Goldbach’s conjecture

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ABSTRACT

This paper presents an elementary proof of the strong Goldbach conjecture. We will show that the conjecture can be derived from an appropriate structuring of the natural numbers. First, we choose an equivalent but more convenient form of the conjecture. Then, with this form in mind, we create a structure for the natural numbers. That structure leads to a distribution property which immediately implies the conjecture. As an additional result of this property we can even obtain a strengthened form of the conjecture. Moreover, further results are achieved by generalizing the structuring. Thus, it turns out that the statement of the strong Goldbach conjecture is the special case of a general principle.

1. INTRODUCTION

In the course of the various attempts to solve the strong and the weak Goldbach conjecture – both formulated by Goldbach and Euler in their correspondence in 1742 – a substantially wrong-headed route was taken, mainly due to the fact that two underlying aspects of the strong (or binary) conjecture were overlooked. First, that focusing exclusively on the additive character of the statement does not take into account its real content, and second, that a principle known as emergence lies beneath the statement, a principle any existing proof of the conjecture must consider.

Let us discuss some of the most important milestones in the different approaches to the problem.

When a proof could not be achieved even for the sum of three primes (the weak conjecture for odd numbers) without additional assumptions, in the twenties of the previous century mathematicians began to search for the maximum number of primes necessary to represent any natural number greater than 1 as their sum. At the beginning, there were proofs that required hundreds of thousands (!) of primes (L. Schnirelmann [2]). In 1937 the weak conjecture was proven (I. Vinogradov [4]), but only above a constant large enough to make available sufficient primes as summands.

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2 first submission to the Annals of Mathematics on March 24, 2013
Almost an entire century passed before the representation for all integers > 1 could be reduced to the maximum of five or six summands of primes, respectively (T. Tao [3]). In 2013 the huge gap of numbers for the weak Goldbach version was closed, using numerical verification combined with a complex estimative proof (H. Helfgott [1]).

However, the so-called Hardy-Littlewood circle method, which was employed and constantly improved upon in those approaches, did not reflect the primes’ actual role in the problem as originally formulated by Goldbach and Euler, by continuously examining ‘how many’ prime numbers are available as summands. As this method does not work for the binary Goldbach conjecture, concern for the original problem has gradually been sidelined up to the present day, even though a solution would definitively resolve the issue of integers represented as the sum of primes.

2. THE STRONG GOLDBACH CONJECTURE

**Theorem 2.1** (Strong Goldbach conjecture (SGB)). *Every even integer greater than 2 can be expressed as the sum of two primes.*

Moreover, we claim

**Theorem 2.2** (SSGB). *Every even integer greater than 6 can be expressed as the sum of two different primes.*

*Proof.* In order to prove Theorem 2.1 and Theorem 2.2, we proceed by contradiction. We provide a structuring of the natural numbers starting from 3, based on three characteristics, i.e. covering, equidistance and maximality. From these characteristics we deduce a distribution property for each natural number, and we show that equivalent reformulations of SGB and SSGB contradict the property when they are not true.

**Notations.** Let \( \mathbb{N} \) denote the natural numbers starting from 1, let \( \mathbb{N}_3 \) denote the natural numbers starting from 3 and \( \mathbb{P}_3 \) the prime numbers starting from 3. Furthermore, we denote the projections from \( \mathbb{N} \times \mathbb{N} \times \mathbb{N} \) onto the \( i \)-th factor by \( \pi_i, 1 \leq i \leq 3 \).

We replace SGB and SSGB with the following equivalent representations:

*Every integer greater than 1 is prime or is the arithmetic mean of two different primes, \( p_1 \) and \( p_2 \).*

and

*Every integer greater than 3 is the arithmetic mean of two different primes, \( p_1 \) and \( p_2 \).*
\[ SGB \iff \forall n \in \mathbb{N}, n > 1 : n \text{ prime} \lor \exists p_1, p_2 \in \mathbb{P}_3 \exists d \in \mathbb{N} \text{ with } p_1 + d = n = p_2 - d \quad (2.1) \]

\[ SSGB \iff \forall n \in \mathbb{N}, n > 3 : \exists p_1, p_2 \in \mathbb{P}_3 \exists d \in \mathbb{N} \text{ with } p_1 + d = n = p_2 - d \quad (2.2) \]

Now, we give the following definitions:

**Definition 2.3.** Let \( T \) be a non-empty subset of \( \mathbb{N}_3 \times \mathbb{N}_3 \times \mathbb{N}_3 \). A triple structure, or simply, structure \( S \) in \( \mathbb{N}_3 \) is a set defined by
\[
S := \{ (t_1k, t_2k, t_3k) | (t_1, t_2, t_3) \in T; k \in \mathbb{N} \}.
\]

**Definition 2.4.** Let \( S \) be a structure in \( \mathbb{N}_3 \), given by the triples \( (s_1, s_2, s_3) \). Then, a set \( N \subseteq \mathbb{N}_3 \) is covered by the structure \( S \) if every \( n \in N \) can be represented by at least one \( s_i, 1 \leq i \leq 3 \); that is, \( \forall n \in N \exists s_i, 1 \leq i \leq 3 \), such that \( n = s_i \). We say that the structure \( S \) provides a covering of \( N \).

Based on these definitions, we can make the following elementary statement:

**Lemma 2.5.** Let \( S \) be a structure based on the set \( T \). Then, \( \mathbb{N}_3 \) is covered by \( S \) if and only if
\[
\mathbb{P}_3 \cup \{4\} \subseteq \bigcup_{1 \leq i \leq 3} \pi_i(T).
\]

**Proof.** Let the union of the sets \( \pi_i(T), 1 \leq i \leq 3 \), contain all odd primes and the number 4.

Then, every prime number in \( \mathbb{N}_3 \) is represented by a component \( t_ik \) with \( t_i \) prime and \( k = 1 \).

Furthermore, every composite number \( n \), different from the powers of 2, has a prime decomposition \( n = pk \) where \( p \in \mathbb{P}_3 \) and \( k \in \mathbb{N} \), and as such, is represented by a triple component \( t_ik \) of \( S \).

Also in this case, all of the infinitely many odd primes are needed.

For all powers of 2 we choose the representation \( n = t_ik \) where \( t_i = 4 \) and \( k = 1, 2, 4, 8, 16, \ldots \).

So, the whole range of \( \mathbb{N}_3 \) is covered by \( S \). On the other hand, if any odd prime or the number 4 is missing in the union of the sets \( \pi_i(T), 1 \leq i \leq 3 \), at least one of the representations for \( n \) described above is no longer possible.

\[ \square \]
Now, we define the specific structure

\[ S_g := \{(pk, mk, qk) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q)/2\}. \]

In this case, we have \( \pi_1(T) = \mathbb{P}_3 \) and \( 4 \in \pi_2(T) \). So, by Lemma 2.5, \( S_g \) is a structure that covers \( \mathbb{N}_3 \). We call \( S_g \) the \( g \)-structure.

Apart from the covering, we can observe the following characteristics of \( S_g \):

The successive components in the triples of \( S_g \) are equidistant. So, we call these triples as well as the structure \( S_g \) equidistant. We note that the number \( m \) in the triples is determined exclusively by the pairs \( (p, q) \).

Actually, for a complete covering of \( \mathbb{N}_3 \) it would be sufficient if we chose \( (3k, 4k, 5k) \) together with triples \( (pk, mk, qk) \) in which all other odd primes do occur. But for our purpose we use the structure \( S_g \) above that contains all pairs \( (p, q) \) of odd primes with \( p < q \). We call this the maximality of the structure \( S_g \).

The extended form of the triples \( (pk, mk, qk) \) is:

\[
(p_1k_1, m_{11}k_1, q_1k_1), (p_1k_1, m_{12}k_1, q_2k_1), \ldots (p_2k_1, m_{22}k_1, q_2k_1), (p_2k_1, m_{23}k_1, q_3k_1), \ldots (p_1k_2, m_{11}k_2, q_1k_2), (p_1k_2, m_{12}k_2, q_2k_2), \ldots (p_2k_2, m_{22}k_2, q_2k_2), (p_2k_2, m_{23}k_2, q_3k_2), \ldots
\]

\[ \ldots \]

\[ \ldots \]

with all \( p_i, q_j \in \mathbb{P}_3 \) and \( k_l \in \mathbb{N} \), where \( p_i < p_{i+1} = q_i \); \( m_{ij} = (p_i + q_j)/2 \); \( i \leq j \); \( i, j, l \in \mathbb{N} \)

The following examples for \( n = 42 \) illustrate the covering:

\[
(42, 54, 66) = (7 \cdot 6, 9 \cdot 6, 11 \cdot 6)
\]

\[
(18, 30, 42) = (3 \cdot 6, 5 \cdot 6, 7 \cdot 6)
\]

\[
(42, 70, 98) = (3 \cdot 14, 5 \cdot 14, 7 \cdot 14)
\]

\[
(33, 42, 51) = (11 \cdot 3, 14 \cdot 3, 17 \cdot 3)
\]

\[
(41, 42, 43) = (41 \cdot 1, 42 \cdot 1, 43 \cdot 1)
\]

\[
(37, 42, 47) = (37 \cdot 1, 42 \cdot 1, 47 \cdot 1)
\]
Note: According to the definition of $S_g$, SGB is equivalent to saying that for composite numbers $n$ there is always the representation $n = mk$ with $k = 1$.

Having considered the role of $k$ as a multiplier generating composite numbers, we will now identify an additional meaning of the numbers $k$, namely that their own multiples within $\mathbb{N}_3$ are strictly set by our structure $S_g$. This dual role of $k$ in the triple representation is a key point in the proof.

For each $k \in \mathbb{N}$ we define

$$M(k) := \{ n \in \mathbb{N} | n = pk \lor n = mk \lor n = qk; 3 \leq p < q \text{ primes}, \ m = (p + q)/2 \}.$$  

Note: For different numbers $k_1$ and $k_2$, the intersection $M(k_1) \cap M(k_2)$ is equal to the redundancies between the triples $(pk_1, mk_1, qk_1)$ and $(pk_2, mk_2, qk_2)$.

By Lemma 2.5 we have shown that $\mathbb{N}_3 = \bigcup_{k \geq 1} M(k)$. We want to show that $\mathbb{N}_3 = M(1)$ or, equivalently, that for a fixed $k \geq 1$ all multiples $nk, n \geq 3$, are contained in $M(k)$. According to (2.1), this is equivalent to SGB.

Note: Due to the maximality of our structure $S_g$, we can say that the components in the triples $(pk, mk, qk)$ are potential candidates for representing all the multiples $nk$, in contrast for example to the structure $S_c$ which uses only pairs of consecutive primes: $S_c := \{ (pk, mk, qk) | k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q, \exists r \in \mathbb{P}_3, p < r < q; m = (p + q)/2 \}$. In this case, the equidistant triples also cover $\mathbb{N}_3$ completely, but infinitely many multiples of $k$ are obviously not contained in the corresponding $M_c(k)$.

We will now see that the existence of all multiples $nk, n \geq 3$, in a single $M(k)$ arises as an effect of the interaction of all $M(k), k \geq 1$. In other subjects, this effect, which is understood as an underlying principle, is called emergence.

$M(k)$ containing all multiples $nk, n \geq 3$, is equivalent to saying that for this fixed $k$ in $\mathbb{N}_3$ there are triples $(pk, mk, qk)$ exclusively, i.e. no $nk$ exists in an isolated way outside of them. We show that this is a consequence of the three properties, covering, equidistance and maximality, that we identified in our structure $S_g$.

Let $N$ be a subset of the set of all odd numbers in $\mathbb{N}_3$ with at least two elements. For a subset $P \subseteq N \times N$ where all elements of $N$ are contained in pairs $(p, q) \in P$ with $p < q$, we then consider the structure $S_p := \{ (pk, mk, qk) | k, m \in \mathbb{N}; (p, q) \in P; p < m < q \}$. We call the structure $S_p$ maximal if all pairs $(p, q) \in N \times N$ with $p < q$ are used in the triples. Furthermore, we call the structure $S_p$ distance-preserving if for all $p, m, q : (q–m)–(m–p) = c$ with a constant $c$. Specifically, we call $S_p$ equidistant if $c = 0$. We note that in a distance-preserving $S_p$ the component $mk$ is exclusively determined by $pk, qk$. In the case of an equidistant $S_p$, we obtain the arithmetic mean.
In order to get a covering of \( \mathbb{N}_3 \setminus \{ \text{powers of 2} \} \) through the components \( pk, qk \) of \( S_p \), \( N \) must contain all odd primes. In this case, due to the construction of \( S_p \), a maximal and distance-preserving \( S_p \) covers \( \mathbb{N}_3 \) and is equidistant because the triples \((3k, 4k, 5k)\) are contained.

Now, let \( S_p \) be a structure that provides a covering of \( \mathbb{N}_3 \) where \( \mathbb{N}_3 \setminus \{ \text{powers of 2} \} \) is represented by the components \( pk, qk \). That is, we only know that at least all odd primes as \( p \) or \( q \) and the number \( m = 4 \) must be contained in \( S_p \). Trivially, a \( nk \) with \( 3 \leq n \leq 5 \) or \( n > 5 \) prime cannot exist outside the triples \((pk, mk, qk)\) in \( S_p \) for a fixed \( k \geq 1 \). So, let \( nk \) be a decomposition of \( x \in \mathbb{N}_3 \) with a composite number \( n > 5 \) and a number \( k \geq 1 \). \( x = nk \) can be represented by a triple component \( n'k', n' \in \mathbb{P}_3 \) or \( n' = 4 \), with \( k' \) different from \( k \).

Then, this \( nk \) can exist outside the triples \((pk, mk, qk)\) in \( S_p \), \( k \) fixed, only under the following necessary condition:

(A) \( S_p \) is not maximal

or

(B) \( S_p \) is not distance-preserving.

It is sufficient to verify this for \( N = \mathbb{P}_3 \) because, due to (A), it applies to larger sets \( N \) by implication.

Let us assume \( \neg (A) \) and \( \neg (B) \) so that \( S_p \) is a maximal and distance-preserving, i.e. equidistant, structure that covers \( \mathbb{N}_3 \) where \( \mathbb{N}_3 \setminus \{ \text{powers of 2} \} \) is represented by \( pk, qk \). If there were such a \( nk \) outside the triples for each fixed \( k \), we then consider the quadruples \((pk, nk, mk, qk)\). After the covering through \( S_p \), \( nk \) would be left over for each fixed \( k \geq 1 \).

Then, \( nk \) can exist in \( \mathbb{N}_3 \) only if

- \( nk \) lies outside the set that is covered by \( S_p \)
  
  or

- \( nk \) is the arithmetic mean of a pair \((pk, qk)\), \( p < q \), that is not used in \( S_p \)
  
  or

- not all triples \((pk, mk, qk)\) of \( S_p \) are equidistant.

Any of the above cases yields a contradiction. Let us confirm that these three cases are indeed exhaustive. First, either \( S_p \) covers the whole range of \( \mathbb{N}_3 \) or not. In the case of complete covering, either all the triples of \( S_p \) are equidistant or at least one of them is not. In the case in which all triples of \( S_p \) are equidistant, either all the pairs \((p, q)\) of odd primes with \( p < q \) are used in \( S_p \) or not. Then, there are no further cases since these pairs \((p, q)\) are the exclusive parameters in the triples for each \( k \). So, if none of the three cases above applies, we also get a contradiction because then the structure \( S_p \) leaves no space in \( \mathbb{N}_3 \) for that \( nk \) with fixed \( k \).
Keeping in mind the representation (2.3), we give an alternative explanation to obtain the same result:

The structure, given by \((pk, mk, qk)\), on \(\mathbb{N}_3\) with the characteristics of covering, equidistance and maximality leads to the following view on each fixed \(k\) that occurs in this covering. For each number \(k \geq 1\) the multiples \(nk, n \geq 3\), in \(\mathbb{N}_3\) are defined as the occurrences of \(k\) within products with factors \(n \geq 3\). We realize that \(k\) occurs at least in our triple form, that is, it at least occurs together with any two different odd prime factors \(p, q\) and with a third factor \(m\) that is the arithmetic mean of \(p\) and \(q\). So, we always have equidistance in this triple occurrence and ‘at least’ means that there might be other occurrences of \(k\) outside the triple form which are not built by the arithmetic mean of two odd primes. If we now consider this triple occurrence for all \(k\), we get a complete covering of \(\mathbb{N}_3\) through all these triple occurrences. Since for each \(k\) the triple occurrence is maximal, i.e. it is based on all pairs \((p, q)\) of odd primes, we see that each \(k\) always occurs in the triple form, that is, any occurrence of \(k\) in \(\mathbb{N}_3\) is part of a triple occurrence of that \(k\). So, such triple occurrences already represent all occurrences of each \(k\).

The triple occurrence applies to each number \(k \geq 1\). So, it also applies to the \(k\) that is contained in a decomposition \(x = nk, n \geq 3, k \geq 1\), of any \(x \in \mathbb{N}_3\). Hence, this \(nk\) cannot exist outside the triples \((pk, mk, qk)\) where \(k\) is fixed.

It is essential to understand that the representation of a \(nk\), where \(n > 5\) is composite, as \(nk = n'k'\), \(n' \in \mathbb{P}_3\) or \(n' = 4, k' \neq k\), constitutes two distinct occurrences within the covering, one of the number \(k\) and another of the number \(k'\). The occurrences of both, \(k\) and \(k'\), are ruled by the triples separately.

We illustrate this principle by the decompositions of our example \(x = 42\). The decompositions which can be represented by the triple components are: \(42 = 7 \cdot 6 = 6 \cdot 7 = 3 \cdot 14 = 14 \cdot 3 = 21 \cdot 2 = 42 \cdot 1\).
In case of \(nk = 6 \cdot 7\) which can be represented for example by \(n'k' = 7 \cdot 6\), the number \(k = 7\) occurs in a middle component \(mk\) with the factor \(m = 6\), while \(k' = 6\) has its occurrence together with the prime factor \(n' = 7\). This also applies to the decompositions which use the numbers \(k = 1, 2, 3\).

Here we realize the aforementioned dual role of the numbers \(k\) in the triples. While varying them leads to complete covering, this also in turn leads to the distributive property of their multiples when we consider each of them.

**Note:** We realize that \(\mathbb{P}_3\) is the smallest subset of odd numbers in \(\mathbb{N}_3\) that enables a complete covering of \(\mathbb{N}_3\) through the triples \((pk, mk, qk)\) where the components \(pk, qk\) already represent all numbers apart from the powers of 2. We will develop a generalization of the numbers \(m\) used in \(S_p\) and of the corresponding condition (B) in the next section.
In terms of the definition of $S_p$, our structure $S_g$ is based on $N = \mathbb{P}_3$. Since $S_g$ provides a covering of $\mathbb{N}_3$ with $\mathbb{N}_3 \setminus \{ \text{powers of 2} \}$ represented by $pk, qk$ and with the condition $\neg(A)$ and $\neg(B)$, we have that no $nk, n \geq 3, k \geq 1$, can exist outside the triples $(pk, mk, qk), k$ fixed. As already noted, this leads immediately to SGB. However, before concluding, in order to obtain an even more comprehensive property for the multiples of each $k$ we will first take a look at the multiples $nk$ where $n \geq 5$ is prime. We do this by extending the argument above:

If for each $k$ there were a $nk, n \geq 5$ prime, that does not appear as a middle component in the triples $(pk, mk, qk)$, we then consider the quadruples $(pk, nk, mk, qk)$ with pairwise distinct primes $3 \leq p < n, m < q$. After the covering of $\mathbb{N}_3$ by all $(pk, mk, qk)$, $nk$ would be represented as the first and third triple component, but it would be left over between first and third components $pk, qk$ for every fixed $k \geq 1$. As in the case of $n$ composite, this yields a contradiction to the properties of the structure.

Now, we can establish our main result:

Our structure $S_g$ with the characteristics of covering, equidistance and maximality implies the following property for the numbers $k$ in the decompositions $x = nk, n \geq 3, k \geq 1$, of any $x \in \mathbb{N}_3$:

(G) The multiples $nk, n \geq 3$, of each fixed $k \geq 1$ are exclusively set by any two odd prime factors $p, q$ with their arithmetic mean $m$, being either a prime factor too or a composite factor.

These two types of triples $(pk, mk, qk)$ form an equidistant distribution of all $nk, n \geq 4$, with respect to $pk, qk$ for each fixed $k \geq 1$.

Referring to (2.1), we now assume that there is an integer $n > 1$ which is not prime and is not the arithmetic mean of two different primes. We then consider the multiple $nk$ for any $k \geq 1$ (i.e. $n$ is a composite factor) and note that $nk$ belongs to none of the triples $(pk, mk, qk)$. This causes a contradiction to (G) and proves SGB.

In the previous paragraph, we already considered all triples $(pk, mk, qk)$ where $m$ is composite. In fact, we have an equivalence between the existence of all these triples and SGB. Thus, (G) is equivalent to SGB together with the exclusive existence of the triples $(pk, mk, qk)$ where $m$ is prime.

Referring to (2.2), we now assume that there is a prime $n > 3$ which is not the arithmetic mean of two other primes. Obviously, $n$ lies between two other odd primes. We then consider the multiple $nk$ for any $k \geq 1$ (i.e. $n$ is a prime factor) and note that $nk$ belongs to none of the triples $(pk, mk, qk)$ where $m$ is prime and $p < n < q$. This causes a contradiction to (G) and, together with SGB, proves SSGB.

\[\Box\]
**Note:** If we additionally allow the values to be \( p = 1 \) and \( q = 3 \), we find that all multiples of any natural number \( k \) occur exclusively in the symmetric form \((pk, mk, qk)\).

### 3. GENERALIZATION AND FURTHER RESULTS

In order to generalize the structure used in the proof of SGB, we will now consider functions \( f : \mathbb{P}_3 \times \mathbb{P}_3 \rightarrow \mathbb{Z} \) with integers \( f(p, q) \). To achieve useful results, we define the following restrictions on \( f \):

First, we restrict \( f \) with the condition (f1): For all pairs \((p, q) \in \mathbb{P}_3 \times \mathbb{P}_3 \) with \( p < q \) the triples \((p, q, f(p, q))\) have the same numerical ordering and the difference between the two distances of successive components remains constant. That is, the resulting triples \((t_1, t_2, t_3)\) satisfy: \( t_1 < t_2 < t_3 \) and \((t_3 - t_2) - (t_2 - t_1) = c \) with a constant \( c \).

Additionally, we set the condition (f2): \( \exists (p, q) \in \mathbb{P}_3 \times \mathbb{P}_3, p < q, \) with \( f(p, q) = 4 \). So, we have the situation that the powers of 2 are contained when we consider the triples \((pk, qk, f(p, q)k)\) for all \( k \geq 1 \). Therefore, in analogy to the Lemma 2.5, the whole range of \( \mathbb{N}_3 \) is covered by the components of these triples.

Here too we use the maximality based on all pairs \((p, q)\) of odd primes, \( p < q \), and we note that these pairs \((p, q)\) are the exclusive parameters in the triples \((pk, qk, f(p, q)k)\) for each \( k \). Since \( f \) is distance-preserving, for each fixed \( k \) the triples \((pk, qk, f(p, q)k)\) distribute their components uniformly. Based on the function \( f \) with the conditions (f1), (f2) we define a \( f \)-specific distance-preserving structure which covers \( \mathbb{N}_3 \) by

\[
S_f := \{(pk, qk, f(p, q)k) \mid k \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; f(p, q) \in \mathbb{N}_3\}
\]

and call it the \( f \)-structure.

From this we obtain the following consequences:

Any function \( f \) determines exactly one of three possible classes of numbers: only even numbers or only odd numbers or both. We call this the \( f \)-class. In case of odd numbers, \( f \) cannot satisfy (f2) so that the \( f \)-structure would not yield a complete covering of \( \mathbb{N}_3 \). So, \( f \) is restricted to be a function which generates either only even numbers or both even and odd numbers. Furthermore, a \( f \)-structure provides a distribution exclusively for \( f(p, q)k \) by the triples \((pk, qk, f(p, q)k)\) for each \( k \). If, for example, \( f \) produces only even numbers, only even multiples of \( k \) are being distributed through the structure. In that case, we would have no statement regarding the odd multiples which are not prime.

For the definition of \( S_f \) we replaced the numbers \( m \) of the \( g \)-structure by the numbers \( f(p, q) \in \mathbb{N}_3 \) determined by a function \( f \) with (f1), (f2). As in the previous section, we then apply the reasoning of contradiction to some \( nk \) outside the triples \((pk, qk, f(p, q)k)\), where \( n \geq 3 \) belongs to the \( f \)-class.
So, based on the $f$-structure $S_f$ we obtain the following property as a generalization of (G):

(F) The multiples $nk$, $n \geq 3$, of each fixed $k \geq 1$ are exclusively set by any two odd prime factors $p, q$ and by the factor $f(p, q)$ where numerical ordering and distances of the three factors are given by (f1).

The triples $(pk, qk, f(p, q)k)$ form a numerically ordered and distance-preserving distribution of all $nk, n \geq 4$, $n$ of the $f$-class, with respect to $pk, qk$ for each fixed $k \geq 1$.

A few observations on the special case when $f$ is the arithmetic mean:

In our proof of SGB, we used the function $f$ that determines the arithmetic mean: $f(p, q) = (p + q)/2 = m$. $f$ generates even and odd numbers and satisfies the conditions (f1) and (f2) for building a $f$-structure on $\mathbb{N}_3$. In this case, $c = 0$ so that the distance-preserving $f$ yields equi-distance. The pairs $(pk, qk)$ are expanded into triples $(pk, mk, qk)$ including also the powers of 2 by $(3k, 4k, 5k)$. As is easily verified, the arithmetic mean is the only function fulfilling (f1) and (f2) that has its values in the middle component of the ordered triples and that generates even and odd numbers.

Let us now consider other functions $f$ which satisfy the conditions (f1) and (f2) and build a $f$-structure on $\mathbb{N}_3$, with the outcome that $f(p, q)$ represents all even integers greater than 2. Due to (f2), in this case $f(p, q)$ is always the first component in the ordered triples.

For example, we can state:

**Corollary 3.1.** All even positive integers are of the form $2p - q + 1$ with odd primes $p < q$.

*Proof.* For the number 2 we have: $2 = 2 \cdot 3 - 5 + 1$. For all even numbers in $\mathbb{N}_3$ we apply our concept of the $f$-structure.

As is easily verified, $f(p, q) = 2p - q + 1$ satisfies the conditions (f1) and (f2) for building a $f$-structure on $\mathbb{N}_3$. We consider only those $f(p, q)$ which lie in $\mathbb{N}_3$ and we note that by the Bertrand-Chebyshev theorem we have: $\forall p \in \mathbb{P}_3, p > 3, \exists q \in \mathbb{P}_3, q > p$ such that $f(p, q) \in \mathbb{N}_3$.

In analogy to the proof of SGB, we now assume that there is an even integer $n > 2$ which is not of the form $n = 2p - q + 1$ with two odd primes $p, q$. We then consider the multiple $nk$ for any $k \geq 1$ (i.e. $n$ is a composite even factor) and note that $nk$ belongs to none of the triples $((2p - q + 1)k, pk, qk)$. This causes a contradiction to (F) and proves the corollary.

\[ \text{\□} \]
Note: If we interchange the primes $p, q$ and consider $f'(p, q) = 2q - p + 1$, then $f'$ also satisfies the condition (f1) for building a $f$-structure. But for a complete covering of $\mathbb{N}_3$ the number 4 is missing, and we can easily verify that there are other even numbers in $\mathbb{N}_3$ which cannot be represented by $f'(p, q)$.

Another interesting example is $f(p, q) = 2p - q - 3$ versus $f'(p, q) = 2q - p - 3$. $f$ satisfies all conditions, including the covering, and therefore represents all even numbers, whereas $f'$ satisfies the covering because of $f'(3, 5) = 4$, but it violates numerical ordering and distance-preserving. There are even numbers in $\mathbb{N}_3$ which cannot be represented by $f'(p, q)$.

4. EXAMPLES FOR SGB AND SSGB

4.1. $n = 14$ and $k = 3$.

Let us assume that $n$ is not the arithmetic mean of two primes. For $nk = 42$, we find for example $(pk', mk', qk') = (3\cdot6, 5\cdot6, 7\cdot6)$, which is part of the occurrence of 6 in the structure. But there is no triple $(p\cdot3, m\cdot3, q\cdot3)$ that contains $n\cdot3$. Thus, $n\cdot3$ violates the occurrence of 3 in the structure.

This contradiction can be resolved only if $n = m$, that is, $n$ must be the arithmetic mean of, for example, 11 and 17.

4.2. $n = 9$ and $k = 3$.

Let us assume that $n$ is not the arithmetic mean of two primes. For $nk = 27$, we only find $(pk', mk', qk') = (3\cdot9, m\cdot9, q\cdot9)$, which is part of the occurrence of 9 in the structure. But there is no triple $(p\cdot3, m\cdot3, q\cdot3)$ that contains $n\cdot3$. Thus, $n\cdot3$ violates the occurrence of 3 in the structure.

This contradiction can be resolved only if $n = m$, that is, $n$ must be the arithmetic mean of, for example, 7 and 11.

4.3. $n = 19$ and $k = 3$.

Let us assume that $n$ is not the arithmetic mean of two primes. For $nk = 57$, we find for example $(pk', mk', qk') = (17\cdot3, 18\cdot3, 19\cdot3)$, which is part of the occurrence of 3 in the structure. But there is no triple $(p\cdot3, m\cdot3, q\cdot3)$ with $p < 19 < q$ that contains $n\cdot3$. Thus, $n\cdot3$ violates the occurrence of 3 in the structure.

This contradiction can be resolved only if $n = m$, that is, $n$ must be the arithmetic mean of 7 and 31.
5. REMARKS

5.1. Due to the unpredictable way that the primes are distributed, all studies on the representation of natural numbers as the sum of primes are problematic when they use approaches based on this distribution.

Despite tremendous efforts over the centuries, the best result so far was five summands. I was always convinced that the solution must lie in the constructive characteristics of the prime numbers and not in their distribution.

5.2. The statement in the binary Goldbach conjecture actually is nothing more than the symmetric structure \((p_k, m_k, q_k)\) used in the proof. As we have shown, it is in fact a specific case of a general distribution principle within the natural numbers. Furthermore, we note that the property of the prime numbers and their infinitude are merely used to guarantee the complete covering of \(\mathbb{N}_3\) through the structure.

In order to discard the usual interpretation of the conjecture that focuses on the sums of primes and thus opposes their multiplicative character, we have tackled the problem differently after shifting to the triple form: Instead of searching for primes which determine the needed arithmetic mean equal to a given \(n\), we have approached the issue from the opposite direction. Based on the multiplicative prime decomposition, we identify \(nk\) as the component of a structure, in this case determined by the arithmetic mean.

A key point in the proof is the dual role of the numbers \(k\): as multiplier they generate composite numbers, while their own multiples within \(\mathbb{N}_3\) are strictly set by the used triple.

The existence of all multiples \(nk, n \geq 3\), in a single \(M(k)\) becomes visible only when we consider all \(M(k), k \geq 1\), simultaneously. In other subject areas, the effect of the formation of new properties after the transition from single items to a whole system is called emergence. (‘The whole is more than the sum of its parts.’)

5.3. By using an appropriate model for the environment in which the problem is given, other arithmetic questions in number theory may also be solved in an elementary way (unless there is no solution on the basis of the underlying axiom system).
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