Forcing (G,D)-number of a Graph

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Abstract: In [7], we introduced the new concept (G,D)-set of graphs. Let $G = (V, E)$ be any graph. A (G,D)-set of a graph $G$ is a subset $S$ of vertices of $G$ which is both a dominating and geodominating (or geodetic) set of $G$. The minimum cardinality of all (G,D)-sets of $G$ is called the (G,D)-number of $G$ and is denoted by $\gamma_{G}(G)$. In this paper, we introduce a new parameter called forcing (G,D)-number of a graph $G$. Let $S$ be a $\gamma_{G}$-set of $G$. A subset $T$ of $S$ is said to be a forcing subset for $S$ if $S$ is the unique $\gamma_{G}$-set of $G$ containing $T$. A forcing subset $T$ of $S$ of minimum cardinality is called a minimum forcing subset of $S$. The forcing (G,D)-number of $S$ denoted by $f_{G,D}(S)$ is the cardinality of a minimum forcing subset of $S$. The forcing (G,D)-number of $G$ is the minimum of $f_{G,D}(S)$, where the minimum is taken over all $\gamma_{G}$-sets $S$ of $G$ and it is denoted by $f_{G,D}(G)$.

Key Words: (G,D)-number, Forcing (G,D)-number, Smarandachely $k$-dominating set.

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§1. Introduction

By a graph $G= (V, E)$, we mean a finite, undirected connected graph without loops and multiple edges. For graph theoretic terminology, we refer [5]. A set of vertices $S$ in a graph $G$ is said to be a Smarandachely $k$-dominating set if each vertex of $G$ is dominated by at least $k$ vertices of $S$. Particularly, if $k = 1$, such a set is called a dominating set of $G$, i.e., every vertex in $V - D$ is adjacent to at least one vertex in $D$. The minimum cardinality among all dominating sets of $G$ is called the domination number $\gamma(G)$ of $G$[6]. A u-v geodesic is a u-v path of length $d(u,v)$. A set $S$ of vertices of $G$ is a geodominating (or geodetic) set of $G$ if every vertex of $G$ lies on an x-y geodesic for some x,y in $S$. The minimum cardinality of a geodominating set is the geodomination (or geodetic) number of $G$ and it is denoted by $g(G)$[1-4]. A (G,D)-set of $G$ is a subset $S$ of $V(G)$ which is both a dominating and geodetic set of $G$. The minimum cardinality of all (G,D)-sets of $G$ is called the (G,D)-number of $G$ and is denoted by $\gamma_{G}(G)$.

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Any (G,D)-set of G of cardinality $\gamma_G$ is called a $\gamma_G$-set of G[7]. In this paper, we introduce a new parameter called forcing (G,D)-number of a graph G. Let S be a $\gamma_G$-set of G. A subset T of S is said to be a forcing subset for S if S is the unique $\gamma_G$-set of G containing T. A forcing subset T of S of minimum cardinality is called a minimum forcing subset of S. The forcing (G,D)-number of S denoted by $f_{G,D}(S)$ is the cardinality of a minimum forcing subset of S. The forcing (G,D)-number of G is the minimum of $f_{G,D}(S)$, where the minimum is taken over all $\gamma_G$-sets S of G and it is denoted by $f_{G,D}(G)$.

§2. Forcing (G,D)-number

**Definition 2.1** Let G be a connected graph and S be a $\gamma_G$-set of G. A subset T of S is called a forcing subset for S if S is the unique $\gamma_G$-set of G containing T. A forcing subset T of S of minimum cardinality is called a minimum forcing subset for S. The forcing (G,D)-number of S denoted by $f_{G,D}(S)$ is the cardinality of a minimum forcing subset of S. The forcing (G,D)-number of G is the minimum of $f_{G,D}(S)$, where the minimum is taken over all $\gamma_G$-sets S of G and it is denoted by $f_{G,D}(G)$. That is, $f_{G,D}(G) = \min\{f_{G,D}(S) : S \text{ is any } \gamma_G\text{-set of } G\}$.

**Example 2.2** In the following figure,

![Graph](image)

Fig.2.1

$S_1 = \{u, x\}$ and $S_2 = \{v, y\}$ are the only two $\gamma_G$-sets of G. $\{u\}, \{x\}$ and $\{u, x\}$ are forcing subsets of $S_1$. Therefore, $f_{G,D}(S_1) = 1$. Similarly, $\{v\}, \{y\}$ and $\{v, y\}$ are the forcing subsets of $f_{G,D}(S_2)$. Therefore, $f_{G,D}(S_2) = 1$. Hence $f_{G,D}(G) = \min\{1, 1\} = 1$. For G, we have, $0 < f_{G,D}(G) = 1 < \gamma_G(G) = 2$.

**Remark 2.3** 1. For every connected graph G, $0 \leq f_{G,D}(G) \leq \gamma_G(G)$.

2. Here the lower bound is sharp, since for any complete graph $S = V(G)$ is a unique $\gamma_G$-set. So, $T = \emptyset$ is a forcing subset for S and $f_{G,D}(K_p) = 0$.

3. Example 2.2 proves the bounds are strict.

**Theorem 2.4** Let G be a connected graph. Then,

(i) $f_{G,D}(G) = 0$ if and only if G has a unique $\gamma_G$-set;

(ii) $f_{G,D}(G) = 1$ if and only if G has at least two $\gamma_G$-sets, one of which, say, S has forcing (G,D)-number equal to 1;
(iii) \( f_{G,D}(G) = \gamma_G(G) \) if and only if every \( \gamma_G \)-set \( S \) of \( G \) has the property, \( f_{G,D}(S) = |S| = \gamma_G(G) \).

**Proof** (i) Suppose \( f_{G,D}(G) = 0 \). Then, by Definition 2.1, \( f_{G,D}(S) = 0 \) for some \( \gamma_G \)-set \( S \) of \( G \). So, empty set is a minimum forcing subset for \( S \). But, empty set is a subset of every set. Therefore, by Definition 2.1, \( S \) is the unique \( \gamma_G \)-set of \( G \). Conversely, let \( S \) be the unique \( \gamma_G \)-set of \( G \). Then, empty set is a minimum forcing subset of \( S \). So, \( f_{G,D}(G) = 0 \).

(ii) Assume \( f_{G,D}(G) = 1 \). Then, by (i), \( G \) has at least two \( \gamma_G \)-sets. \( f_{G,D}(G) = \min\{f_{G,D}(S) : S \text{ is any } \gamma_G \text{-set of } G \} \). So, \( f_{G,D}(S) = 1 \) for at least one \( \gamma_G \)-set \( S \). Conversely, suppose \( G \) has at least two \( \gamma_G \)-sets satisfying the given condition. By (i), \( f_{G,D}(G) \neq 0 \). Further, \( f_{G,D}(G) \geq 1 \). Therefore, by assumption, \( f_{G,D}(G) = 1 \).

(iii) Let \( f_{G,D}(G) = \gamma_G(G) \). Suppose \( S \) is a \( \gamma_G \)-set of \( G \) such that \( f_{G,D}(S) < |S| = \gamma_G(G) \). So, \( S \) has a forcing subset \( T \) such that \(|T| < |S|\). Therefore, \( f_{G,D}(G) = \min\{f_{G,D}(S) : S \text{ is a } \gamma_G \text{-set of } G \} \leq |T| < |S| = \gamma_G(G) \). This is a contradiction. So, every \( \gamma_G \)-set \( S \) of \( G \) satisfies the given condition. The converse is obvious. Hence the result. \( \square \)

**Corollary 2.5** \( f_{G,D}(P_n) = 0 \) if \( n \equiv 1(\text{mod}3) \).

**Proof** Let \( P_n = (v_1, v_2, \ldots, v_{3k+1}) \), \( k \geq 0 \). Now, \( S = \{v_1, v_4, v_7, \ldots, v_{3k+1}\} \) is the unique \( \gamma_G \)-set of \( P_n \). So, by Theorem 2.4, \( f_{G,D}(P_n) = 0 \). \( \square \)

**Observation 2.6** Let \( G \) be any graph with at least two \( \gamma_G \)-sets. Suppose \( G \) has a \( \gamma_G \)-set \( S \) satisfying the following property:

\( S \) has a vertex \( u \) such that \( u \in S' \) for every \( \gamma_G \)-set \( S' \) different from \( S \) \( (I) \).

Then, \( f_{G,D}(G) = 1 \).

**Proof** As \( G \) has at least two \( \gamma_G \)-sets, by Theorem 2.4, \( f_{G,D}(G) \neq 0 \). If \( G \) satisfies \( (I) \), then we observe that \( f_{G,D}(S) = 1 \). So, by Definition 2.1, \( f_{G,D}(G) = 1 \). \( \square \)

**Corollary 2.7** Let \( G \) be any graph with at least two \( \gamma_G \)-sets. Suppose \( G \) has a \( \gamma_G \)-set \( S \) such that \( S \cap S' = \emptyset \) for every \( \gamma_G \)-set \( S' \) different from \( S \). Then \( f_{G,D}(G) = 1 \).

**Proof** Given that \( G \) has a \( \gamma_G \)-set \( S \) such that \( S \cap S' = \emptyset \) for every \( \gamma_G \)-set \( S' \) different from \( S \). Then, we observe that \( S \) satisfies property \( (I) \) in Observation 2.6. Hence, we have, \( f_{G,D}(G) = 1 \). \( \square \)

**Corollary 2.8** Let \( G \) be any graph with at least two \( \gamma_G \)-sets. If pair wise intersection of distinct \( \gamma_G \)-sets of \( G \) is empty, then \( f_{G,D}(G) = 1 \).

**Proof** The proof proceeds along the same lines as in Corollary 2.7. \( \square \)

**Corollary 2.9** \( f_{G,D}(C_n) = 1 \) if \( n = 3k \), \( k > 1 \).

**Proof** Let \( n = 3k \), \( k > 1 \). Let \( V(C_n) = \{v_1, v_2, \ldots, v_{3k}\} \). Note that the only \( \gamma_G \)-sets of \( C_n \) are \( S_1 = \{v_1, v_4, \ldots, v_{3(k-1)+1}\} \), \( S_2 = \{v_2, v_5, \ldots, v_{3(k-1)+2}\} \) and \( S_3 = \{v_3, v_6, \ldots, v_{3k}\} \).
Further, we have, $S_1 \cap S_2 = S_1 \cap S_3 = S_2 \cap S_3 = \emptyset$. That is, pair wise intersection of distinct $\gamma_G$-sets of $C_n$ is empty. Hence, from Corollary 2.8, we have $f_{G,D}(C_n) = 1$ if $n = 3k$. \hfill \Box

**Definition 2.10** A vertex $v$ of $G$ is said to be a $(G,D)$-vertex of $G$ if $v$ belongs to every $\gamma_G$-set of $G$.

**Remark 2.11** 1. All the extreme vertices of a graph $G$ are $(G,D)$-vertices of $G$.

2. If $G$ has a unique $\gamma_G$-set $S$, then every vertex of $S$ is a $(G,D)$-vertex of $G$.

**Lemma 2.12** Let $G = (V,E)$ be any graph and $u \in V(G)$ be a $(G,D)$-vertex of $G$. Suppose $S$ is a $\gamma_G$-set of $G$ and $T$ is a minimum forcing subset of $S$, then $u \notin T$.

**Proof** Since $u$ is a $(G,D)$-vertex of $G$, $u$ is in every $\gamma_G$-set of $G$. Given that $S$ is a $\gamma_G$-set of $G$ and $T$ is a minimum forcing subset of $S$, Suppose $u \in T$. Then, there exists a $\gamma_G$-set $S'$ of $G$ different from $S$ such that $T - \{u\} \subseteq S'$. Otherwise, $T - \{u\}$ is a forcing subset of $S$. Since $u \in S'$, $T \subseteq S'$. This contradicts the fact that $T$ is a minimum forcing subset of $S$. Hence, from the above arguments, we have $u \notin T$. \hfill \Box

**Corollary 2.13** Let $W$ be the set of all $(G,D)$-vertices of $G$. Suppose $S$ is a $\gamma_G$-set of $G$ and $T$ is a forcing subset of $S$. If $W$ is non-empty, then $T \neq S$.

**Definition 2.14** Let $G$ be a connected graph and $S$ be a $\gamma_G$-set of $G$. Suppose $T$ is a minimum forcing subset of $S$. Let $E = S - T$ be the relative complement of $T$ in its relative $\gamma_G$-set $S$. Then, $\mathcal{L}$ is defined by

$$
\mathcal{L} = \{E | E \text{ is a relative complement of a minimum forcing subset } T \text{ in its relative } \gamma_G - \text{ set } S \text{ of } G\}.
$$

**Theorem 2.15** Let $G$ be a connected graph and $\zeta = \text{The intersection of all } E \in \mathcal{L}$. Then, $\zeta$ is the set of all $(G,D)$-vertices of $G$.

**Proof** Let $W$ be the set of all $(G,D)$-vertices of $G$.

**Claim** $W = \zeta$, the intersection of all $E \in \mathcal{L}$. Let $v \in W$. By Definition 2.10, $v$ is in every $\gamma_G$-set of $G$. Let $S$ be a $\gamma_G$-set of $G$ and $T$ be a minimum forcing subset of $S$. Then, $v \in S$.

From Lemma 2.12, we have, $v \notin T$. So, $v \in E = S - T$. Hence, $v \in E$ for every $E \in \mathcal{L}$. That is, $v \in \zeta$. Conversely, let $v \in \zeta$. Then, $v \in E = S - T$, where $T$ is a minimum forcing subset of the $\gamma_G$-set $S$. So, $v \in S$ for every $\gamma_G$-set $S$ of $G$. That is, $v \in W$. \hfill \Box

**Corollary 2.16** Let $S$ be a $\gamma_G$-set of a graph $G$ and $T$ is a minimum forcing subset of $S$. Then, $W \cap T = \emptyset$.

**Remark** The above result holds even if $G$ has a unique $\gamma_G$-set.

**Corollary 2.18** Let $W$ be the set of all $(G,D)$-vertices of a graph $G$. Then, $f_{G,D}(G) \leq \gamma_G(G) - |W|$. 


Remark 2.19 In the above corollary, the inequality is strict. For example, consider the following graph G.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{graph1}
\caption{Fig. 2.2}
\end{figure}

For G, \( S_1 = \{v_1, v_4, v_5\}, S_2 = \{v_1, v_3, v_5\}, S_3 = \{v_1, v_4, v_6\} \) are the only distinct \( \gamma_G \)-sets. Therefore, \( \gamma_G(G) = 3 \). But, \( f_{G,D}(S_1) = 2 \) and \( f_{G,D}(S_2) = f_{G,D}(S_3) = 1 \). So, \( f_{G,D}(G) = \min\{f_{G,D}(S) : S \text{ is a } \gamma_G \text{-set of } G\} = 1 \). Also, \( W = \{1\} \). Now, \( \gamma_G(G) - |W| = 3 - 1 = 2 \). Hence \( f_{G,D}(G) \leq \gamma_G(G) - |W| \).

Also the upper bound is sharp. For example, consider the following graph G.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{graph2}
\caption{Fig. 2.3}
\end{figure}

For G, \( S_1 = \{v_1, v_4, v_5\}, S_2 = \{v_1, v_3, v_6\} \) are different \( \gamma_G \)-sets. Therefore, \( \gamma_G(G) = 3 \). But, \( f_{G,D}(S_1) = f_{G,D}(S_2) = 2 \). So, \( f_{G,D}(G) = \min\{f_{G,D}(S) : S \text{ is a } \gamma_G \text{-set of } G\} = 2 \). Also, \( W = \{1\} \). Now, \( \gamma_G(G) - |W| = 3 - 1 = 2 \). Hence, \( f_{G,D}(G) = \gamma_G(G) - |W| \).

Corollary 2.20 \( f_{G,D}(G) \leq \gamma_G(G) - k \) where \( k \) is the number of extreme vertices of G.

Proof The result follows from \( |W| \geq k \). \( \square \)

Theorem 2.21 For a complete graph \( G = K_p \), \( f_{G,D}(G) = 0 \) and \( |W| = p \).
Proof $V(K_p)$ is the unique $\gamma_{G}$-set of $K_p$. Hence by Theorem 2.4, $f_{G,D}(K_p) = 0$. By Remark 2.11, $W = V(G)$ with $|W| = p$. □

References