The Forcing Domination Number of Hamiltonian Cubic Graphs

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Abstract: A set of vertices S in a graph G is called to be a Smarandachely dominating k-set, if each vertex of G is dominated by at least k vertices of S. Particularly, if k = 1, such a set is called a dominating set of G. The Smarandachely domination number γ_k(G) of G is the minimum cardinality of a Smarandachely dominating set of G. For abbreviation, we denote γ_1(G) by γ(G). In 1996, Reed proved that the domination number γ(G) of every n-vertex graph G with minimum degree at least 3 is at most 3n/8. Also, he conjectured that γ(H) ≥ ⌈n/3⌉ for every connected 3-regular n-vertex graph H. In [?], the authors presented a sequence of Hamiltonian cubic graphs whose domination numbers are sharp and in this paper we study forcing domination number for those graphs.

Key Words: Smarandachely dominating k-set, dominating set, forcing domination number, Hamiltonian cubic graph.

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§1. Introduction

Throughout this paper, all graphs considered are finite, undirected, loopless and without multiple edges. We refer the reader to [12] for terminology in graph theory.

Let G be a graph, with n vertices and e edges. Let N(v) be the set of neighbors of a vertex v and N[v] = N(v) ∪ {v}. Let d(v) = |N(v)| be the degree of v. A graph G is r-regular if d(v) = r for all v. Particularly, if r = 3 then G is called a cubic graph. A vertex in a graph G dominates itself and its neighbors. A set of vertices S in a graph G is called to be a Smarandachely dominating k-set, if each vertex of G is dominated by at least k vertices of S. Particularly, if k = 1, such a set is called a dominating set of G. The Smarandachely domination number γ_k(G) of G is the minimum cardinality of a Smarandachely dominating set of G. For abbreviation, we denote γ_1(G) by γ(G). A subset F of a minimum dominating set S is a forcing subset for S if S is the unique minimum dominating set containing F. The forcing domination number f(G, γ) of S is the minimum cardinality among the forcing subsets of S, and the forcing domination number f(G, γ) of G is the minimum forcing domination number among...
the minimum dominating sets of $G$ ([1], [2], [5]-[7]). For every graph $G$, $f(G, \gamma) \leq \gamma(G)$. Also, the forcing domination number of several classes of graphs are determined, including complete multipartite graphs, paths, cycles, ladders and prisms. The forcing domination number of the cartesian product $G$ of $k$ copies of the cycle $C_{2k+1}$ is studied.

The problem of finding the domination number of a graph is NP-hard, even when restricted to cubic graphs. One simple heuristic is the greedy algorithm, ([11]). Let $d_g$ be the size of the dominating set returned by the greedy algorithm. In 1991 Parekh [9] showed that $d_g \leq n + 1 - \sqrt{2n+1}$. Also, some bounds have been discovered on $\gamma(G)$ for cubic graphs. Reed [10] proved that $\gamma(G) \leq \frac{2}{3}n$. He conjectured that $\gamma(H) \geq \left[\frac{n}{4}\right]$ for every connected $3$-regular (cubic) $n$-vertex graph $H$. Reed’s conjecture is obviously true for Hamiltonian cubic graphs. Fisher et al. [3]-[4] repeated this result and showed that if $G$ has girth at least 5 then $\gamma(G) \leq \frac{n}{14}n$. In the light of these bounds on $\gamma$, in 2004 Seager considered bounds on $d_g$ for cubic graphs and showed that ([11]):

For any graph of order $n$, $\left[\frac{n}{1+\Delta G}\right] \leq \gamma(G)$ (see [4]) and for a cubic graph $G$, $d_g \leq \frac{4}{9}n$.

In this paper, we would like to study the forcing domination number for Hamiltonian cubic graphs. In [8], the authors showed that:

Lemma A. If $r \equiv 2$ or $3 \pmod{4}$, then $\gamma(G') = \gamma(G)$.

Lemma B. If $r \equiv 0$ or $1 \pmod{4}$, then $\gamma(G') = \gamma(G) - 1$.

Theorem C. If $r \equiv 1 \pmod{4}$, then $\gamma(G_0) = m \left[\frac{n}{4}\right] - \left[\frac{m}{4}\right]$.

§2. Forcing domination number

Remark 2.1 Let $G = (V, E)$ be the graph with $V = \{v_1, v_2, ..., v_n\}$ for $n = 2r$ and $E = \{v_i v_j \mid |i-j| = 1 \text{ or } r\}$. So $G$ has two vertices $v_1$ and $v_n$ of degree two and $n-2$ vertices of degree three. By the graph $G$ is the graph described in Fig.1.

![Fig.1. The graph G.](image)

For the following we put $N_p[x] = \{z \mid z \text{ is only dominated by } x\} \cup \{x\}$.

Remark 2.2 Suppose that the graphs $G'$ and $G''$ are two induced subgraphs of $G$ such that $V(G') = V(G) - \{v_1, v_n\}$ and $V(G'') = V(G) - \{v_1\}$ (or $V(G'') = V(G) - \{v_2r\}$).

Remark 2.3 Let $G_0$ be a graph of order $mn$ that $n = 2r$, $V(G_0) = \{v_{11}, v_{12}, ..., v_{1n}, v_{21}, v_{22}, ..., v_{2n}, v_{m1}, v_{m2}, ..., v_{mn}\}$ and $E = \bigcup_{i=1}^{m} \{v_{ij} v_{il} \mid j - l = 1 \text{ or } r\} \cup \{v_{in} v_{i+1,1} \mid i = 1, 2, ..., m-1\} \cup \{v_{11} v_{mn}\}$. By the graph $G_0$ is 3-regular graph. Suppose that the graph $G_1$
is an induced subgraph of $G_0$ with the vertices $v_{i1}, v_{i1}, ..., v_{in}$. By the graph $G_0$ is the graph described in Fig. 2.

![Fig. 2. The graph $G_0$.](image)

**Proposition 2.4** If $r \equiv 0 \pmod{4}$, then $f(G, \gamma) \leq 2$, otherwise $f(G, \gamma) = 1$.

**Proof** First we suppose that $r \equiv 1 \pmod{4}$. It is easy to see that $f(G, \gamma) > 0$, because $G$ has at least two minimum dominating set. Suppose $F = \{v_1\} \subset S$ where $S$ is a minimum dominating set. Since $\gamma(G) = 2\lfloor r/4 \rfloor + 1$, for two vertices $v_x$ and $v_y$ in $S$, $|N[v_x] \cup N[v_y]| \geq 6$. This implies that $\{v_2, v_{r+1}\} \cap S = \emptyset$, then $v_{r+3} \in S$. A same argument shows that $v_5 \in S$. Thus $S$ must be contains $\{v_{r+7}, v_9, ..., v_{2r-2}, v_r\}$, therefore $f(G, \gamma) = 1$.

If $r \equiv 2 \pmod{4}$, we consider $S = \{v_2, v_6, v_{10}, ..., v_r, v_{r+4}, v_{r+8}, ..., v_{2r-6}, v_{2r-2}\}$. Assign the set $F = \{v_2\}$ then it follows $f(G, \gamma) \leq 1$, because $|N_p[x]| = 4$ to each vertex $x \in S$. On the other hand since $G$ has at least two minimum dominating set. Hence $f(G, \gamma) = 1$.

If $r \equiv 3 \pmod{4}$, for $S = \{v_1, v_5, v_9, ..., v_{r-2}, v_{r+3}, v_{r+7}, ..., v_{2r-4}, v_{2r}\}$, the set $F = \{v_1\}$ shows that $f(G, \gamma) \leq 1$. Further, since $G$ has at least two minimum dominating set, then it follows $f(G, \gamma) = 1$.

Finally let $r \equiv 0 \pmod{4}$, we consider $S = \{v_{i1}, v_5, v_9, ..., v_{r-2}, v_{r+3}, v_{r+7}, ..., v_{2r-5}, v_{2r-1}\}$. If $F = \{v_{i1}, v_{r+1}\}$, a simple verification shows that $f(G, \gamma) \leq 2$. \hfill \Box

**Proposition 2.5** If $r \equiv 1 \pmod{4}$ then $f(G', \gamma) = 0$.

**Proof** By Lemma 6, we have $\gamma(G') = 2\lfloor r/4 \rfloor$. Now, we suppose that $S$ is an arbitrary minimum dominating set for $G'$. Obviously for each vertex $v_x \in S$, $|N_p[v_x]| = 4$, so $\{v_{r-1}, v_{r+2}\} \subset S$. But $\{v_{2r-2}, v_{r-2}\} \cap S = \emptyset$ therefore $v_{2r-3} \in S$. Thus $S$ must be contains $\{v_{r-5}, v_{r-9}, ..., v_{r+10}, v_{r+6}\}$, then $S$ is uniquely determined and it follows that $f(G', \gamma) = 0$. \hfill \Box

**Proposition 2.6** If $r \equiv 0 \pmod{4}$ then $f(G'', \gamma) = 0$.

**Proof** Let $r \equiv 0 \pmod{4}$ and $S$ be an arbitrary minimum dominating set for $G''$ with $V(G'') = V(G) - \{v_1\}$. If $\{v_{2r}, v_{2r-1}\} \cap S \neq \emptyset$. Without loss of generality, we assume that $v_{2r} \in S$ then $S$ must be contains $\{v_{r+2}, v_{r-2}, v_{r-6}, ..., v_{10}, v_6, v_{2r-4}, v_{2r-8}, ..., v_{r+8}\}$. On the other hand by Lemma 6, $\gamma(G'') = 2\lfloor r/4 \rfloor$ (Note that by Proof of Lemma 6 one can see
\( \gamma(G') = \gamma(G'') \) where \( r \equiv 0 \pmod{4} \). So the vertices \( v_3, v_4, v_{r+4} \) and \( v_{r+5} \) must be dominated by one vertex and this is impossible. Thus necessarily \( v_r \in S \), but \( \{v_{r-1}, v_{2r-1}\} \cap S = \emptyset \) which implies \( v_{2r-2} \in S \). Finally the remaining non-dominated vertices \( \{v_{r+1}, v_{r+2}, v_2\} \) is just dominated by \( v_{r+2} \). Therefore the set \( S = \{v_4, v_8, \ldots, v_{r-4}, v_r, v_{r+2}, v_{r+6}, \ldots, v_{2r-2}\} \) is uniquely determined which implies \( f(G'', \gamma) = 0 \). □

§3. Main Results

**Theorem 3.1** If \( r \equiv 2 \) or \( 3 \pmod{4} \), then \( f(G_0, \gamma) = m \).

*Proof* Let \( r \equiv 2 \pmod{4} \) and \( S \) be a minimum dominating set for \( G_0 \). If there exists \( i \in \{1, 2, \ldots, m\} \) such that \( S \cap \{v_{i}, v_{in}\} \neq \emptyset \) then it implies \( |S \cap G_i| > 2 |r/4| + 1 \). Moreover \( \gamma(G_0) = m(2 |r/4| + 1) \). From this it immediately follows that there exists \( j \in \{1, 2, \ldots, m\} - \{i\} \) such that \( |S \cap G_j| < 2 |r/4| + 1 \) and this is contrary to Lemma A. Hence \( S \cap \{v_{i1}, v_{in}\} = \emptyset \) for \( 1 \leq i \leq m \). On the other hand \( f(G_i, \gamma) = 1 \) for \( 1 \leq i \leq m \) which implies \( f(G_0, \gamma) = m \).

Now we suppose that \( r \equiv 3 \pmod{4} \) and \( S \) is minimum dominating set for \( G_0 \), such that \( F = \{v_{i1} \mid 1 \leq i \leq m\} \subseteq S \). Since \( v_{i1} \in S \) and \( \gamma(G_0) = 2 |r/4| + 2 \) then \( \{v_{i2}, v_{i3}\} \cap S = \emptyset \) and this implies \( v_{i(r+3)} \in S \). With similar description, we have \( \{v_{i5, v_{i9}}, \ldots, v_{i(r-2), v_{i(r+6)}, v_{i(r+11)}, \ldots, v_{i(2r-4)}\} \subseteq S \). But for the remaining non-dominated vertices \( v_{ir}, v_{i(2r)} \) and \( v_{i(2r-1)} \) necessarily implies that \( v_{i(2r)} \in S \). Hence \( S \) is the unique minimum dominating set containing \( F \). Thus \( f(G_0, \gamma) \leq m \). A trivial verification shows that \( f(G', \gamma), f(G'', \gamma) \geq 1 \) for \( i \in \{1, 2, \ldots, m\} \), therefore \( f(G_0, \gamma) = m \). □

**Theorem 3.2** \( f(G_0, \gamma) = \begin{cases} 1 & \text{if } m \equiv 0 \pmod{3} \\ 2 & \text{otherwise} \end{cases} \) for \( r \equiv 1 \pmod{4} \).

*Proof* If \( m \equiv 0 \pmod{3} \), we suppose that \( F = \{v_{i1}\} \subseteq S \) and \( S \) is a minimum dominating set for \( G_0 \). By Theorem C, we have \( \gamma(G_0) = m \lceil n/4 \rceil - \lfloor n/3 \rfloor \), then \( v_{3i1} \in S \). Here, we use the proof of Propositions 4 and 5. From this the sets \( S \cap V(G_1), S \cap V(G_2), S \cap V(G_3) \) uniquely characterize. By continuing this process the set \( S \) uniquely obtain, then \( f(G_0, \gamma) = 1 \).

If \( m \equiv 1 \) or \( 2 \pmod{3} \), then the set \( F = \{v_{i1}, v_{mn}\} \) uniquely characterize the minimum dominating set for \( G_0 \), therefore \( f(G_0, \gamma) = 2 \). □

**Theorem 3.3** \( f(G_0, \gamma) = \begin{cases} \lceil \frac{m}{4} \rceil + 1 & \text{if } m \equiv 0 \pmod{3} \\ \lfloor \frac{m}{3} \rfloor + 3 & \text{otherwise} \end{cases} \) for \( r \equiv 0 \pmod{4} \).

*Proof* If \( m \equiv 0 \pmod{3} \) the set \( F = \{v_{21}, v_{2(r+4)}, v_{5(r+4)}, v_{8(r+4)}, \ldots, v_{m-1(r+4)}\} \) determine the unique minimum dominating set for \( G_0 \) then \( f(G_0, \gamma) \leq \lfloor m/3 \rfloor + 1 \). But \( \gamma(G_i) = 2 \lfloor r/4 \rfloor \) for \( \lfloor m/3 \rfloor \) of \( G_i \)'s. Hence \( f(G_0, \gamma) = \lfloor m/3 \rfloor + 1 \). The proof of the case \( m \equiv 1 \) or \( 2 \pmod{3} \) is similar to the previous case. □
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