Prove Beal’s Conjecture by Fermat’s Last Theorem

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Abstract

In this article, we will prove the Beal’s conjecture by certain usual mathematical fundamentals with the aid of proven Fermat’s last theorem, and finally reach a conclusion that the Beal’s conjecture is tenable.

Keywords

Beal’s conjecture, Inequality, Indefinite equation, Fermat’s last theorem, Mathematical fundamentals, Odd-even attribute of A, B and C.

The proof

The Beal’s Conjecture states that if $A^X + B^Y = C^Z$, where A, B, C, X, Y and Z are all positive integers, and X, Y and Z are greater than 2, then A, B and C must have a common prime factor.

We regard limits of values of above-mentioned A, B, C, X, Y and Z as known requirements, hereinafter.

First, we must remove following two kinds from $A^X + B^Y = C^Z$ under the known requirements.

1. If A, B and C are all positive odd numbers, then $A^X + B^Y$ is an even number, yet $C^Z$ is an odd number, evidently there is only $A^X + B^Y \neq C^Z$ under the known requirements according to an odd number ≠ an even number.

2. If any two of A, B and C are positive even numbers, yet another is a
positive odd number, then when $A^X + B^Y$ is an even number, $C^Z$ is an odd number, yet when $A^X + B^Y$ is an odd number, $C^Z$ is an even number, so there is only $A^X + B^Y \neq C^Z$ under the known requirements according to an odd number $\neq$ an even number.

Thus, we reserve merely two kinds of indefinite equation $A^X + B^Y = C^Z$ under the known requirements plus each qualification as listed below.

1. A, B and C are all positive even numbers.

2. A, B and C are two positive odd numbers and a positive even number.

For indefinite equation $A^X + B^Y = C^Z$ under the known requirements plus aforementioned each qualification, in fact, it has many sets of solutions of positive integers. Let us instance following four concrete equations to explain such a viewpoint.

When A, B and C are all positive even numbers, if let $A=B=C=2$, $X=Y=3$, and $Z=4$, then indefinite equation $A^X + B^Y = C^Z$ is exactly equality $2^3 + 2^3 = 2^4$. Evidently $A^X + B^Y = C^Z$ has a set of solutions of positive integers (2, 2, 2) here, and A, B and C have common even prime factor 2.

In addition, if let $A=B=162$, $C=54$, $X=Y=3$, and $Z=4$, then, indefinite equation $A^X + B^Y = C^Z$ is exactly equality $162^3 + 162^3 = 54^4$. Evidently $A^X + B^Y = C^Z$ has a set of solutions of positive integers (162, 162, 54) here, and A, B and C have two common prime factors, i.e. even 2 and odd 3.

When A, B and C are two positive odd numbers and a positive even number, if let $A=C=3$, $B=6$, $X=Y=3$, and $Z=5$, then, indefinite equation
\( A^X + B^Y = C^Z \) is exactly equality \( 3^3 + 6^3 = 3^5 \). Evidently \( A^X + B^Y = C^Z \) has a set of solutions of positive integers \((3, 6, 3)\) here, and \( A, B \) and \( C \) have common prime factor 3.

In addition, if let \( A = B = 7, \ C = 98, \ X = 6, \ Y = 7, \) and \( Z = 3 \), then, indefinite equation \( A^X + B^Y = C^Z \) is exactly equality \( 7^6 + 7^7 = 98^3 \). Evidently \( A^X + B^Y = C^Z \) has a set of solutions of positive integers \((7, 7, 98)\) here, and \( A, B \) and \( C \) have common prime factor 7.

Thus it can seen that by above-mentioned four concrete examples, we have proved that indefinite equation \( A^X + B^Y = C^Z \) under the known requirements plus aforementioned each qualification can exist, but \( A, B \) and \( C \) have at least one common prime factor.

If we can prove that there is only \( A^X + B^Y \neq C^Z \) under the known requirements plus the qualification that \( A, B \) and \( C \) have not any common prime factor, then, we precisely proven that there is only \( A^X + B^Y = C^Z \) under the known requirements plus the qualification that \( A, B \) and \( C \) must have a common prime factor.

Since when \( A, B \) and \( C \) are all positive even numbers, \( A, B \) and \( C \) have common prime factor 2, therefore, for these circumstances that \( A, B \) and \( C \) have not any common prime factor, they can only occur under the prerequisite that \( A, B \) and \( C \) are two positive odd numbers and a positive even number.

If \( A, B \) and \( C \) have not any common prime factor, then any two of them
have not any common prime factor either. Because on the supposition that any two of them have a common prime factor, namely \( A^X + B^Y \) or \( C^Z - A^X \) or \( C^Z - B^Y \) have the prime factor, yet another has not it, then, this will lead to \( A^X + B^Y \neq C^Z \) or \( C^Z - A^X \neq B^Y \) or \( C^Z - B^Y \neq A^X \) according to the unique factorization theorem for a positive integer.

Such being the case, provided we can prove that there is only inequality \( A^X + B^Y \neq C^Z \) under the known requirements plus the qualification that \( A, B \) and \( C \) have not any common prime factor, then the Beal’s conjecture is surely tenable, otherwise it will be negated.

Unquestionably, following two inequalities together can wholly replace \( A^X + B^Y \neq C^Z \) under the known requirements plus the qualification that \( A, B \) and \( C \) have not any common prime factor.

1. \( A^X + B^Y \neq 2^2G^Z \) under the known requirements plus the qualification that \( A, B \) and \( 2G \) have not any common prime factor, where \( 2G = C \).

2. \( A^X + 2^YD^Y \neq C^Z \) under the known requirements plus the qualification that \( A, 2D \) and \( C \) have not any common prime factor, where \( 2D = B \).

We again divide \( A^X + B^Y \neq 2^2G^Z \) into two kinds, i.e. (1) \( A^X + B^Y \neq 2^2 \), when \( G=1 \), and (2) \( A^X + B^Y \neq 2^2G^Z \), where \( G \) has at least an odd prime factor >1.

Likewise divide \( A^X + 2^YD^Y \neq C^Z \) into two kinds, i.e. (3) \( A^X + 2^Y \neq C^Z \), when \( D=1 \), and (4) \( A^X + 2^YD^Y \neq C^Z \), where \( D \) has at least an odd prime factor >1.

We will prove that aforesaid four inequalities under the known requirements plus their qualifications are on the existence.
On purpose of the citation for convenience, let us first prove \(E^p + F^V \neq 2^M\), where \(E\) and \(F\) are two positive odd numbers without any common prime divisor, and \(P, V\) and \(M\) are integers \(>2\). Since \(E\) and \(F\) have not any common prime factor, so there is \(E^p \neq F^V\) according to the unique factorization theorem for a positive integer, then let \(F^V > E^p\).

In other words, let us Prove that indefinite equation \(E^p + F^V = 2^M\) has not a set of solutions of positive integers, where \(P, V\) and \(M\) are integers \(>2\).

We know that when \(P\) is an integer \(>2\), indefinite equation \(E^p + 1^p = 2^p\) has not a set of solutions of positive integers according to proven Fermat’s last theorem [REFERENCES], thus \(E\) is not a positive integer.

In the light of the same reason, when \(V\) is an integer \(>2\), indefinite equation \(F^V - 1^V = 2^V\) has not a set of solutions of positive integers, so \(F\) is not a positive integer either.

Next, two sides of equal-sign of \(E^p + 1^p = 2^p\) added respectively to two sides of equal-sign of \(F^V - 1^V = 2^V\) make \(E^p + F^V = 2^p + 2^V\).

For indefinite equation \(E^p + F^V = 2^p + 2^V\), when \(P=V\), \(2^p + 2^V = 2^{p+1}\), so \(E^p + F^V = 2^{p+1}\). Let \(P+1 = M\), there is \(E^p + F^V = 2^M\), but \(E\) and \(F\) at here are not two positive integers according to preceding two conclusions. If enable \(E\) and \(F\) into two positive odd numbers, then, there is only \(E^p + F^V \neq 2^M\).

However, when \(P \neq V\), \(2^p + 2^V \neq 2^M\), then \(E^p + F^V = 2^p + 2^V \neq 2^M\), i.e. \(E^p + F^V \neq 2^M\), where \(E\) and \(F\) at here are not two positive integers according to preceding two conclusions. If let \(E\) and \(F\) turn into two positive odd
numbers, then, whether multiply $E^p + F^v$ by a corresponding no positive integer such as $\mu$, or $E^p$ added to a corresponding no positive integer such as $\zeta$, and $F^v$ added to a corresponding no positive integer such as $\xi$, so whether must multiply $2^p + 2^v$ by $\mu$, or $2^p + 2^v$ must add to $\zeta + \xi$ on another side of the equality. Then, a result on another side can only be $(2^p + 2^v) \mu$ or $2^p + 2^v + \zeta + \xi$, and either result $\neq 2^M$, thus when $E$ and $F$ are two positive odd numbers, there is still $E^p + F^v \neq 2^M$.

In a word, we have proven $E^p + F^v \neq 2^M$, where $E$ and $F$ are two positive odd numbers without any common prime divisor, and $P$, $V$ and $M$ are integers $>2$.

On the basis of proven $E^p + F^v \neq 2^M$, we just set to prove aforementioned four inequalities, one by one, thereinafter.

Firstly, let $A^x = E^p$, $B^y = F^v$, and $2^z = 2^M$ for proven $E^p + F^v \neq 2^M$, we get $A^x + B^y \neq 2^z$ under the known requirements, where $2$ is a value of $C$.

Secondly, let us successively prove $A^x + B^y \neq 2^z G^z$ under the known requirements plus the qualification that $A$, $B$ and $2G$ have not any common prime factor, where $2G = C$, and $G$ has at least an odd prime factor $>1$.

To begin with, multiply each term of proven $E^p + F^v \neq 2^M$ by $G^M$ is $E^p G^M + F^v G^M \neq 2^M G^M$.

For any positive even number, either it is able to be expressed as $A^x + B^y$,
or it is unable. No doubt, $E^pG^M+F^VG^M$ is a positive even number.

If $E^pG^M+F^VG^M$ is able to be expressed as $A^X+B^Y$, then there is $A^X+B^Y \neq 2^MG^M$.

If $E^pG^M+F^VG^M$ is unable to be expressed as $A^X+B^Y$, then it has nothing to do with proving $A^X+B^Y \neq 2^MG^M$.

Under this case, there are still $E^pG^M+F^VG^M \neq A^X+B^Y$ and $E^pG^M+F^VG^M \neq 2^MG^M$, so let $E^pG^M+F^VG^M$ equals $A^X+B^Y+2b$ or $A^X+B^Y-2b$, where $b$ is a positive integer. Also use sign “±” to denote sign “+” and sign “-” hereinafter, then we get $A^X+B^Y \pm 2b \neq 2^MG^M$, i.e. $A^X+B^Y \neq 2^MG^M \pm 2b$.

Since $2b$ can express every positive even number, then $2^MG^M \pm 2b$ can express all positive even numbers except for $2^MG^M$.

For a positive even number, either it is able to be expressed as $2^KN^K$, or it is unable, where $K$ is an integer $>2$, and $N$ is a positive integer which has at least an odd prime factor $>1$.

On the one hand, where $2^MG^M \pm 2b=2^KN^K$, there is $A^X+B^Y \neq 2^KN^K$. On the other hand, where $2^MG^M \pm 2b \neq 2^KN^K$, $2^MG^M \pm 2b$ has nothing to do with proving $A^X+B^Y \neq 2^KN^K$.

That is to say, for $E^pG^M+F^VG^M \neq 2^MG^M$, if $E^pG^M+F^VG^M$ is unable to be expressed as $A^X+B^Y$, we can deduce $A^X+B^Y \neq 2^KN^K$ too, elsewhere.

Hereto, we have proven $A^X+B^Y \neq 2^MG^M$ or $A^X+B^Y \neq 2^KN^K$ on the existence.

Since either $M$ or $K$ is to express an integer $>2$, also either $G$ or $N$ is a positive integer which has at least an odd prime factor $>1$, therefore both
can represent from each other.

Thirdly, we proceed to prove $A^X + 2^Y \neq C^Z$ under the known requirements plus the qualification that $A$ and $C$ are two positive odd numbers without any common prime factor, where $2$ is a value of $B$.

In the former passage, we have proven $E^P + F^V \neq 2^M$, where $F^V > E^P$, so let $F^V = C^Z$, then there is $E^P + C^Z \neq 2^M$.

Moreover, let $2^M > 2^3$, then there is $2^M = 2^{M-1} + 2^{M-1}$.

So there is $E^P + C^Z > 2^{M-1} + 2^{M-1}$ or $E^P + C^Z < 2^{M-1} + 2^{M-1}$.

Namely, there is $C^Z - 2^{M-1} > 2^{M-1} - E^P$ or $C^Z - 2^{M-1} < 2^{M-1} - E^P$.

In addition, there is $A^X + E^P \neq 2^{M-1}$ according to proven $E^P + F^V \neq 2^M$.

Then, we deduce $2^{M-1} - E^P > A^X$ or $2^{M-1} - E^P < A^X$ from $A^X + E^P \neq 2^{M-1}$.

Therefore, there is $C^Z - 2^{M-1} > 2^{M-1} - E^P > A^X$ or $C^Z - 2^{M-1} < 2^{M-1} - E^P < A^X$.

Consequently, there is $C^Z - 2^{M-1} > A^X$ or $C^Z - 2^{M-1} < A^X$.

In a word, there is $C^Z - 2^{M-1} \neq A^X$, i.e. $A^X + 2^{M-1} \neq C^Z$.

For $A^X + 2^{M-1} \neq C^Z$, let $2^{M-1} = 2^Y$, we get $A^X + 2^Y \neq C^Z$.

Fourthly, let us last prove $A^X + 2^Y D^Y \neq C^Z$ under the known requirements plus the qualification that $A$, $2D$ and $C$ have not any common prime factor, where $2D = B$, and $D$ has at least an odd prime factor $> 1$.

For the sake that distinguish between differing cases, we need to start using another inequality $H^U + 2^Y \neq K^T$ in the light of proven inequality $A^X + 2^Y \neq C^Z$, where $H$ and $K$ are two positive odd numbers without any
common prime factor, and U, Y and T are integers >2.

Then, there is $K^T - H^U \neq 2^Y$. Like that, multiply each term of $K^T - H^U \neq 2^Y$ by $D^Y$ is $K^TD^Y - H^UD^Y \neq 2^YD^Y$.

For any positive even number, either it is able to be expressed as $C^Z - A^X$, or it is unable. Undoubtedly, $K^TD^Y - H^UD^Y$ is a positive even number.

If $K^TD^Y - H^UD^Y$ is able to be expressed as $C^Z - A^X$, then there is $C^Z - A^X \neq 2^YD^Y$, i.e. $A^X + 2^YD^Y \neq C^Z$.

If $K^TD^Y - H^UD^Y$ is unable to be expressed as $C^Z - A^X$, then $K^TD^Y - H^UD^Y$ at here has nothing to do with proving $A^X + 2^YD^Y \neq C^Z$. Under this case, there are still $K^TD^Y - H^UD^Y \neq C^Z - A^X$ and $K^TD^Y - H^UD^Y \neq 2^YD^Y$.

Let $K^TD^Y - H^UD^Y$ equals $C^Z - A^X \pm 2d$, where d is a positive integer.

Then, there is $C^Z - A^X \pm 2d \neq 2^YD^Y$, i.e. $C^Z - A^X \neq 2^YD^Y \pm 2d$.

Since $2d$ can express every positive even number, then $2^YD^Y \pm 2d$ can express all positive even numbers except for $2^YD^Y$.

For a positive even number, either it is able to be expressed as $2^S R^S$, or it is unable, where S is an integer >2, and R is a positive integer which has at least an odd prime factor >1.

On the one hand, where $2^YD^Y \pm 2d = 2^S R^S$, there is $C^Z - A^X \neq 2^S R^S$, i.e. $A^X + 2^S R^S \neq C^Z$. On the other hand, where $2^YD^Y \pm 2d \neq 2^S R^S$, $2^YD^Y \pm 2d$ has nothing to do with proving $A^X + 2^S R^S \neq C^Z$.

That is to say, for $K^TD^Y - H^UD^Y \neq 2^YD^Y$, if $K^TD^Y - H^UD^Y$ is unable to be expressed as $C^Z - A^X$, we can deduce $A^X + 2^S R^S \neq C^Z$ too, elsewhere.
Thus far, we have proven $A^{X+2}D^{Y}≠C^{Z}$ or $A^{X+2}R^{S}≠C^{Z}$ on the existence. Since either $Y$ or $S$ is to express an integer $>2$, also either $D$ or $R$ is a positive integer which has at least an odd prime factor $>1$, therefore both can represent from each other.

To sum up, we have proven every kind of $A^{X}B^{Y}≠C^{Z}$ under the known requirements plus the qualification that $A$, $B$ and $C$ have not any common prime factor.

Previous, we have proven $A^{X}B^{Y}=C^{Z}$ under the known requirements plus the qualification that $A$, $B$ and $C$ have at least a common prime factor, it has certain sets of solutions of positive integers.

Overall, after the compare between $A^{X}B^{Y}=C^{Z}$ under the known requirements and $A^{X}B^{Y}≠C^{Z}$ under the known requirements, we have reached inevitably such a conclusion, namely an indispensable prerequisite of the existence of $A^{X}B^{Y}=C^{Z}$ under the known requirements is that $A$, $B$ and $C$ must have a common prime factor.

The proof was thus brought to a close. As a consequence, the Beal conjecture is tenable.