A Proof of the Collatz Conjecture

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Abstract

If every positive integer is able to be operated into 1 by set operational rule of the Collatz conjecture, then begin with 1, we can get all positive integers by another operational rule on quite the contrary to the set operational rule after pass infinite many operations. Thereby, we try to substitute such a proof of equal value for directly proving the Collatz conjecture by mathematical induction, and achieved our anticipated goal.

Keywords

Mathematical induction, Classify integers, The bunch of integers' chains, Leftward operational rule, Rightward operational rule, Operational routes.

Basic Concepts

The Collatz conjecture is also known variously as 3n+1 conjecture, the Ulam conjecture, Kakutani's problem, the Thwaites conjecture, Hasse's algorithm or the Syracuse problem, etc.

The Collatz conjecture asserts that Take any positive integer n, if n is an even number, then divide n by 2 to get an integer; if n is an odd number, then multiply n by 3 and add 1 to obtain an even number.

Repeat the above process indefinitely, then no matter which positive integer you start with, you will always eventually reach a result of 1.

We consider the way of aforesaid two steps as the leftward operational rule for any positive integer you given. Also consider another operational rule on quite the contrary to the leftward operational rule as the rightward operational rule for any positive integer we successively got. Let us consider two such operational rules as set each other's contrary operational rules.

The rightward operational rule states that for any positive integer we successively got such as n, if n is an odd number, then multiply n by 2 to obtain an even number. If n is an even number, then, on the one hand, multiply n by 2 to obtain another even number. On the other hand, if the difference of n minus 1 is able to be divided by 3 and get an odd number, then must operate the step as such, and proceed from here to operate even more; if it is not so, need not to do the step.

We regard a segment within an operational course according to either operational rule as an operational route. If either end of the operational route is positive integer P, then we term it a P-operational route.

Begin with 1, proceed to operate every positive integer successively got by the rightward operational rule, then the course automatically forms a bunch of operational routes which consist of positive integers plus arrow's signs. We term the bunch of operational routes as a bunch of integers' chains.

Since a direct origin of each positive integer at the bunch of integers' chains is unique, thus each positive integer therein is unique too. Thus any segment of the bunch of integers' chains is none of the repeat.

Comparatively speaking, in greater limits, integers on the left of the bunch of integers' chains are smaller integers in contrast to on the right.

Overall, from left to right integers at the bunch of integers' chains are getting more and more, greater and greater, up to infinite infinities. Please, see a beginning of the bunch of integers' chains as first illustration.

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                              …            …    452→…
                             168↑        680↑  226↑→75→…
                            84↑        340↑→113↑     227→…
                          42↑         170↑          682↑→…
     1                   21↑         85↑          341↑
1→2→4↑→8→16↓→32→64↑→128→256↑→512→1024↑→2048→…
              5→10↓→20→40↓→80→160↓→320→…
                   3↓      13↓     53→106↓→212→…      …
                    6↓      26↓          35→70↓→……   30↑
                     12↓      52↓→17↓        23→46↑→15↑
                      24↓     104↓  34↓→11→22↓→44→…
                       …        …    68↓      7↓
                                      136↓→…   14↓
                                        45↓      28↓→…
                                         …        9→18→…
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First Illustration

Annotation: ↓ and ↑ must rightwards tilt, but each page is narrow, thus it can only so.

No matter which positive integer, it is surely at the bunch of integers' chains so long as it is able to be operated into 1 by the leftward operational rule; conversely, it is so too . That is to say, positive integers successively got at the bunch of integers' chains by rightward operational rule and positive integers which can operate into 1 by the leftward operational rule are one-to-one correspondence.

Therefore, whether a positive integer suits the conjecture? We need merely to ascertain it whether is at the bunch of integers' chains.

If every positive integer is able to be operated into 1 by the leftward operational rule, then, there are all positive integers at the bunch of integers' chains after pass infinite many operations by the rightward operational rule. Conversely, if we can that prove all positive integers exist at the bunch of integers' chains after pass infinite many operations by the rightward operational rule, then, every positive integer is surely able to be operated into 1 by the leftward operational rule.

Because of this, we have an intention to prove that the bunch of integers' chains contains all positive integers by mathematical induction.

Nevertheless, we need also to first determine an axiom, so that after an anticipative result arises out, we just use it to give an affirmation.

Axiom* For positive integer P orderly proving, if there is positive integer $L<P$ at a P-operational route or at another operational route which intersects with a P-operational route, known L suits the conjecture, then P suits the conjecture. For example, if $P=31+3^2\eta$ and $\eta\geq 0$, a P-operational route is $27+2^3\eta \rightarrow 82+3*2^3\eta\rightarrow 41+3*2^2\eta\rightarrow 124+3^2*2^2\eta\rightarrow 62+3^2*2\eta\rightarrow 31+3^2\eta > 27+2^3\eta$, where $27+2^3\eta \leq L$, then $31+3^2\eta$ suit the conjecture.

Also if $P=5+12\mu$ and $\mu\geq 0$, a P-operational route is $5+12\mu\rightarrow 16+36\mu\rightarrow 8+18\mu\rightarrow 4+9\mu<5+12\mu$, where $4+9\mu \leq L$, then $5+12\mu$ suit the conjecture.

In addition, if $P=63+3*2^8\varphi$ and $\varphi \geq 0$, a P-operational route intersects with another, they are is $63+3*2^8\varphi \rightarrow 190+3^2*2^8\varphi \rightarrow 95+3^2*2^7\varphi \rightarrow 286+3^3*2^7\varphi \rightarrow 143+3^3*2^6\varphi \rightarrow 430+3^4*2^6\varphi \rightarrow 215+3^4*2^5\varphi \rightarrow 646+3^5*2^5\varphi \rightarrow 323+3^5*2^4\varphi \rightarrow 970+3^6*2^4\varphi \rightarrow 485+3^6*2^3\varphi \rightarrow 1456+3^7*2^3\varphi \rightarrow 728+3^7*2^2\varphi \rightarrow 364+3^7*2\varphi \rightarrow 182+3^7\varphi \rightarrow \ldots$

$\uparrow 121+3^6*2\varphi \leftarrow 242+3^6*2^2\varphi \leftarrow 484+3^6*2^3\varphi \leftarrow 161+3^5*2^3\varphi \leftarrow 322+3^5*2^4\varphi \leftarrow 107+3^4*2^4\varphi \leftarrow 214+3^4*2^5\varphi \leftarrow 71+3^3*2^5\varphi \leftarrow 142+3^3*2^6\varphi \leftarrow 47+3^2*2^6\varphi < 63+3*2^8\varphi$, where $47+3^2*2^6\varphi \le L$, then $63+3*2^8\varphi$ suit the conjecture.

The Proof

Let us set about the proof that the bunch of integers' chains contains all positive integers by mathematical induction, thereinafter.

1. From the above-listed first illustration, you can directly see and discover that begin with 1, orderly operate positive integers successively got by the rightward operational rule, we got a part of positive integers including consecutive positive integers from 1 to 18 therein, and they formed a beginning of the bunch of integers' chains.

2. Suppose that after further operate positive integers successively got by the rightward operational rule, there are consecutive positive integers $\le n$ within positive integers successively got at the bunch of integers' chains, where n is a positive integer ≥ 18.

3. Prove that after continue to operate positive integers successively got by the rightward operational rule, we can get consecutive positive integers $\le 2n$ within positive integers successively got at the bunch of integers' chains.

Let us divide limits of consecutive positive integers at the number axis into segments, according to greatest integer $2^X n$ at per segment, where $X \geq 0$ and $n \geq 18$, so as to accord with the proof. The illustration follows.

1———n———2n——————4n———————8n———→

Second Illustration

Proof * Since there are consecutive positive integers ≤ n within positive integers already got at the bunch of integers' chains, thus multiply every positive integer ≤n by 2 by the rightward operational rule, we get all even numbers between n and 2n+1, irrespective of repeated even numbers ≤n.

Then all even numbers just got between n and 2n+1 are at the bunch of integers' chains.

Next, we must seek an origin of each kind of odd numbers between n and 2n+1, whether there is such an origin, it is able to be smaller than the kind of odd numbers.

First, let us divide odd numbers between n and 2n+1 into two kinds, i.e. 5+4k and 7+4k, where k is a natural number, then any odd number between n and 2n+1 must belong in one of the two kinds. By now we list the two kinds of odd numbers in correspondence with k, below.

k: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16…

5+4k: 9, 13, 17, 21, 25, 29, 33, 37, 41, 45, 49, 53, 57, 61, 65, 69 …

7+4k: 11, 15, 19, 23, 27, 31, 35, 39, 43, 47, 51, 55, 59, 63, 67, 71 …

From 5+4k→16+12k→8+6k→4+3k<5+4k, 5+4k suit the conjecture according to the axiom, so 5+4k are at the bunch of integers' chains.

For 7+4k, let us again divide them into three kinds, i.e. 11+12c, 15+12c and 19+12c, where c ≥ 0. From 7+8c→22+24c→ 11+12c > 7+8c, 11+12c suit the conjecture according to the axiom, so 11+12c are at the bunch of integers' chains. By now list remainder two kinds of odd numbers in correspondence with c, below.

c:	0,	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12, …
15+12c:	15,	27,	39,	51,	63,	75,	87,	99,	111,	123,	135,	147,	159 …
19+12c:	19,	31,	43,	55,	67,	79,	91,	103,	115,	127,	139,	151,	163 …

Hereinafter, we will operate 15+12c and 19+12c by the leftward operational rule, also discover and affirm satisfactory results at certain operational branches. Firstly, let us operate 15+12c below.

15+12c→46+36c→23+18c→70+54c→35+27c ♣

d=2e+1: 29+27e (**1**)   e=2f: 142+486f→71+243f ♥

♣35+27c↓→c=2d+1: 31+27d↑→d=2e: 94+162e→47+81e↑→e=2f+1:64+81f (**2**)

c=2d: 106+162d→53+81d↓→d=2e+1:67+81e↓→e=2f+1:74+81f (**3**)

d=2e:160+486e♦   e=2f: 202+486f→101+243f ♠

g=2h+1: 200+243h (**4**)   …

♥ 71+243f↓→f=2g+1:157+243g↑→g=2h: 472+1458h→236+729h↑→ …

f=2g: 214+1458g→107+729g↓→g=2h+1: 418+729h↓→…

g=2h: 322+4374h→… …

g=2h: 86+243h (**5**)

♠101+243f↓→f=2g+1:172+243g↑→g=2h+1:1246+1458h→…

F=2g: 304+1458g→152+729g ↓→…

…

…

♦160+486e→80+243e↓→e=2f+1: 970+1458f→485+729f↑→ … …

e=2f:40+243f↓→f=2g+1:850+1458g→ 425+729g↑→ …

f=2g: 20+243g↓→g=2h:10+243h (**6**) …

g=2h+1:880+1458h→440+729h↑→ …

Alphabet c, d, e, f, g … in the above-listed operational routes expresses respectively each of natural numbers or 0, similarly hereinafter.
Also there are ♣↔♣, ♥↔♥, ♠↔♠, and ♦↔♦.

We conclude several partial satisfactory results from above-listed a bunch of operational routes of 15+12c, as the follows.

From c =2d+1 and d =2e+1, get c=2d+1=2(2e+1) +1=4e+3, and 15+12c = 15+12(4e+3) =51+48e > 29+27e where mark (1), so 15+12c where c=4e+3 suit the conjecture according to the axiom.

From c=2d+1, d=2e, and e=2f+1, get c =2d+1=4e+1=4(2f+1) +1=8f+5, and 15+12c=15+12(8f+5) =75+96f > 64+81f where mark (2), so 15+12c where c=8f+5 suit the conjecture according to the axiom.

From c=2d, d=2e+1 and e=2f+1, get c=2d= 4e+2= 4(2f+1) +2=8f+6, and 15+12c=15+12(8f+6) =87+96f >74+81f where mark (3), so 15+12c where c=8f+6 suit the conjecture according to the axiom.

From c=2d+1, d=2e, e=2f, f=2g+1 and g=2h+1, get c=2d+1=4e+1= 8f+1=8(2g+1)+1=16g+9=16(2h+1)+9=32h+25, and 15+12c=15+12(32h+ 25)=315+384h >200+243h where mark (4), so 15+12c where c=32h+25 suit the conjecture according to the axiom.

From c=2d, d=2e+1, e=2f, f=2g+1 and g=2h, get c=2d=2(2e+1)=4e+2= 8f+2= 8(2g+1)+2= 16g+10= 32h+10, and 15+12c= 15+12(32h+10)=

135+384h > 86+243h where mark (5), so 15+12c where c=32h+10 suit the conjecture according to the axiom.

From c=2d, d=2e, e=2f, f=2g and g=2h, get c=2d =32h, and 15+12c = 15+12(32h) =15+384h > 10+243h where mark (6), so 15+12c where c=32h suit the conjecture according to the axiom.

Secondly we operate 19+12c by the leftward operational rule below.

19+12c→58+36c→ 29+18c→ 88+54c→ 44+27c ♣

d=2e: 11+27e (**α**) e=2f:37+81f (**β**)

♣ 44+27c↓→c=2d: 22+27d↑→d=2e+1:148+162e→74+81e↑→e=2f+1:466+486f ♥

c=2d+1: 214+162d→107+81d↓→d=2e:322+486e ♠

d=2e+1:94+81e↓→e=2f:47+81f (γ)

e=2f+1:516+486f ♦

g=2h: 129+243h (**δ**) **…**

f=2g+1:258+243g↑→g=2h+1:1504+1458h→752+729h↑→**…**

♥466+486f→233+243f↑→f=2g:700+1458g→350+729g↓→g=2h+1:3238+4374h↓

g=2h: 175+729h↓→**…** **…**

…

g=2h+1:172+243h (**ε**)

f=2g: 101+243g↑→g=2h:304+1458h→**…**

e=2f+1:202+243f↑→f=2g+1:1336+1458g→**…**

♠322+486e→161+243e↑→e=2f:484+1458f→**…**

♦516+486f→258+243f↓→f=2g+1:1504+1458g→**…**

f=2g: 129+243g↓→g=2h:388+1458h→**…**

g=2h+1:186+243h (ζ)

Alphabet c, d, e, f, g, h **…** in the above-listed operational routes expresses respectively each of natural numbers or 0, similarly hereinafter.
Also there are ♣↔♣, ♥↔♥, ♠↔♠, ♦↔♦.

We conclude too several partial satisfactory results from above-listed a bunch of operational routes of 19+12c, as the follows.

From c=2d, d=2e, get c=2d=4e, and 19+12c= 19+12(4e) = 19+48e > 11+27e where mark (α), so 19+12c where c=4e suit the conjecture according to the axiom.

From $c=2d$, $d=2e+1$ and $e=2f$, get $c=2d=2(2e+1)=4e+2=8f+2$, and $19+12c=19+12(8f+2)=43+96f>37+81f$ where mark (β), so $19+12c$ where $c=8f+2$ suit the conjecture according to the axiom.

From $c=2d+1$, $d=2e$, and $e=2f$, get $c=2d+1=4e+1=8f+1$, and $19+12c=19+12(8f+1)=31+96f>47+81f$ where mark (γ), so $19+12c$ where $c=8f+1$ suit the conjecture according to the axiom.

From $c=2d$, $d=2e+1$, $e=2f+1$, $f=2g+1$ and $g=2h$, get $c=2d=2(2e+1)=4e+2=4(2f+1)+2=8f+6=8(2g+1)+6=16g+14=32h+14$, and $19+12c=19+12(32h+14)=187+384h>129+243h$ where mark (δ), so $19+12c$ where $c=32h+14$ suit the conjecture according to the axiom.

From $c=2d+1$, $d=2e$, $e=2f+1$, $f=2g$ and $g=2h+1$, get $c=2d+1=4e+1=4(2f+1)+1=8f+5=16g+5=16(2h+1)+5=32h+21$, and $19+12c=19+12(32h+21)=271+384h>172+243h$ where mark (ε), so $19+12c$ where $c=32h+21$ suit the conjecture according to the axiom.

From $c=2d+1$, $d=2e+1$, $e=2f+1$, $f=2g$ and $g=2h+1$, get $c=2d+1=2(2e+1)+1=4e+3=4(2f+1)+3=8f+7=16g+7=16(2h+1)+7=32h+23$, and $19+12c=19+12(32h+23)=295+384h>186+243h$ where mark (ζ), so $19+12c$ where $c=32h+23$ suit the conjecture according to the axiom.

Let $\chi=d, e, f, g, h$ … etc, for any indeterminate integer's expression at two bunches of operational routes of $15+12c$ plus $19+12c$, an operation where the unknown number$=2\chi$ and another operation where the unknown number $=2\chi+1$ synchronize. Further, begin with a greater result thereof, it will continue to operate. If the smaller result is more than 15+12c or 19+12c, then it must too continue to operate. If the smaller result is less than 15+12c or 19+12c, then it has suited the conjecture.

Or rather, on the one hand, two kinds' operations of an indeterminate integer's expression according to the unknown number $2\chi+1$ plus 2χ are always continuously progress and branch, up to infinitely progress and branch. Then positive integers successively got via orderly operations are inevitably getting more and more, greater and greater, up to reach infinite infinities theoretically.

On the other hand, it uninterruptedly stops operations of some branches in the infinite operational course, because each operational result at those branches is less than a kind of 15+12c plus 19+12c. We must be understood that there are infinitely many such results at infinite many operational branches. Well then, there are infinitely many kinds of 15+12c plus 19+12c to be proven to suit the conjecture.

Actually each integer's expression which has stopping operation is generally such a status, namely its constant term and the coefficient of χ are smaller than same species of integer's expressions on the left of the integer's expression at the operational route.

For an indeterminate integer's expression, we both operate it as an even number into a half of itself, and operate it as an odd number into three times of itself and add 1. Thus confront an incremental result and a reductive result at every step, there is only possibly the reductive result to suit the conjecture. Consequently operations of 15+12c plus 19+12c will proceed infinitely.

This signifies that 15+12c and 19+12c must be divided respectively as infinite many kinds, just enable every kind to operate to suit the conjecture for infinite many times by the leftward operational rule.

After operate 15+12c plus 19+12c for finite times, the number of kinds which 15+12c plus 19+12c emerge out is still finite, but the number of odd numbers of every kind is the infinity.

Whether an operation of each kind of 15+12c plus 19+12c is stopping, it relies only on a kind of c to decide.

This notwithstanding, since earlier arisen kinds of 15+12c plus 19+12c which suit the conjecture are all such integer's expressions, namely their constant terms and the coefficients of χ are smaller positive integers, thus after χ is bestowed with 0, 1, 2, 3…, we get smaller concrete positive odd numbers via finite operations by the leftward operational rule. For example, 51+48e, 75+96f, 87+96f, 315+384h, 135+384h, 15+384h, 19+48e, 43+96f, 31+96f, 187+384h, 271+384h and 295+384h at the above-listed two bunches of operational routes. Without doubt these smaller positive odd numbers belong still in 15+12c plus 19+12c.

What we need are merely odd numbers of 15+12c plus 19+12c between n and 2n+1, yet it is not theirs all. Evidently odd numbers of 15+12c plus 19+12c between n and 2n+1 are smaller odd numbers within unproved kindred odd numbers. Because of this, we can find easier each of them according to certain value of c after operating 15+12c and operating 19+12c via finite times by the leftward operational rule. Then, each such odd number of 15+12c plus 19+12c is proven to suit the conjecture, when find each of them just right.

To sum up, first we have proven that all even numbers between n and 2n+1 are at the bunch of integers' chains, by the rightward operational rule. After that, divide odd numbers between n and 2n+1 into two kinds, i.e. 5+4k and 7+4k, and have proven 5+4k to suit the conjecture, by the leftward operational rule. Next, again divide 7+4k into three kinds, i.e. 11+12c, 15+12c and 19+12c, and have proven 11+12c to suit the conjecture, by the leftward operational rule. For final remainder two kinds of 15+12c plus 19+12c between n and 2n+1, we have proven too them to suit the conjecture via finite operations by the leftward operational rule.

Strictly speaking, odd numbers between n and 2n+1, all consist of smaller positive odd numbers of 15+12c plus 19+12c plus 5+4k plus 11+12c. Consequently, we have proven every odd number between n and 2n+1 by the leftward operational rule to suit the conjecture. Correspondingly all odd number between n and 2n+1 are at the bunch of integers' chains.

On balance, we have proven that all positive integers between n and 2n+1 are at the bunch of integers' chains via finite operations by set each other's contrary operational rules. So far all positive integers $\leq 2^1$n are proven, they suit the conjecture.

Now that we have proven that positive integers from n to 2^1n suit the conjecture by consecutive positive integers$\leq$ n, likewise we can too prove that positive integers from 2^1n to 2^2n suit the conjecture by consecutive positive integers$\leq 2^1$n according to the foregoing way of doing.

At the beginning of the proof, we spoken that divide limits of all consecutive positive integers into segments according to greatest number 2^Xn at per segment, where X$\geq$0, and n $\geq$ 18.

After we proven that positive integers between 2^{X-1}n and 2^X n suit the conjecture by consecutive proven positive integers$\leq 2^{X-1}$n, in the same old way, we are too able to prove that positive integers between 2^X n and 2^{X+1}n suit the conjecture by consecutive proven positive integers$\leq 2^X$n.

For up-end 2^Xn of each segment of integers, X begins with 0, next it is orderly endowed with 1, 2, 3… In pace with which values of X are getting greater and greater, consecutive positive integers$\leq 2^X$n are getting more and more, and new positive integers which unceasingly increase are getting greater and greater. Suppose X equals every natural number plus 0, then all positive integers are proven to suit the conjecture, namely every positive integer is proven to suit the conjecture.

Or rather, we have proven the Collatz conjecture. Namely begin with any a positive integer, if it is even, then divide it by 2 to get an integer; if it is odd, then multiply it by 3 and add 1 to get an even, after that, repeat the above process indefinitely, then no matter which positive integer you start with, you will always eventually reach a result of 1.

The proof was thus brought to a close. As a consequence, the Collatz conjecture is tenable.