Why Yang-Mills Magnetic Monopoles Appear to Confine their Gauge Fields and have Composite Features, Similarly to Baryons

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Abstract: We develop in detail, the classical magnetic monopoles of Yang-Mills gauge theory, and show how these classical monopoles, when analyzed using Gauss' / Stokes' theorem, appear to confine their gauge fields, and also, appear to be composite objects. Of course, baryons, which include the protons and neutrons at the heart of nuclear physics, also confine their gauge fields and are similarly-composite objects. This raises the question whether the magnetic monopoles of Yang-Mills theory are in some fashion related to the observed physical baryons. Because this exposition is classical, we also discuss the extent to which classical field theory can be used to effectively analyze baryons and confinement, and what would need to also be considered in a complete quantum field development.

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1. Introduction: The Field Strength Curvature Tensor in Gauge Theory, and a Review of Gauge-Covariant Derivatives

In 1918, [1], [2] Hermann Weyl first conceived the idea that electrodynamics might be unified with Einstein's recently-developed geometric view of gravitation [3], by analyzing a "twisting" of vectors under parallel transport to measure the geometric curvature of a gauge space. While Weyl first conceived of this as a local "gauge" symmetry, in 1929 [4] he corrected his original misconception into the modern view of a local "phase" symmetry. Notwithstanding, the original misnomer "gauge" is still used to name Weyl's theory, perhaps as a reminder to posterity that even the most foundational physical theories are sometimes properly-conceived in the abstract but misconceived in some details that need to be worked out over time.

In gravitational theory, the Riemann curvature tensor $R^{\sigma}{}_{\alpha\mu\nu}$ may of course be *defined* as a measure of the degree to which the gravitationally-covariant derivative $\partial_{;\mu}$ is non-commuting when it operates on an arbitrary vector A_σ, that is, as $R^{\sigma}{}_{\alpha\mu\nu}A_\sigma \equiv \left[\partial_{;\mu},\partial_{;\nu}\right]A_\alpha$. What Weyl in essence found, is that the antisymmetric, second rank, field strength tensor / bivector $F_{\mu\nu}$ which appears in electromagnetic theory may be defined as a measure of the extent to which the gauge-covariant derivative D_μ is not self-commuting when it operates on an arbitrary scalar field φ. That is, $F_{\mu\nu}$ may be *defined* analogously to $R^{\sigma}{}_{\alpha\mu\nu}$, as a type of curvature in "gauge space," by:

$$F_{\mu\nu}\varphi \equiv i\left[D_\mu,D_\nu\right]\varphi = iD_\mu\left(D_\nu\varphi\right)-iD_\nu\left(D_\mu\varphi\right). \tag{1.1}$$

It is instructive to review how the explicit relationship between the field strength $F_{\mu\nu}$ and a gauge / vector potential G_μ then arises from this definition (1.1).

Gauge-covariant derivatives, like covariant derivatives in Riemannian geometry, take a form that depends on the representation of the object they act upon. Taking the gauge field as the defining (fundamental) representation, the form of the gauge-covariant derivatives in (1.1) is $D_{F\mu}=\partial_\mu - iG_\mu$, where the subscripted *F* denotes "fundamental." But in other situations to be reviewed, it is a bit more complicated than this. (In general, for compactness, we scale the interaction charge strength *g* into the gauge field via $gG_\mu \to G_\mu$. This *g* can always be extracted back out when explicitly needed.) So, applying $D_{F\mu}=\partial_\mu - iG_\mu$ in (1.1), we may write:

$$\begin{aligned} &iD_{F\mu}\left(D_{F\nu}\varphi\right)=i\left(\partial_\mu - iG_\mu\right)\left(\left(\partial_\nu - iG_\nu\right)\varphi\right)=i\partial_\mu\left(\partial_\nu\varphi - iG_\nu\varphi\right)+G_\mu\left(\partial_\nu\varphi - iG_\nu\varphi\right)\\ &= i\partial_\mu\partial_\nu\varphi+\partial_\mu G_\nu\varphi+G_\nu\partial_\mu\varphi+G_\mu\partial_\nu\varphi-iG_\mu G_\nu\varphi \end{aligned}, \tag{1.2}$$

as well as the reverse-signed, transposed-indexed:

$$-iD_{F\nu}\left(D_{F\mu}\varphi\right)=-i\partial_\nu\partial_\mu\varphi-\partial_\nu G_\mu\varphi-G_\mu\partial_\nu\varphi-G_\nu\partial_\mu\varphi+iG_\nu G_\mu\varphi. \tag{1.3}$$

Using (1.2) and (1.3) in (1.1) then yields:

$$F_{\mu\nu}\varphi \equiv i\left[D_{F\mu},D_{F\nu}\right]\varphi = iD_{F\mu}\left(D_{F\nu}\varphi\right) - iD_{F\nu}\left(D_{F\mu}\varphi\right) = i\left[\partial_\mu,\partial_\nu\right]\varphi + \partial_{[\mu}G_{\nu]}\varphi - i\left[G_\mu,G_\nu\right]\varphi . \quad (1.4)$$

In flat spacetime where $R^\sigma{}_{\alpha\mu\nu}A_\sigma \equiv \left[\partial_{;\mu},\partial_{;\nu}\right]A_\alpha = \left[\partial_\mu,\partial_\nu\right]A_\alpha = 0$ and removing the arbitrary operand field φ, the above becomes the more familiar:

$$F_{\mu\nu} = \partial_{[\mu}G_{\nu]} - i\left[G_\mu,G_\nu\right] = \left(\partial_{[\mu} - iG_{[\mu}\right)G_{\nu]} = D_{F[\mu}G_{\nu]} . \quad (1.5)$$

Again, $D_{F\mu} \equiv \partial_\mu - iG_\mu$ above is the gauge-covariant derivative when it acts upon gauge field objects G_ν in the fundamental representation, but in general, when operating on other representations, it is a bit more complicated.

If the gauge fields commute, i.e., if $\left[G_\mu,G_\nu\right]=0$, then (1.5) reduces to $F_{\mu\nu} = \partial_{[\mu}G_{\nu]} = \partial_\mu G_\nu - \partial_\nu G_\mu$ and the gauge theory is known as an *abelian* gauge theory. If the gauge fields do *not* commute, $\left[G_\mu,G_\nu\right]\neq 0$, then (1.5) becomes the field strength for a *non-abelian* gauge theory, often also referred to as Yang-Mills [5] gauge theory.

Using differential forms, we may write the abelian field strength as:

$$F = \tfrac{1}{2!}F_{\mu\nu}dx^\mu \wedge dx^\nu = F_{\mu\nu}dx^\mu dx^\nu = \left(\partial_\mu G_\nu - \partial_\nu G_\mu\right)dx^\mu dx^\nu = \partial_\mu G_\nu dx^\mu \wedge dx^\nu = dG . \quad (1.6)$$

In general, the wedge product $dx^\mu \wedge dx^\nu = dx^\mu dx^\nu - dx^\nu dx^\mu = \left[dx^\mu, dx^\nu\right]$ is antisymmetric under adjacent index interchange, and the differential elements are anticommuting, $dx^\mu dx^\nu = -dx^\nu dx^\mu$. So, by inspection from (1.5) in view of (1.6), the non-abelian field strength is:

$$F = dG - i\left[G,G\right] \equiv DG . \quad (1.7)$$

Here, compacted into differential forms, the gauge-covariant derivative is not separable from its operand as was $D_{F\mu} = \partial_\mu - iG_\mu$ when operating on G_ν in (1.1) to (1.5), but rather involves the commutator of *G* with the operand which, in this case, just so happens to also be *G*. This in fact reveals the more-general form of the gauge-covariant derivative as we shall review next, and so we have removed the *F* subscript

Now, focusing on non-abelian gauge theories, we introduce a set of Hermitian generators $t^i = t^{\dagger i}$ which form a closed group under multiplication via $\left[t^i,t^j\right] = if^{ijk}t^k$, where f^{ijk} are the group structure constants and are antisymmetric under the transposition of any two adjacent indexes. For any simple group SU(N), the internal symmetry indexes $i,j,k = 1...N^2-1$. We may then define $F_{\mu\nu} \equiv t^k F^k{}_{\mu\nu}$ and $G_\mu \equiv t^i G^i{}_\mu$ and use these in (1.5) to expand:

$$F_{\mu\nu} = t^k F^k{}_{\mu\nu} = \partial_{[\mu} G_{\nu]} - i\left[G_\mu, G_\nu\right] = t^k \partial_{[\mu} G^k{}_{\nu]} - i\left[t^i, t^j\right] G^i{}_\mu G^j{}_\nu = t^k \partial_{[\mu} G^k{}_{\nu]} + f^{ijk} t^k G^i{}_\mu G^j{}_\nu \,. \quad (1.8)$$

Factoring out t^k this simplifies to the recognizable:

$$F^k{}_{\mu\nu} = \partial_{[\mu} G^k{}_{\nu]} + f^{ijk} G^i{}_\mu G^j{}_\nu \,. \quad (1.9)$$

Now, let us momentarily consider the situation where the t^i are one half (½) times the three (3) Pauli spin matrix generators of SU(2), $t^i = \frac{1}{2}\sigma^i$, so that f^{ijk} simply becomes the rank-3 Levi-Civita tensor, $f^{ijk} \to \varepsilon^{ijk}$, which again, is antisymmetric in all indexes. In spacetime, if we were to write $\varepsilon^{ijk} A^i B^j$ for any two vectors A^i and B^j and were to regard i, j, k as indexes for the *space* dimensions *x*, *y*, *z*, then, for example, $\varepsilon^{ij3} A^i B^j = A^1 B^2 - A^2 B^1 = (\mathbf{A}\times\mathbf{B})^3$ is the z-component of the cross product $\mathbf{A}\times\mathbf{B}$, and more generally, $\varepsilon^{ijk} A^i B^j = (\mathbf{A}\times\mathbf{B})^k$. But of course, the i, j, k indexes in (1.9) are not space indexes, but are *internal symmetry* indexes. So rather than using the cross-product symbol "×" which is used for vectors in physical space, and because we still wish to be able compactly represent the fundamentally-antisymmetric character of f^{ijk} in the form of a "cross-like product" in internal symmetry space, we instead employ the wedge symbol "$\wedge$." Although $G^i{}_\mu$ and $G^j{}_\nu$ in (1.9) both are gauge fields *G*, they have different spacetime indexes μ and ν, so we may still think of them as two different vectors just like A^i and B^j above. So analogously to $\varepsilon^{ijk} A^i B^j = (\mathbf{A}\times\mathbf{B})^k$ in the three space dimensions of spacetime, we write $f^{ijk} G^i{}_\mu G^j{}_\nu = \left(G_\mu \wedge G_\nu\right)^k$ in internal symmetry space. Then, we use this in (1.9) to write $F^k{}_{\mu\nu} = \partial_{[\mu} G^k{}_{\nu]} + \left(G_\mu \wedge G_\nu\right)^k$. Because the general form of this equation holds in SU(N) for each of the indexes $k = 1 \ldots N^2 - 1$, we may remove the *k* index throughout to write:

$$F_{\mu\nu} = \partial_{[\mu} G_{\nu]} + G_\mu \wedge G_\nu \,. \quad (1.10)$$

Then, compacting (1.10) to differential forms as in (1.6), we have:

$$\begin{aligned} F &= \tfrac{1}{2!} F_{\mu\nu} dx^\mu \wedge dx^\nu = F_{\mu\nu} dx^\mu dx^\nu = \left(\partial_\mu G_\nu - \partial_\nu G_\mu + G_\mu \wedge G_\nu\right) dx^\mu dx^\nu \\ &= \partial_\mu G_\nu dx^\mu \wedge dx^\nu + \tfrac{1}{2!} G_\mu \wedge G_\nu dx^\mu \wedge dx^\nu = dG + G \wedge G = \left(d + G\wedge\right) G \equiv DG \end{aligned} \,. \quad (1.11)$$

Now, Jaffe and Witten point out at pages 1 and 2 of [6], that:

> "If *A* denotes the U(1) gauge connection, locally a one-form on space-time, then the curvature or electromagnetic field tensor is the two-form $F = dA$ [see (1.6) above], and Maxwell's equations in the absence of charges and currents read $0 = dF = d * F$."

They then proceed to explain that in "non-abelian gauge theory":

> "at the classical level one replaces the gauge group U(1) of electromagnetism by a compact gauge group G. The definition of the curvature arising from the connection must be modified to $F = dA + A \wedge A$ and Maxwell's equations are replaced by the Yang–Mills equations, $0 = d_A F = d_A * F$, where d_A is the gauge-covariant extension of the exterior derivative."

Equation (1.11) is precisely $F = dA + A \wedge A$ with the gauge field simply renamed from *A* to *G*, and what Jaffe and Witten write above is a condensed explanation for what we have laid out above in equations (1.1) through (1.11). When we use the generalized one-form *G* and two-form *F* without any particular generator set t^i, then the differential forms equation is written as $F = dG - i[G,G]$ in (1.7). But when one does introduce a set of group generators t^i and the antisymmetric structure contestants $f^{ijk} \to \wedge$, the differential forms equation is $F = dG + G \wedge G$ in (1.11). To display the particular $i = 1...N^2 - 1$ field components for a compact simple gauge group SU(N), this equation is $F^i = dG^i + (G \wedge G)^i$. So $F = dG - i[G,G]$ (commutator form) and $F = dG + G \wedge G$ (wedge form) are just alternative ways of saying the same thing. But a benefit of the wedge form is that we may write $F = (d + G\wedge)G \equiv DG$ so as to define a gauge-covariant derivative $D \equiv (d + G\wedge)$ $(= d_A)$ in a form which is fully-separable from its operand, and which is generally applicable to *any and all operands*. We will find it useful in general to develop both these forms.

Indeed, the reason we have gone through the exercise of (1.8) through (1.11), is to explore the question of how one generally performs $d_A = D$, independently of its operand, "where d_A is the gauge-covariant extension of the exterior derivative." That is, we want to be able to generalize the taking of these derivatives, and especially, to ascertain the correct way to derive the equations $*J = d_A * F = D * F$ and $P = d_A F = DF$ which specify the electric and magnetic three-form charge densities $*J$ and P.

Specifically, as already stated, if we write equation (1.11) as $F = (d + G\wedge)G \equiv DG$ with $D \equiv (d + G\wedge)$, we find that $D \equiv (d + G\wedge)$ *is in fact the generalized definition of the gauge-covariant derivative which tells us how to take higher-rank gauge derivatives, independent of the representation of the operand*. Thus, the Maxwell equations for Yang-Mills theory, in differential forms, where t^i and f^{ijk} are specified, with index *i* suppressed, for SU(N), where we use the duality operator *, and with $F = dG + G \wedge G$, are merely the $i = 1...N^2 - 1$ equations:

$$\begin{aligned} *J &= D*F = D*DG = (d+G\wedge)*F = d*F + G\wedge *F = d*(dG+G\wedge G) + G\wedge *(dG+G\wedge G) \\ &= d*dG + d*(G\wedge G) + G\wedge *dG + G\wedge *(G\wedge G) \\ P &= DF = DDG = (d+G\wedge)F = dF + G\wedge F = d(dG+G\wedge G) + G\wedge (dG+G\wedge G) \\ &= ddG + d(G\wedge G) + G\wedge dG + G\wedge G\wedge G \end{aligned} \quad .(1.12)$$

The duality operator was first developed by Reinich [7] later elaborated by Wheeler [8], and it uses the Levi-Civita as laid out in [9] at pages 87-89.

In this paper, we shall develop the classical Yang-Mills magnetic monopole P in detail, and shall show how this monopole, when analyzed using Gauss' / Stokes' theorem, appears to confine its gauge fields. Of course, baryons, which include the protons and neutrons at the heart of nuclear physics, also confine their gauge fields. So this raises the question whether the magnetic monopoles of Yang-Mills theory are in some fashion related to baryons.

2. Classical Field Equations for the Yang-Mills Magnetic Monopole

To further develop the monopole P, first, akin to the derivation (1.1) through (1.5), we calculate the commutator:

$$\begin{aligned} &\left[D_\sigma, F_{\mu\nu}\right]\varphi = D_\sigma\left(F_{\mu\nu}\varphi\right) - F_{\mu\nu}D_\sigma\varphi = \left(\partial_\sigma - iG_\sigma\right)\left(F_{\mu\nu}\varphi\right) - F_{\mu\nu}\left(\partial_\sigma - iG_\sigma\right)\varphi \\ &= \partial_\sigma F_{\mu\nu}\varphi + F_{\mu\nu}\partial_\sigma\varphi - iG_\sigma F_{\mu\nu}\varphi - F_{\mu\nu}\partial_\sigma\varphi + iF_{\mu\nu}G_\sigma\varphi = \partial_\sigma F_{\mu\nu}\varphi - i\left[G_\sigma, F_{\mu\nu}\right]\varphi \end{aligned} . \quad (2.1)$$

We *can* use $D_\sigma = D_{F\sigma} = \partial_\sigma - iG_\sigma$ in the above, precisely because this is a commutator, and so the gauge field will be commuted with the operand $F_{\mu\nu}$ as in $F = dG - i[G,G]$ a.k.a. $F = dG + G\wedge G$. Removing φ we see that (2.1) contains the useful identity:

$$\left[D_\sigma, F_{\mu\nu}\right] = \partial_\sigma F_{\mu\nu} - i\left[G_\sigma, F_{\mu\nu}\right] = D_\sigma F_{\mu\nu} . \quad (2.2)$$

Then, combining (2.2) with (1.1) in the form $F_{\mu\nu} = i\left[D_\mu, D_\nu\right]$ first yields:

$$D_\sigma F_{\mu\nu} = i\left[D_\sigma, \left[D_\mu, D_\nu\right]\right] \quad (2.3)$$

containing an anticommuting succession of gauge-covariant derivatives. This in turn means that the index-cyclical combination:

$$P_{\sigma\mu\nu} = D_\sigma F_{\mu\nu} + D_\mu F_{\nu\sigma} + D_\nu F_{\sigma\mu} = i\left(\left[D_\sigma, \left[D_\mu, D_\nu\right]\right] + \left[D_\mu, \left[D_\nu, D_\sigma\right]\right] + \left[D_\nu, \left[D_\sigma, D_\mu\right]\right]\right) = 0 , \quad (2.4)$$

by the Jacobian identity. So we see that the *Yang-Mills magnetic monopoles vanish, just like those of abelian gauge theory*. Consequently, we can append $P=0$ from (2.4) to (1.12), and so write $P = DF = DDG = 0$.

But there is another zero in the monopole P of (1.12), and that is the zero which comes from $ddG = 0$. This is rooted in the geometric relationship $dd = \mathbf{0}$ of exterior calculus in spacetime: "the exterior derivative of an exterior derivative is zero." In general in this paper, we shall highlight the zero of $dd = \mathbf{0}$ to distinguish it from the (not highlighted) zero of the Jacobian identity $DDG = 0$ which is established by the combination of (1.12) and (2.4). The highlighted zero in $dd = \mathbf{0}$ is a "subset" identity contained within (1.12), which we may now rewrite as:

$$0 = P = DF = DDG = \mathbf{0} + d\left(G \wedge G\right) + G \wedge dG + G \wedge G \wedge G. \tag{2.5}$$

Of course, in an abelian gauge theory such as Maxwell's electrodynamics where $\left[G_\mu, G_\nu\right] = 0$ so that $F_{\mu\nu} = \partial_{[\mu} G_{\nu]}$ in (1.5) thus $F = dG$, the Magnetic monopole densities are themselves specified by $P_{\text{abelian}} = dF = ddG = \mathbf{0}$. This means that the Yang-Mills monopole density in (2.5), although it too is equal to zero, contains a number of non-zero terms embedded within, as well as the term $ddG = \mathbf{0}$ which we associate with the vanishing monopoles of electrodynamics. This will be very important to keep in mind as we develop this monopole, because this "abelian subset" embedding of $ddG = \mathbf{0}$ within (2.5) will be directly responsible for *confining* the gauge fields within the Yang-Mills monopole, and will lead us to consider whether there is some connection between Yang-Mills monopoles and baryons.

Next let us ascertain the commutator form for the monopole (2.5). Via the exact same type of calculation we used to turn (1.5) a.k.a. (1.7) into (1.11), one may demonstrate that $P = DF = dF - i[G, F]$ is equivalent to $P = DF = \left(d + G \wedge\right)F$. So, combining the former, $P = DF = dF - i[G, F]$, with $F = DG = dG - i[G, G]$ a.k.a. $F = DG = \left(d + G \wedge\right)G$ from (1.7) and (1.11) , we may translate (2.5) into the commutator expression:

$$\begin{aligned} P = DF = DDG &= dF - i[G,F] = d\left(dG - i[G,G]\right) - i\left[G, dG - i[G,G]\right] \\ &= ddG - id[G,G] - i[G,dG] - \left[G,[G,G]\right] \\ &= \mathbf{0} - id[G,G] - i[G,dG] - \left[G,[G,G]\right] = 0 \end{aligned} \quad . \tag{2.6}$$

Let us now expand (2.6) above into tensor components term-by-term, and then do some reconsolidation of terms. For P and $-id[G,G]$ we have:

$$P = \tfrac{1}{3!} P_{\sigma\mu\nu} dx^\sigma \wedge dx^\mu \wedge dx^\nu = P_{\sigma\mu\nu} dx^\sigma dx^\mu dx^\nu, \tag{2.7}$$

$$\begin{aligned}
&-id[G,G]=-\tfrac{1}{3!}i\left(\partial_\sigma\left[G_\mu,G_\nu\right]+\partial_\mu\left[G_\nu,G_\sigma\right]+\partial_\nu\left[G_\sigma,G_\mu\right]\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=-\tfrac{1}{2!}i\partial_\sigma\left[G_\mu,G_\nu\right]dx^\sigma\wedge dx^\mu\wedge dx^\nu=-3i\partial_\sigma\left[G_\mu,G_\nu\right]dx^\sigma dx^\mu dx^\nu=-6i\partial_\sigma\left(G_\mu G_\nu\right)dx^\sigma dx^\mu dx^\nu\\
&=-6i\left(\partial_\sigma G_\mu G_\nu+G_\mu\partial_\sigma G_\nu\right)dx^\sigma dx^\mu dx^\nu=-6i\left(\partial_\sigma G_\mu G_\nu-G_\sigma\partial_\mu G_\nu\right)dx^\sigma dx^\mu dx^\nu\\
&=-i\left(\partial_\sigma G_\mu G_\nu-G_\sigma\partial_\mu G_\nu\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=-\tfrac{1}{3!}i\left(\partial_{[\sigma}G_{\mu]}G_\nu+\partial_{[\mu}G_{\nu]}G_\sigma+\partial_{[\nu}G_{\sigma]}G_\mu\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&\quad+\tfrac{1}{3!}i\left(G_\sigma\partial_{[\mu}G_{\nu]}+G_\mu\partial_{[\nu}G_{\sigma]}+G_\nu\partial_{[\sigma}G_{\mu]}\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=-idGG+iGdG
\end{aligned}\quad. \tag{2.8}$$

The sign reversal in the third line of (2.8) reveals the identity $d[G,G]=dGG-GdG$, in contrast to scalar product rule $d(a\cdot b)=da\cdot b+a\cdot db$. For $-i[G,dG]$ we further have:

$$\begin{aligned}
&-i[G,dG]=-\tfrac{1}{3!}i\left(\left[G_\sigma,\partial_\mu G_\nu\right]+\left[G_\mu,\partial_\nu G_\sigma\right]+\left[G_\nu,\partial_\sigma G_\mu\right]\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=-\tfrac{1}{2!}i\left[G_\sigma,\partial_\mu G_\nu\right]dx^\sigma\wedge dx^\mu\wedge dx^\nu=-3i\left[G_\sigma,\partial_\mu G_\nu\right]dx^\sigma dx^\mu dx^\nu\\
&=-3i\left[G_\sigma\partial_\mu G_\nu-\partial_\mu\left(G_\nu G_\sigma\right)\right]dx^\sigma dx^\mu dx^\nu=-3i\left[G_\sigma\partial_\mu G_\nu-G_\nu\partial_\mu G_\sigma-\partial_\mu G_\nu G_\sigma\right]dx^\sigma dx^\mu dx^\nu\\
&=-3i\left[G_\sigma\partial_\mu G_\nu-G_\sigma\partial_\mu G_\nu-\partial_\mu G_\nu G_\sigma\right]dx^\sigma dx^\mu dx^\nu=3i\partial_\sigma G_\mu G_\nu dx^\sigma dx^\mu dx^\nu\\
&=\tfrac{1}{2}i\partial_\sigma G_\mu G_\nu dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=\tfrac{1}{3!}i\left(\partial_\sigma G_\mu G_\nu+\partial_\mu G_\nu G_\sigma+\partial_\nu G_\sigma G_\mu\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=\tfrac{1}{2}\tfrac{1}{3!}i\left(\partial_{[\mu}G_{\nu]}G_\sigma+\partial_{[\nu}G_{\sigma]}G_\mu+\partial_{[\sigma}G_{\mu]}G_\nu\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=\tfrac{1}{2}idGG
\end{aligned} \tag{2.9}$$

in which the GdG cancel out by a similar sign reversal in the third and fourth lines. Finally:

$$\begin{aligned}
&-\left[G,[G,G]\right]=-\tfrac{1}{3!}\left(\left[G_\sigma,\left[G_\mu,G_\nu\right]\right]+\left[G_\mu,\left[G_\nu,G_\sigma\right]\right]+\left[G_\nu,\left[G_\sigma,G_\mu\right]\right]\right)dx^\sigma\wedge dx^\mu\wedge dx^\nu\\
&=-\tfrac{1}{2!}\left[G_\sigma,\left[G_\mu,G_\nu\right]\right]dx^\sigma\wedge dx^\mu\wedge dx^\nu=-3\left[G_\sigma,\left[G_\mu,G_\nu\right]\right]dx^\sigma dx^\mu dx^\nu\\
&=-6\left[G_\sigma,G_\mu G_\nu\right]dx^\sigma dx^\mu dx^\nu=-12G_\sigma G_\mu G_\nu dx^\sigma dx^\mu dx^\nu
\end{aligned}\quad. \tag{2.10}$$

In (2.6), we then use (2.7) to (2.10) and $d[G,G]=dGG-GdG$ from (2.8) to restructure, thus reducing and consolidating the monopole as much as is possible, into:

$$\begin{aligned}P &= \mathbf{0} - id\left[G,G\right] - i\left[G,dG\right] - \left[G,\left[G,G\right]\right] \\ &= \mathbf{0} - idGG + iGdG + \tfrac{1}{2}idGG - \left[G,\left[G,G\right]\right] \\ &= \mathbf{0} - \tfrac{1}{2}idGG + \tfrac{1}{2}iGdG + \tfrac{1}{2}iGdG - \left[G,\left[G,G\right]\right] \\ &= \mathbf{0} - \tfrac{1}{2}id\left[G,G\right] + \tfrac{1}{2}iGdG - \left[G,\left[G,G\right]\right] = 0\end{aligned} \quad . \tag{2.11}$$

Now, of central interest in the discussion to follow, the monopole contains a Gauss/Stokes-integrable term $d\left[G,G\right]$ (and the **0**) together with the two non-integrable terms GdG and $\left[G,\left[G,G\right]\right]$. From (2.8) we may further extract:

$$\begin{aligned}&iGdG = \tfrac{1}{3!}i\left(G_\sigma\partial_{[\mu}G_{\nu]} + G_\mu\partial_{[\nu}G_{\sigma]} + G_\nu\partial_{[\sigma}G_{\mu]}\right)dx^\sigma \wedge dx^\mu \wedge dx^\nu \\ &= \tfrac{1}{2!}iG_\sigma\partial_{[\mu}G_{\nu]}dx^\sigma \wedge dx^\mu \wedge dx^\nu = 3iG_\sigma\partial_{[\mu}G_{\nu]}dx^\sigma dx^\mu dx^\nu = 6iG_\sigma\partial_\mu G_\nu dx^\sigma dx^\mu dx^\nu\end{aligned} \quad . \tag{2.12}$$

So expanding the P result in (2.11) back into tensor form making use of (2.12) and some intermediate results within (2.8) and (2.10), we obtain:

$$\begin{aligned}P &= P_{\sigma\mu\nu}dx^\sigma dx^\mu dx^\nu \\ &= \mathbf{0} - \tfrac{1}{2}id\left[G,G\right] + \tfrac{1}{2}iGdG - \left[G,\left[G,G\right]\right] \\ &= \mathbf{0} + \left(-3i\partial_\sigma\left(G_\mu G_\nu\right) + 3iG_\sigma\partial_\mu G_\nu - 12G_\sigma G_\mu G_\nu\right)dx^\sigma dx^\mu dx^\nu = 0\end{aligned} \quad , \tag{2.13}$$

Finally, it will be of great interest to use Gauss' / Stokes Theorem $\iint dX = \oint X$ for any differential form X, to ascertain the classical surface flux associated with this non-abelian magnetic monopole. Here, we work from (2.13) to write:

$$\begin{aligned}\iiint P &= \iiint P_{\sigma\mu\nu}dx^\sigma dx^\mu dx^\nu \\ &= \mathbf{0} - 3i\iiint\partial_\sigma\left(G_\mu G_\nu\right)dx^\sigma dx^\mu dx^\nu + \iiint\left(3iG_\sigma\partial_\mu G_\nu - 12G_\sigma G_\mu G_\nu\right)dx^\sigma dx^\mu dx^\nu \\ &= \mathbf{0} - 3i\oiint G_\mu G_\nu dx^\mu dx^\nu + \iiint\left(3iG_\sigma\partial_\mu G_\nu - 12G_\sigma G_\mu G_\nu\right)dx^\sigma dx^\mu dx^\nu \\ &= \mathbf{0} - \tfrac{3}{2}i\oiint\tfrac{1}{2!}\left[G_\mu, G_\nu\right]dx^\mu \wedge dx^\nu \\ &\quad + \tfrac{1}{2}i\iiint\tfrac{1}{3!}\left(G_\sigma\partial_{[\mu}G_{\nu]} + G_\mu\partial_{[\nu}G_{\sigma]} + G_\nu\partial_{[\sigma}G_{\mu]}\right)dx^\sigma \wedge dx^\mu \wedge dx^\nu \\ &\quad - \iiint\tfrac{1}{3!}\left(\left[G_\sigma,\left[G_\mu,G_\nu\right]\right] + \left[G_\mu,\left[G_\nu,G_\sigma\right]\right] + \left[G_\nu,\left[G_\sigma,G_\mu\right]\right]\right)dx^\sigma \wedge dx^\mu \wedge dx^\nu \\ &= \oiint dG - \tfrac{3}{2}i\oiint\left[G,G\right] + \tfrac{1}{2}i\iiint GdG - \iiint\left[G,\left[G,G\right]\right] = 0 \\ &\qquad = \mathbf{0} - \tfrac{3}{2}i\oiint\left[G,G\right] + \tfrac{1}{2}i\iiint GdG - \iiint\left[G,\left[G,G\right]\right] = 0\end{aligned} \quad . \tag{2.14}$$

In the final two lines above, we have used $\iiint \mathbf{0} = \iiint ddG = \oiint dG = \mathbf{0}$, which is the Gauss' / Stokes' integral form of the exterior calculus relationship $dd = \mathbf{0}$. By writing (2.14) using the not-highlighted 0 of the Jacobian identity (2.4) as:

$$\begin{aligned} &\oiint dG - \tfrac{3}{2} i \oiint [G,G] = -\tfrac{1}{2} i \iiint GdG + \iiint \left[G,[G,G]\right] \\ &\quad = \mathbf{0} - \tfrac{3}{2} i \oiint [G,G] = -\tfrac{1}{2} i \iiint GdG + \iiint \left[G,[G,G]\right] \end{aligned}, \tag{2.15}$$

we clearly see the relationship between what is contained within the three-dimensional volume $\iiint$ and what net flows through the closed two-dimensional surface $\oiint$ enclosing that volume. Now, we wish to interpret what is being said by (2.15).

3. Confinement of Gauge fields within Yang-Mills Magnetic Monopoles

We start with the term $\oiint dG = \mathbf{0}$ which is embedded in (2.15). In electrodynamics, Gauss' law for magnetism and Faraday's law are both contained within:

$$\iiint P = \iiint dF = \iiint ddG = \oiint F = \oiint F^{\mu\nu} dx_\mu dx_\nu = \oiint dG = \mathbf{0}. \tag{3.1}$$

At rest, this tells us that while magnetic fields may flow across some surfaces, there is never a *net* flux of a magnetic field through any *closed* two dimensional surface. In the form $P = dF = ddG = \mathbf{0}$, this simply says there are no observed magnetic charges. So how might we interpret the presence of $\oiint dG = \mathbf{0}$ as *one of the terms* among a number of *non-vanishing* terms in equations (2.14) and (2.15) for the Yang-Mills magnetic monopoles?

To find out, let us return to the *non-abelian*, *Yang-Mills* field strength (1.5) and rewrite this using the differential forms equation:

$$\begin{aligned} &\oiint F = \tfrac{1}{2!} \oiint F_{\mu\nu} dx^\mu \wedge dx^\nu = \tfrac{1}{2!} \oiint \partial_{[\mu} G_{\nu]} dx^\mu \wedge dx^\nu - \tfrac{1}{2!} i \oiint \left[G_\mu, G_\nu\right] dx^\mu \wedge dx^\nu \\ &= \oiint dG - i \oiint [G,G] = \mathbf{0} - i \oiint [G,G] \end{aligned}. \tag{3.2}$$

We may then use (3.2) to rewrite (2.15) as follows:

$$\oiint F = -i \oiint [G,G] = -\tfrac{1}{3} i \iiint GdG + \tfrac{2}{3} \iiint \left[G,[G,G]\right] \neq 0. \tag{3.3}$$

So if (3.1) tells us that there is no net flux of magnetic flux over of any closed surface in abelian electrodynamics, (3.3) tells us that there *is* a net flux across closed surfaces of whatever the *analog* is to a magnetic field, in Yang-Mills gauge theory.

Now, we have a puzzle here: any time we see a term $\oiint F$, we know that we are talking about a magnetic monopole, and that whatever is contained within the associated volume integral is a magnetic charge. Indeed, (3.3) may be thought of as *the very definition of a magnetic charge*, which in (3.3) is *not* zero. At the same time, we found in (2.4) a.k.a. (2.6) that $P = DF = DDG = 0$, which is to say, that the magnetic charge density is zero, just as it is in electrodynamics. So if $P = DF = DDG = 0$ but $\oiint F \neq 0$, how do we reconcile the former equation which says the magnetic charge density is zero with the latter equation which says there is a non-zero magnetic charge?

One way to think this through, is take the Yang-Mills electric charge field equation (1.12) $*J = D*F$, revert this (merely for pedagogic simplicity) to its abelian form $*J = d*F$ which contains Gauss' law for electricity, and then apply Gauss' / Stokes' Theorem to obtain $\oiint *F = \iiint *J \quad \left(= \iiint d*F\right)$. Just as $\oiint F$ in the rest frame represents a net flux of magnetic field through a closed surface, $\oiint *F$ in the rest frame represents a net flux of electric field through a closed surface. And this $\oiint *F$ then becomes the very definition of the *electric* charge. But here, electric charge density is defined by $*J$ inside $\iiint *J$, while in (3.3) magnetic charge density is defined by $-\frac{1}{3}iGdG + \frac{2}{3}\left[G,\left[G,G\right]\right]$ inside $-\frac{1}{3}i\iiint GdG + \frac{2}{3}\iiint\left[G,\left[G,G\right]\right]$. That is, we have a magnetic charge density $-\frac{1}{3}iGdG + \frac{2}{3}\left[G,\left[G,G\right]\right]$ which we need to think about in comparison to an electric charge density $*J$.

The answer to this puzzle is that the magnetic charge density is *not* the P of $P = DF = DDG = 0$, it is the $P' \equiv -\frac{1}{3}iGdG + \frac{2}{3}\left[G,\left[G,G\right]\right]$ in (3.3). The magnetic charge as defined by the enclosure within $\oiint F$ is a three-form just like $*J$ and P, but it is not an *elementary* three-form. Rather, it is a three form constructed from $-\frac{1}{3}iGdG$ which includes some dynamical behavior of the gauge fields inside the volume integral, and from $\frac{2}{3}\left[G,\left[G,G\right]\right]$ which represents pure gauge field amalgams subsisting within the volume integral. That is, the magnetic charge is a *composite three-form* built out of gauge fields, rather than an elementary three form like the abelian electric charge. Indeed, we may take this a step further:

In electrodynamics, the three-form $*J$ which in tensor language is related to the electric source current density vector J^α by $*J_{\sigma\mu\nu} = \left(-g\right)^{.5}\varepsilon_{\alpha\sigma\mu\nu}J^\alpha$, is a *true electric source* which then gives rise to gauge fields in abelian gauge theory via $*J = d*F = d*dG$, and per (1.12), via $*J = D*F = D*DG$ in Yang-Mills gauge theory. On the other hand, the $P' = -\frac{1}{3}iGdG + \frac{2}{3}\left[G,\left[G,G\right]\right]$ in (3.3), if written as a tensor (see (2.14)) and converted over to a one form via the same general identity $*P'_{\sigma\mu\nu} = \left(-g\right)^{.5}\varepsilon_{\alpha\sigma\mu\nu}P'^\alpha$, will result in a *faux magnetic source* which is constructed solely out of gauge fields G, which themselves are sourced by $*J = D*F = D*DG$. So, there is only one *elementary* source J, not two sources J and P. From this one source J, gauge fields G are emitted. From these gauge fields G, a faux magnetic source

$P' = -\frac{1}{3} iGdG + \frac{2}{3}\left[G,\left[G,G\right]\right]$ is assembled. And finally, from this faux magnetic source, $\oiint F \neq 0$ flows across closed surfaces as laid out in (3.3). The electric source J^{α}, whether in abelian or non-abelian gauge theory, has its own independent existence, and it is the source of any and all gauge fields. But the faux magnetic source charge in (3.3) has *no independent existence* apart from the gauge fields *G*. Rather, it is built out of the gauge fields. *So the Yang-Mills monopoles are composite, not elementary, objects.* And, by the way, so too are baryons.

Having resolved the puzzle of how to reconcile $P = DF = DDG = 0$ with $\oiint F \neq 0$, we next pose the following question: what happens to the total flux $\oiint F$ in (3.2) under the local gauge-like transformation $F^{\mu\nu} \to F^{\mu\nu}{}' = F^{\mu\nu} - \partial^{[\nu} G^{\mu]}$? In differential forms, this transformation is $F \to F' = F - dG$, which means, precisely because $\oiint dG = \mathbf{0}$, that:

$$\oiint F \to \oiint F' = \oiint \left(F - dG\right) = \oiint F\,, \tag{3.4}$$

So, the net surface flux in the monopole equation (3.3) is *invariant* under the transformation $F^{\mu\nu} \to F^{\mu\nu}{}' = F^{\mu\nu} - \partial^{[\nu} G^{\mu]}$, which means that the gauge field is *not observable* with respect to net flux across closed surfaces of the monopole. The abelian expression $\oiint dG = \mathbf{0}$, expanded to show the Riemann tensor, may be written as $\oiint F = \oiint dG = \iiint R^{\tau}{}_{\nu\sigma\mu} G_{\tau} dx^{\sigma} dx^{\mu} dx^{\nu} = \mathbf{0}$, and explicitly shows how individual gauges fields G_{τ} couple with the spacetime geometry as represented by $R^{\tau}{}_{\nu\sigma\mu}$. This represents an *absence* of monopoles in electrodynamics, and yields the *symmetry principle* (3.4) for the behavior of magnetic monopole monopoles in Yang-Mills theory generally.

But if the non-zero flux in the Yang-Mills monopole equation (3.3) is invariant under the gauge-like transformation $F^{\mu\nu} \to F^{\mu\nu}{}' = F^{\mu\nu} - \partial^{[\nu} G^{\mu]}$ which means that the gauge fields G^{μ} are not net observables over a closed monopole surface, *this would seem to suggest that the Yang-Mills monopole inherently confine their gauge fields*. This is another hint that the monopole equation (3.3) could be the classical field equation for a baryon, in integral form.

The final point is that because the faux magnetic source $P' = -\frac{1}{3} iGdG + \frac{2}{3}\left[G,\left[G,G\right]\right]$ is constructed out of gauge fields, and because the gauge fields are in turn sourced by $*J = D*F = D*DG$, and because electric sources may be represented in vector form in terms of Dirac fermion wavefunctions ψ via $J^{\mu} = \overline{\psi}\gamma^{\mu}\psi$, it should be possible in principle, and would certainly be desirable in practice, to rewrite the *faux* magnetic source $-\frac{1}{3} iGdG + \frac{2}{3}\left[G,\left[G,G\right]\right]$ in terms of the *true* source currents J^{μ} from which they arise, and then to rewrite the $J^{\mu} = \overline{\psi}\gamma^{\mu}\psi$ in terms of their fermion wavefunctions ψ. The upshot of all this, is that while $\oiint F$ in (3.3) is presently expressed in terms of gauge fields as $\oiint F\left(G\right)$, once we obtain the gauge fields $G\left(J\right)$

in terms of sources and the sources $J(\psi)$ in terms of fermions, we will end up with $\oiint F\left(G\left(J(\psi)\right)\right)$. Then, we would need to apply the Exclusion Principle of Fermi-Dirac-Pauli statistics to maintain the ψ in distinct quantum eigenstates, which would give us the opportunity, for example, to introduce a color degree of freedom to do so and thus make a connection to SU(3)$_C$ Chromodynamics, with $\oiint F\left(G\left(J\left(\psi_R,\psi_G,\psi_B\right)\right)\right)$. So this means that the Yang-Mills monopoles are not only composite objects, but are composite objects which contain fermions and gauge fields, and that these fermions will need to obey some form of quantum exclusion, which may include SU(3)$_C$. And, by the way, all of the same the same is true of baryons, and as to fermion exclusion, quarks.

It is for these reasons, that it may be fruitful to entertain the prospect that (3.3) is not only the classical field equation for a Yang-Mills magnetic monopole, but may be synonymous with the classical field equation for a baryon.

4. Can a Classical Field Equation Really Teach us Anything Useful about Confinement and Baryons?

Given that (3.3) is a classical field equation, we must pose the question whether such a classical equation can really have anything of interest to say about baryons and confinement, which have many features that arise only out of quantum field theory. For example, it might be observed that a classical analysis which seeks to discuss baryons and confinement in no way takes account of quantum field theory with operator-valued fields. This, it might be argued, is despite the fact that there are many reasons to believe confinement and the existence of a mass gap are related to the running of the coupling constant, which is an inherently quantum effect.

Certainly, (3.3) above is a completely classical field equation, not yet taking into account any aspects (or the need to prove existence) of a non-trivial relativistic quantum Yang–Mills theory on $\Re^4$ [6]. And, of course, there are many reasons to believe that confinement is related to the running of the strong coupling constant, which is an inherently quantum effect, and which manifests in asymptotic freedom at "ultraviolet" energy and infrared slavery at low energy [10]. However, just like electrodynamics, Yang-Mills gauge theory has a classical formulation and (is expected once quantum Yang-Mills existence is proven, to have) a quantum field formulation. This means that (3.3) may reveal inherently-confining attributes for the magnetic monopoles of Yang-Mills gauge theory which appear at the classical level and which are rooted in the relationship $dd = \mathbf{0}$ of Riemannian spacetime exterior geometry, as well as inherently-composite attributes expressed by $\oiint F\left(G\left(J(\psi)\right)\right)$. That opens up the question how these same attributes translate through to quantum Yang-Mills theory.

Specifically, *if* in fact (3.3) for $\oiint F$ is an equation for baryon-like gauge field confinement properties of Yang-Mills magnetic monopoles based upon their abelian-subset behaviors rooted in the classical equation $ddG = \mathbf{0}$ and its integral form $\oiint dG = \mathbf{0}$, and if the composite faux magnetic charge $P' = -\frac{1}{3}iGdG + \frac{2}{3}\left[G,\left[G,G\right]\right]$ in (3.3) in some way represents a baryon charge, then the classical baryons that would be represented by (3.3) do not suddenly become "not baryons" in quantum field theory. Rather, there would *two sets of behaviors* that need to be studied: a) how these monopoles behave in a classical formulation, which includes (3.3) and (3.4) above, and b) how these monopoles additionally behave in quantum field theory. So if we can demonstrate that the classical behaviors appear to be confining and appear to involve a non-elementary, composite charge that includes some amalgam of fermions and gauge fields, one should expect that this will "bleed" through to yield quantum amplitudes and running couplings and color symmetries that buttress, not defy, these classical behaviors, just as abelian magnetic monopoles do not suddenly appear and ordinary magnetic fields do not suddenly net flow through closed surfaces, once one goes from classical to quantum electrodynamics.

Further, one might take the perspective that the *cause* for confinement and baryon compositeness is the classical field equation (3.3) for a Yang-Mills monopole which has the symmetry (3.4), and that one of the *effects* of this is that in a quantum field treatment of these baryon monopoles, the strong coupling will weaken for ultraviolet and strengthen for infrared probes. And, it can be argued that this is a more natural approach than simply trying to figure out how to "glue" together disparate quarks into baryons *without knowing to begin with what sorts of covariant objects baryons actually are in spacetime*. Indeed, if the hints of baryons and confinement that arise in (3.3) and (3.4) are correct, then we would need to start thinking of baryons as third-rank antisymmetric tensors and related three-forms in spacetime governed by the classical equation (3.3) with the symmetry (3.4), and then see how that connects to everything else we know about baryons. The "let's glue together the quarks" approach, notwithstanding many opportunities to do so, has thus far failed to explain why QCD "must have 'quark confinement, that is, even though the theory is described in terms of elementary fields, such as the quark fields, that transform non-trivially under SU(3), the physical particle states—such as the proton, neutron, and pion—are SU(3)-invariant." ([6] at page 3.) This SU(3)-invariance of *physical particle states* is a *symmetry principle*, and while not every classical symmetry carries through to quantum field theory, for example, the chiral anomaly (e.g., [11], section IV.7), there is no apparent *a priori* reason to believe that whatever classical symmetries are found for these monopoles (such as (3.4)) will *only* manifest in the classical but not the quantum field theory. At the very least, the question for study becomes "do these symmetries carry over from classical to quantum field theory, and if not, why not, and in what manner are they altered?"

Additionally, approaching confinement starting from a classical treatment of baryons has validating precedent in the MIT Bag Model reviewed in, e.g., [**Error! Bookmark not defined.**], section 18. Irrespective of the specifics of any particular bag-type model of confinement, the MIT Bag Model very correctly makes one very important point: *focus carefully on what flows and does not flow across any closed two-dimensional surface.* And it does so using the *classical* formulation of Gauss' / Stokes' theorem. This is why the integral form of Maxwell's equations in classical field theory may well be a very sensible starting point studying confinement, because from the Bag Model viewpoint, confinement is all about what passes and does not pass through closed surfaces containing the extended field configuration within the baryon volume.

Finally, it is certainly unrealistic to expect that a classical-only treatment of baryons based on Yang-Mills magnetic monopoles will explain *all* of the observed phenomenology of baryons. It cannot and will not. Only a proper quantum field treatment may be expected to do so. Yet, at the same time, that there are some important physics insights to be gained even from a classical treatment of the Yang-Mills monopole equation (3.3). And, we know that if we can fully develop a classical theory on its own terms, and then obtain its Lagrangian density $\mathfrak{L}(\phi)$ and action $S(\phi)$ in terms of its fields ϕ, that we can convert over to a quantum field theory via the path integration $Z=\int D\phi \exp i\int \mathfrak{L} d^4x=\int D\phi \exp iS$. While carrying out the path integration of a non-linear theory such as Yang-Mills gauge theory (and especially gravitational theory) is still an exceptionally challenging problem, that does not mean one ought not make the effort to find the correct road for doing so. This begins by finding the right classical theory to quantize.

So what is most important is for researchers in particle, baryon and nuclear theory to be aware of the possibility of modelling baryons as Yang-Mills magnetic monopoles to gain possible insight into confinement and related QCD symmetries, so that this possible connection can be further developed, vetted, and empirically-tested by anyone who finds it interesting or promising.

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