!_k$**

Given $n!_k$, the expansion of the expression is:

$n(n - k) \cdots (n - ak)$

So there are $a + 1$ terms.

Using a simple example, e.g. for $k = 5$, we can construct a table of the number of terms;

<table>
<thead>
<tr>
<th>n</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>no. of terms</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>n</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>no. of terms</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

From this we see that there are $\lceil \frac{n}{k} \rceil$ terms.

Therefore a full expression for the Smarandacheial function is

$$SM(n, k) = (-1)^{\lceil \frac{n - m^*}{k} \rceil} n!_k(n - m^*)!_k$$

$n!_k$ for $\gcd(n, k) < k$

In this case, we have no easy relation. We can however spot an interesting and deep relation with these $k$-factorials - their relation to a neighbouring $n!_k$ with $\gcd(n, k) = k$.

To demonstrate this connection, we will examine $15!_5$.

This is $15.10.5 = 750$. 

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**On $k$-factorials and Smarandacheials**

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Now $16!_5$ is 16.11.6.1. There seems to be nothing else we can do. However, we can write this as:

$$\left(\frac{16}{15}\right)\left(\frac{11}{10}\right)\left(\frac{6}{5}\right)15!_5$$

Still nothing, but then we see that $\frac{16}{15} = 1 + \frac{1}{15}$, and so on, and so we get:

$$\left(1 + \frac{1}{15}\right)\left(1 + \frac{1}{10}\right)\left(1 + \frac{1}{5}\right)15!_5$$

If we expand the brackets, hey presto (I have skipped a few steps here)

$$16!_5 = 15!_5 + 15.10 + 15.5 + 10.5 + 15 + 10 + 5 + 1$$

This is generalizable into

$$(n + x)!_k = \left[n!_k + \sum_{d \in S} \frac{x^d n!_k}{d}\right] x$$

where $x$ is less than $k$, $S$ is the distinct power set of components of $n!_k$ (e.g. for $15!_5$, $S = 15, 10, 5$), and $d_i$ is the number of elements of $S$ involved in $d$.

For example, $18!_5$ gives

$$18!_5 = [15!_5 + 3(15.10 + 15.5 + 10.5) + 9(15 + 10 + 5) + 27] \times 3$$

$18!_6 = 5616$, and the sum on the RHS is $750 + 825 + 270 + 27 = 1872$, and $1872 \times 3 = 5616$.

References