On the Smarandache function and the divisor product sequences

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Abstract Let $n$ be any positive integer, $P_d(n)$ denotes the product of all positive divisors of $n$. The main purpose of this paper is using the elementary and analytic methods to study the mean value properties of a new arithmetical function $S(P_d(n))$, and give an interesting asymptotic formula for it.

Keywords Smarandach function, Divisor product sequences, Composite function, mean value, Asymptotic formula.

§1. Introduction

For any positive integer $n$, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer $m$ such that $n$ divide $m!$. That is, $S(n) = \min\{m : m \in \mathbb{N}, n|m!\}$. And the Smarandache divisor product sequences $\{P_d(n)\}$ is defined as the product of all positive divisors of $n$. That is, $P_d(n) = \prod_{d|n} d = n^{\frac{1}{2}}$, where $d(n)$ is the Dirichlet divisor function.

For examples, $P_d(1) = 1, P_d(2) = 2, P_d(3) = 3, P_d(4) = 8, \cdots$. In problem 25 of reference [1], Professor F.Smarandache asked us to study the properties of the function $S(n)$ and the sequence $\{P_d(n)\}$. About these problems, many scholars had studied them, and obtained a series interesting results, see references [2], [3], [4], [5] and [6]. But at present, none had studied the mean value properties of the composite function $S(P_d(n))$, at least we have not seen any related papers before. In this paper, we shall use the elementary methods to study the mean value properties of $S(P_d(n))$, and give an interesting asymptotic formula for it. That is, we shall prove the following conclusion:

**Theorem.** For any fixed positive integer $k$ and any real number $x \geq 1$, we have the asymptotic formula

$$
\sum_{n \leq x} S(P_d(n)) = \frac{\pi^4}{72} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} b_i \cdot \frac{x^2}{\ln^i x} + O \left( \frac{x^2}{\ln^{k+1} x} \right),
$$

where $b_i$ ($i = 2, 3, \cdots, k$) are computable constants.
§2. Some simple lemmas

To complete the proof of the theorem, we need the following several simple lemmas. First we have

**Lemma 1.** For any positive integer $\alpha$, we have the estimate

$$S(p^\alpha) \leq \alpha p.$$

Especially, when $\alpha \leq p$, we have $S(p^\alpha) = \alpha p$, where $p$ is a prime.

**Proof.** See reference [3].

**Lemma 2.** For any positive integer $n$, let $n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denotes the factorization of $n$ into prime powers, then we have

$$S(n) = \max_{1 \leq i \leq k} \{S(p_i^{\alpha_i})\}.$$

**Lemma 3.** Let $P(n)$ denotes the greatest prime divisor of $n$, if $P(n) > \sqrt{n}$, then we have $S(n) = P(n)$.

**Proof.** The proof of Lemma 2 and Lemma 3 can be found in reference [4].

§3. Proof of the theorem

In this section, we shall use the above lemmas to complete the proof of our theorem. For any positive integer $n$, it is clear that from the definition of $P_d(n)$ we have

$$P_d^2(n) = \left( \prod_{r \mid n} r^{r} \right) \cdot \left( \prod_{r \mid n} \frac{n}{r} \right) = \sum_{r \mid n} 1 = d(n).$$

So we have the identity $P_d(n) = n^{\frac{d(n)}{2}}$. Let $n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denotes the factorization of $n$ into prime powers. First we separate all integers $n$ in the interval $[1, x]$ into two subsets $A$ and $B$ as follows:

$$A = \{ n : n \leq x, P(n) \leq \sqrt{n} \}, \quad B = \{ n : n \leq x, P(n) > \sqrt{n} \}.$$

If $n \in A$, then from Lemma 1 and Lemma 2, and note that $P_d(n) = n^{\frac{d(n)}{2}}$ we have

$$P_d(n) = n \cdot \frac{d(n)}{2} = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k}.$$ 

Therefore,

$$S(P_d(n)) = S \left( \frac{\alpha_1 d(n)}{2} \frac{\alpha_2 d(n)}{2} \cdots \frac{\alpha_k d(n)}{2} \right) = \max_{1 \leq i \leq k} \left\{ S \left( \frac{\alpha_i d(n)}{2} \right) \right\} \leq \max_{1 \leq i \leq k} \left\{ \frac{\alpha_i d(n)}{2} \right\} \leq \frac{d(n)}{2} \sqrt{n \ln n}.$$
From reference [10] we know that

\[ \sum_{n \leq x} d(n) = x \ln x + O(x). \]

So we have the estimate

\[ \sum_{n \in A} S(P_d(n)) \leq \sum_{n \in A} \frac{d(n)}{2} \sqrt{n} \ln n \ll \sum_{n \leq x} d(n) \sqrt{x} \ln x \ll x^{\frac{3}{2}} \ln^2 x. \]  \hspace{1cm} (1)

If \( n \in B \), let \( n = n_1 p \), where \( n_1 < \sqrt{n} < p \). It is clear that \( d(n_1) < \sqrt{n} < p \) and \( d(n) = 2d(n_1) \). So from Lemma 3 we have

\[ \sum_{n \in B} S(P_d(n)) = \sum_{n_1 p \leq x} S \left( \frac{d(n_1 p)}{2} \right) = \sum_{n_1 p \leq x, n_1 < p} S \left( \frac{d(n_1 p)}{2} \right) = \sum_{n \leq \sqrt{x}} d(n) \sum_{p \leq \frac{x}{n}} p + O \left( \sum_{n \leq \sqrt{x}} d(n) \cdot \frac{n}{\ln n} \right) = \sum_{n \leq \sqrt{x}} d(n) \sum_{p \leq \frac{x}{n}} p + O(x). \]  \hspace{1cm} (2)

From the Abel’s summation formula (see Theorem 4.2 of [10]) and the Prime Theorem (see Theorem 3.2 of [11]) we have

\[ \pi(x) = \sum_{i=1}^{k} \frac{a_i \cdot x}{\ln^i x} + O \left( \frac{x}{\ln^{k+1} x} \right), \]

where \( a_i \) (\( i = 1, 2, \cdots, k \)) are computable constants and \( a_1 = 1 \). We have

\[ \sum_{p \leq \frac{x}{n}} p = \frac{x}{n} \pi \left( \frac{x}{n} \right) - \int_{\frac{2}{x}}^{\frac{x}{n}} \pi(y)dy = \frac{x^2}{2n^2 \ln x} + \sum_{i=2}^{k} \frac{c_i \cdot x^2 \ln^n x}{n^2 \ln^2 x} + O \left( \frac{x^2}{n^2 \ln^{k+1} x} \right), \]  \hspace{1cm} (3)

where \( c_i \) (\( i = 2, 3, \cdots, k \)) are computable constants.

Note that

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \]

and

\[ \sum_{n=1}^{\infty} \frac{d(n)}{n^2} = \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right)^2 = \frac{\pi^4}{36}. \]  \hspace{1cm} (4)
from (2), (3) and (4) we obtain
\[
\sum_{n \in B} S(P_d(n)) = \frac{x^2}{2 \ln x} \sum_{n \leq \sqrt{x}} \frac{d(n)}{n^2} + \sum_{n \leq \sqrt{x}} \sum_{i=2}^{k} c_i \cdot \frac{x^2 d(n) \ln^i n}{n^2 \ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right)
\]
\[
= \frac{\pi^4}{72} \frac{x^2}{\ln x} + \sum_{i=2}^{k} b_i \cdot \frac{x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),
\]
where \(b_i (i = 2, 3, \cdots, k)\) are computable constants.

Now combining (1) and (5) we may immediately get the asymptotic formula
\[
\sum_{n \leq x} S(P_d(n)) = \sum_{n \in A} S(P_d(n)) + \sum_{n \in B} S(P_d(n))
\]
\[
= \frac{\pi^4}{72} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} b_i \cdot \frac{x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),
\]
where \(b_i (i = 2, 3, \cdots, k)\) are computable constants. This completes the proof of Theorem.

References