The Forcing Weak Edge Detour Number of a Graph

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Abstract: For two vertices $u$ and $v$ in a graph $G = (V, E)$, the distance $d(u, v)$ and detour distance $D(u, v)$ are the length of a shortest or longest $u-v$ path in $G$, respectively, and the Smarandache distance $d^i_S(u, v)$ is the length $d(u, v) + i(u, v)$ of a $u-v$ path in $G$, where $0 \leq i(u,v) \leq D(u,v) - d(u,v)$. A $u-v$ path of length $d^i_S(u,v)$, if it exists, is called a Smarandachely $u-v$ i-detour. A set $S \subseteq V$ is called a Smarandachely i-detour set if every edge in $G$ has both its ends in $S$ or it lies on a Smarandachely i-detour joining a pair of vertices in $S$. In particular, if $i(u,v) = 0$, then $d^i_S(u,v) = d(u,v)$; and if $i(u,v) = D(u,v) - d(u,v)$, then $d^i_S(u,v) = D(u,v)$. For $i(u,v) = D(u,v) - d(u,v)$, such a Smarandachely i-detour set is called a weak edge detour set in $G$. The weak edge detour number $dn_w(G)$ of $G$ is the minimum order of its weak edge detour sets and any weak edge detour set of order $dn_w(G)$ is a weak edge detour basis of $G$. For any weak edge detour basis $S$ of $G$, a subset $T \subseteq S$ is called a forcing subset for $S$ if $S$ is the unique weak edge detour basis containing $T$. A forcing subset for $S$ of minimum cardinality is a minimum forcing subset of $S$. The forcing weak edge detour number of $S$, denoted by $f dn_w(S)$, is the cardinality of a minimum forcing subset for $S$. The forcing weak edge detour number of $G$, denoted by $f dn_w(G)$, is $f dn_w(G) = \min \{ f dn_w(S) \}$, where the minimum is taken over all weak edge detour bases $S$ in $G$. The forcing weak edge detour numbers of certain classes of graphs are determined. It is proved that for each pair $a, b$ of integers with $0 \leq a \leq b$ and $b \geq 2$, there is a connected graph $G$ with $f dn_w(G) = a$ and $dn_w(G) = b$.

Key Words: Smarandache distance, Smarandachely i-detour set, weak edge detour set, weak edge detour number, forcing weak edge detour number.

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§1. Introduction

For vertices $u$ and $v$ in a connected graph $G$, the distance $d(u,v)$ is the length of a shortest $u-v$ path in $G$. A $u-v$ path of length $d(u,v)$ is called a $u-v$ geodesic. For a vertex $v$ of $G$, the eccentricity $e(v)$ is the distance between $v$ and a vertex farthest from $v$. The minimum eccentricity among the vertices of $G$ is the radius, $radG$ and the maximum eccentricity among the vertices of $G$ is its diameter, $diamG$ of $G$. Two vertices $u$ and $v$ of $G$ are antipodal if $d(u,v)$

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For vertices $u$ and $v$ in a connected graph $G$, the **detour distance** $D(u, v)$ is the length of a longest $u$–$v$ path in $G$. A $u$–$v$ path of length $D(u, v)$ is called a **$u$–$v$ detour**. It is known that the distance and the detour distance are metrics on the vertex set $V(G)$. The **detour eccentricity** $e_D(v)$ of a vertex $v$ in $G$ is the maximum detour distance from $v$ to a vertex of $G$. The **detour radius**, $\text{rad}_DG$ of $G$ is the minimum detour eccentricity among the vertices of $G$, while the **detour diameter**, $\text{diam}_DG$ of $G$ is the maximum detour eccentricity among the vertices of $G$. These concepts were studied by Chartrand et al. [2].

A vertex $x$ is said to lie on a $u$–$v$ detour $P$ if $x$ is a vertex of $P$ including the vertices $u$ and $v$. A set $S \subseteq V$ is called a **detour set** if every vertex $v$ in $G$ lies on a detour joining a pair of vertices of $S$. The **detour number** $dn(G)$ of $G$ is the minimum order of a detour set and any detour set of order $dn(G)$ is called a **detour basis** of $G$. A vertex $v$ that belongs to every detour basis of $G$ is a **detour vertex** in $G$. If $G$ has a unique detour basis $S$, then every vertex in $S$ is a detour vertex in $G$. These concepts were studied by Chartrand et al. [3].

In general, there are graphs $G$ for which there exist edges which do not lie on a detour joining any pair of vertices of $V$. For the graph $G$ given in Figure 1.1, the edge $v_1v_2$ does not lie on a detour joining any pair of vertices of $V$. This motivated us to introduce the concept of **weak edge detour set** of a graph [5].

![Figure 1: G](image)

The **Smarandache distance** $d_S(u, v)$ is the length $d(u, v) + i(u, v)$ of a $u$–$v$ path in $G$, where $0 \leq i(u, v) \leq D(u, v) - d(u, v)$. A $u$–$v$ path of length $d_S(u, v)$, if it exists, is called a **Smarandachely $u$–$v$ i-detour**. A set $S \subseteq V$ is called a **Smarandachely i-detour set** if every edge in $G$ has both its ends in $S$ or it lies on a Smarandachely i-detour joining a pair of vertices in $S$. In particular, if $i(u, v) = 0$, then $d_S(u, v) = d(u, v)$ and if $i(u, v) = D(u, v) - d(u, v)$, then $d_S(u, v) = D(u, v)$. For $i(u, v) = D(u, v) - d(u, v)$, such a Smarandachely i-detour set is called a **weak edge detour set** in $G$. The **weak edge detour number** $dn_w(G)$ of $G$ is the minimum order of its weak edge detour sets and any weak edge detour set of order $dn_w(G)$ is called a **weak edge detour basis** of $G$. A vertex $v$ in a graph $G$ is a **weak edge detour vertex** if $v$ belongs to every weak edge detour basis of $G$. If $G$ has a unique weak edge detour basis $S$, then every vertex in $S$ is a weak edge detour vertex of $G$. These concepts were studied by A. P. Santhakumaran and S. Athikeyanathan [5].

To illustrate these concepts, we consider the graph $G$ given in Figure 1.2. The sets $S_1 = \{u, x\}$, $S_2 = \{u, y\}$ and $S_3 = \{u, z\}$ are the detour bases of $G$ so that $dn(G) = 2$ and the sets $S_4 = \{u, v, y\}$ and $S_5 = \{u, x, z\}$ are the weak edge detour bases of $G$ so that $dn_w(G) = 3$. The vertex $u$ is a detour vertex and also a weak edge detour vertex of $G$. 
The following theorems are used in the sequel.

**Theorem 1.1** ([5]) For any graph $G$ of order $p \geq 2$, $2 \leq d_{w}(G) \leq p$.

**Theorem 1.2** ([5]) Every end-vertex of a non-trivial connected graph $G$ belongs to every weak edge detour set of $G$. Also if the set $S$ of all end-vertices of $G$ is a weak edge detour set, then $S$ is the unique weak edge detour basis for $G$.

**Theorem 1.3** ([5]) If $T$ is a tree with $k$ end-vertices, then $d_{w}(T) = k$.

**Theorem 1.4** ([5]) Let $G$ be a connected graph with cut-vertices and $S$ a weak edge detour set of $G$. Then for any cut-vertex $v$ of $G$, every component of $G - v$ contains an element of $S$.

Throughout this paper $G$ denotes a connected graph with at least two vertices.

§2. **Forcing Weak Edge Detour Number of a Graph**

First we determine the weak edge detour numbers of some standard classes of graphs so that their forcing weak edge detour numbers will be determined.

**Theorem 2.1** Let $G$ be the complete graph $K_{p}$ ($p \geq 3$) or the complete bipartite graph $K_{m,n}$ ($2 \leq m \leq n$). Then a set $S \subseteq V$ is a weak edge detour basis of $G$ if and only if $S$ consists of any two vertices of $G$.

**Proof** Let $G$ be the complete graph $K_{p}$ ($p \geq 3$) and $S = \{u, v\}$ be any set of two vertices of $G$. It is clear that $D(u, v) = p - 1$. Let $xy \in E$. If $xy = uv$, then both its ends are in $S$. Let $xy \neq uv$. If $x \neq u$ and $y \neq v$, then the edge $xy$ lies on the $u-v$ detour $P: u, x, y, \ldots, v$ of length $p - 1$. If $x = u$ and $y = v$, then the edge $xy$ lies on the $u-v$ detour $P: u = x, y, \ldots, v$ of length $p - 1$. Hence $S$ is a weak edge detour set of $G$. Since $|S| = 2$, $S$ is a weak edge detour basis of $G$.

Now, let $S$ be a weak edge detour basis of $G$. Let $S'$ be any set consisting of two vertices of $G$. Then as in the first part of this theorem $S'$ is a weak edge detour basis of $G$. Hence $|S| = |S'| = 2$ and it follows that $S$ consists of any two vertices of $G$.

Let $G$ be the complete bipartite graph $K_{m,n}$ ($2 \leq m \leq n$). Let $X$ and $Y$ be the bipartite sets of $G$ with $|X| = m$ and $|Y| = n$. Let $S = \{u, v\}$ be any set of two vertices of $G$.

**Case 1** Let $u \in X$ and $v \in Y$. It is clear that $D(u, v) = 2m - 1$. Let $xy \in E$. If $xy = uv$, then
both of its ends are in \( S \). Let \( xy \neq uv \) be such that \( x \in X \) and \( y \in Y \). If \( x \neq u \) and \( y \neq v \), then the edge \( xy \) lies on the \( u-v \) detour \( P: u, y, x, \ldots, v \) of length \( 2m - 1 \). If \( x = u \) and \( y \neq v \), then the edge \( xy \) lies on the \( u-v \) detour \( P: u = x, y, \ldots, v \) of length \( 2m - 1 \). Hence \( S \) is a weak edge detour set of \( G \).

**Case 2** Let \( u, v \in X \). It is clear that \( D(u, v) = 2m - 2 \). Let \( xy \in E \) be such that \( x \in X \) and \( y \in Y \). If \( x \neq u \), then the edge \( xy \) lies on the \( u-v \) detour \( P: u, y, x, \ldots, v \) of length \( 2m - 2 \). If \( x = u \), then the edge \( xy \) lies on the \( u-v \) detour \( P: u = x, y, \ldots, v \) of length \( 2m - 2 \). Hence \( S \) is a weak edge detour set of \( G \).

**Case 3** Let \( u, v \in Y \). It is clear that \( D(u, v) = 2m \). Then, as in Case 2, \( S \) is a weak edge detour set of \( G \). Since \( |S| = 2 \), it follows that \( S \) is a weak edge detour basis of \( G \).

Now, let \( S \) be a weak edge detour basis of \( G \). Let \( S' \) be any set consisting of two vertices of \( G \). Then as in the first part of the proof of \( K_{m,n} \), \( S' \) is a weak edge detour basis of \( G \). Hence \( |S| = |S'| = 2 \) and it follows that \( S \) consists of any two vertices adjacent or not.

**Theorem 2.2** Let \( G \) be an odd cycle of order \( p \geq 3 \). Then a set \( S \subseteq V \) is a weak edge detour basis of \( G \) if and only if \( S \) consists of any two adjacent vertices of \( G \).

**Proof** Let \( S = \{ u, v \} \) be any set of two adjacent vertices of \( G \). It is clear that \( D(u, v) = p-1 \). Then every edge \( e \neq uv \) of \( G \) lies on the \( u-v \) detour and both the ends of the edge \( uv \) belong to \( S \) so that \( S \) is a weak edge detour set of \( G \). Since \( |S| = 2 \), \( S \) is a weak edge detour basis of \( G \).

Now, assume that \( S \) is a weak edge detour basis of \( G \). Let \( S' \) be any set of two adjacent vertices of \( G \). Then as in the first part of this theorem \( S' \) is a weak edge detour basis of \( G \). Hence \( |S| = |S'| = 2 \). Let \( S = \{ u, v \} \). If \( u \) and \( v \) are not adjacent, then since \( G \) is an odd cycle, the edges of \( u-v \) geodesic do not lie on the \( u-v \) detour in \( G \) so that \( S \) is not a weak edge detour set of \( G \), which is a contradiction. Thus \( S \) consists of any two adjacent vertices of \( G \).

**Theorem 2.3** Let \( G \) be an even cycle of order \( p \geq 4 \). Then a set \( S \subseteq V \) is a weak edge detour basis of \( G \) if and only if \( S \) consists of any two adjacent vertices or two antipodal vertices of \( G \).

**Proof** Let \( S = \{ u, v \} \) be any set of two vertices of \( G \). If \( u \) and \( v \) are adjacent, then \( D(u, v) = p-1 \) and every edge \( e \neq uv \) of \( G \) lies on the \( u-v \) detour and both the ends of the edge \( uv \) belong to \( S \). If \( u \) and \( v \) are antipodal, then \( D(u, v) = p/2 \) and every edge \( e \) of \( G \) lies on a \( u-v \) detour in \( G \). Thus \( S \) is a weak edge detour set of \( G \). Since \( |S| = 2 \), \( S \) is a weak edge detour basis of \( G \).

Now, assume that \( S \) is a weak edge detour basis of \( G \). Let \( S' \) be any set of two adjacent vertices or two antipodal vertices of \( G \). Then as in the first part of this theorem \( S' \) is a weak edge detour basis of \( G \). Hence \( |S| = |S'| = 2 \). Let \( S = \{ u, v \} \). If \( u \) and \( v \) are not adjacent and \( u \) and \( v \) are antipodal, then the edges of the \( u-v \) geodesic do not lie on the \( u-v \) detour in \( G \) so that \( S \) is not a weak edge detour set of \( G \), which is a contradiction. Thus \( S \) consists of any two adjacent vertices or two antipodal vertices of \( G \).

**Corollary 2.4** If \( G \) is the complete graph \( K_p \) (\( p \geq 3 \)) or the complete bipartite graph \( K_{m,n} \) (\( 2 \leq m \leq n \)) or the cycle \( C_p \) (\( p \geq 3 \)), then \( dn_w(G) = 2 \).
Proof} This follows from Theorems 2.1, 2.2 and 2.3.

Every connected graph contains a weak edge detour basis and some connected graphs may contain several weak edge detour bases. For each weak edge detour basis $S$ in a connected graph $G$, there is always some subset $T$ of $S$ that uniquely determines $S$ as the weak edge detour basis containing $T$. We call such subsets ”forcing subsets” and we discuss their properties in this section.

**Definition 2.5** Let $G$ be a connected graph and $S$ a weak edge detour basis of $G$. A subset $T \subseteq S$ is called a forcing subset for $S$ if $S$ is the unique weak edge detour basis containing $T$. A forcing subset for $S$ of minimum cardinality is a minimum forcing subset of $S$. The forcing weak edge detour number of $S$, denoted by $fdn_w(S)$, is the cardinality of a minimum forcing subset for $S$. The forcing weak edge detour number of $G$, denoted by $fdn_w(G)$, is $fdn_w(G) = \min \{fdn_w(S)\}$, where the minimum is taken over all weak edge detour bases $S$ in $G$.

**Example 2.6** For the graph $G$ given in Figure 2.1(a), $S = \{u, v, w\}$ is the unique weak edge detour basis so that $fdn_w(G) = 0$. For the graph $G$ given in Figure 2.1(b), $S_1 = \{u, v, x\}$, $S_2 = \{u, v, y\}$ and $S_3 = \{u, v, w\}$ are the only weak edge detour bases so that $fdn_w(G) = 1$. For the graph $G$ given in Figure 2.1(c), $S_1 = \{u, w, x\}$, $S_2 = \{u, w, y\}$, $S_3 = \{v, w, x\}$ and $S_4 = \{v, w, y\}$ are the four weak edge detour bases so that $fdn_w(G) = 2$.

![Figure 3: G](image-url)

The following theorem is clear from the definitions of weak edge detour number and forcing weak edge detour number of a connected graph $G$.

**Theorem 2.7** For every connected graph $G$, $0 \leq fdn_w(G) \leq dn_w(G)$.

**Remark 2.8** The bounds in Theorem 2.7 are sharp. For the graph $G$ given in Figure 2.1(a), $fdn_w(G) = 0$. For the cycle $C_3$, $fdn_w(C_3) = dn_w(C_3) = 2$. Also, all the inequalities in Theorem 2.7 can be strict. For the graph $G$ given in Figure 2.1(b), $fdn_w(G) = 1$ and $dn_w(G) = 3$ so that $0 < fdn_w(G) < dn_w(G)$.

The following two theorems are easy consequences of the definitions of the weak edge detour number and the forcing weak edge detour number of a connected graph.

**Theorem 2.9** Let $G$ be a connected graph. Then

a) $fdn_w(G) = 0$ if and only if $G$ has a unique weak edge detour basis,
b) $\text{fdn}_w(G) = 1$ if and only if $G$ has at least two weak edge detour bases, one of which is a unique weak edge detour basis containing one of its elements, and
c) $\text{fdn}_w(G) = \text{dn}_w(G)$ if and only if no weak edge detour basis of $G$ is the unique weak edge detour basis containing any of its proper subsets.

**Theorem 2.10** Let $G$ be a connected graph and let $\mathcal{F}$ be the set of relative complements of the minimum forcing subsets in their respective weak edge detour bases in $G$. Then $\bigcap_{F \in \mathcal{F}} F$ is the set of weak edge detour vertices of $G$. In particular, if $S$ is a weak edge detour basis of $G$, then no weak edge detour vertex of $G$ belongs to any minimum forcing subset of $S$.

**Theorem 2.11** Let $G$ be a connected graph and $W$ be the set of all weak edge detour vertices of $G$. Then $\text{fdn}_w(G) \leq \text{dn}_w(G) - |W|$.

**Proof** Let $S$ be any weak edge detour basis $S$ of $G$. Then $\text{dn}_w(G) = |S|$, $W \subseteq S$ and $S$ is the unique weak edge detour basis containing $S - W$. Thus $\text{fdn}_w(S) \leq |S - W| = |S| - |W| = \text{dn}_w(G) - |W|$. □

**Remark 2.12** The bound in Theorem 2.11 is sharp. For the graph $G$ given in Figure 2.1(c), $\text{dn}_w(G) = 3$, $|W| = 1$ and $\text{fdn}_w(G) = 2$ as in Example 2.6. Also, the inequality in Theorem 2.11 can be strict. For the graph $G$ given in Figure 2.2, the sets $S_1 = \{v_1, v_4\}$ and $S_2 = \{v_2, v_3\}$ are the two weak edge detour bases for $G$ and $W = \emptyset$ so that $\text{dn}_w(G) = 2$, $|W| = 0$ and $\text{fdn}_w(G) = 1$. Thus $\text{fdn}_w(G) < \text{dn}_w(G) - |W|$.

![Figure 4: G](image)

In the following we determine $\text{fdn}_w(G)$ for certain graphs $G$.

**Theorem 2.13** a) If $G$ is the complete graph $K_p$ ($p \geq 3$) or the complete bipartite graph $K_{m,n}$ ($2 \leq m \leq n$), then $\text{dn}_w(G) = \text{fdn}_w(G) = 2$.
b) If $G$ is the cycle $C_p$ ($p \geq 4$), then $\text{dn}_w(G) = \text{fdn}_w(G) = 2$.
c) If $G$ is a tree of order $p \geq 2$ with $k$ end-vertices, then $\text{dn}_w(G) = k$, $\text{fdn}_w(G) = 0$.

**Proof** a) By Theorem 2.1, a set $S$ of vertices is a weak edge detour basis if and only if $S$ consists of any two vertices of $G$. For each vertex $v$ in $G$ there are two or more vertices adjacent with $v$. Thus the vertex $v$ belongs to more than one weak edge detour basis of $G$. Hence it follows that no set consisting of a single vertex is a forcing subset for any weak edge detour basis of $G$. Thus the result follows.
b) By Theorems 2.2 and 2.3, a set $S$ of two adjacent vertices of $G$ is a weak edge detour basis of $G$. For each vertex $v$ in $G$ there are two vertices adjacent with $v$. Thus the vertex $v$
belongs to more than one weak edge detour basis of $G$. Hence it follows that no set consisting of a single vertex is a forcing subset for any weak edge detour basis of $G$. Thus the result follows.

c) By Theorem 1.3, $dn_w(G) = k$. Since the set of all end-vertices of a tree is the unique weak edge detour basis, the result follows from Theorem 2.9(a).

The following theorem gives a realization result.

**Theorem 2.14** For each pair $a$, $b$ of integers with $0 \leq a \leq b$ and $b \geq 2$, there is a connected graph $G$ with $fdn_w(G) = a$ and $dn_w(G) = b$.

**Proof** The proof is divided into two cases following.

**Case 1:** $a = 0$. For each $b \geq 2$, let $G$ be a tree with $b$ end-vertices. Then $fdn_w(G) = 0$ and $dn_w(G) = b$ by Theorem 2.13(c).

**Case 2:** $a \geq 1$. For each $i (1 \leq i \leq a)$, let $F_i : u_i, v_i, w_i, x_i, u_i$ be the cycle of order 4 and let $H = K_{1,b-a}$ be the star at $v$ whose set of end-vertices is $\{z_1, z_2, \ldots, z_{b-a}\}$. Let $G$ be the graph obtained by joining the central vertex $v$ of $H$ to both vertices $u_i, w_i$ of each $F_i (1 \leq i \leq a)$. Clearly the graph $G$ is connected and is shown in Figure 2.3.

Let $W = \{z_1, z_2, \ldots, z_{b-a}\}$ be the set of all $(b-a)$ end-vertices of $G$. First, we show that $dn_w(G) = b$. By Theorems 1.2 and 1.4, every weak edge detour basis contains $W$ and at least one vertex from each $F_i (1 \leq i \leq a)$. Thus $dn_w(G) \geq (b-a) + a = b$. On the other hand, since the set $S_1 = W \cup \{v_1, v_2, \ldots, v_a\}$ is a weak edge detour set of $G$, it follows that $dn_w(G) \leq |S_1| = b$. Therefore $dn_w(G) = b$.

Next we show that $fdn_w(G) = a$. It is clear that $W$ is the set of all weak edge detour vertices of $G$. Hence it follows from Theorem 2.11 that $fdn_w(G) \leq dn_w(G) - |W| = b - (b-a) = a$. Now, since $dn_w(G) = b$, it is easily seen that a set $S$ is a weak edge detour basis of $G$ if and only if $S$ is of the form $S = W \cup \{y_1, y_2, \ldots, y_a\}$, where $y_i \in \{v_i, x_i\} \subseteq V(F_i) (1 \leq i \leq a)$. Let $T$ be a subset of $S$ with $|T| < a$. Then there is a vertex $y_j (1 \leq j \leq a)$ such that $y_j \notin T$. Let $s_j \in \{v_j, x_j\} \subseteq V(F_j)$ distinct from $y_j$. Then $S' = (S - \{y_j\}) \cup \{s_j\}$ is a weak edge detour basis that contains $T$. Thus $S$ is not the unique weak edge detour basis containing $T$. Thus $fdn_w(S) \geq a$. Since this is true for all weak edge detour basis of $G$, it follows that $fdn_w(G) \geq a$ and so $fdn_w(G) = a$.  

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![Figure 5: G](image-url)
References