Two conjectures on primes and a conjecture on Fermat pseudoprimes to base two

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Abstract. I treated the 2-Poulet numbers in many papers already but they continue to be a source of inspiration for me; in this paper I make a conjecture on primes inspired by the relation between the two prime factors of a 2-Poulet number and I also make a conjecture on Fermat pseudoprimes to base two.

Conjecture 1 (on primes):
For any prime \( p \), \( p \geq 7 \), there exist an infinity of primes \( q \), \( q > p \), such that the number \( r = (q - 1)/(p - 1) \) is a natural number. In other words, for any such prime \( p \) there exist an infinity of natural numbers \( r \) such that \( q = r\cdot p - p + 1 \) is prime.

Conjecture 2 (on primes):
For any prime \( p \), \( p \geq 7 \), there exist an infinity of primes \( q \), \( q > p \), such that the number \( r = (q - 1)/(p - 1) \) is a rational but not natural number. In other words, for any such prime \( p \) there exist an infinity of rational but not natural numbers \( r \) such that \( q = r\cdot p - p + 1 \) is prime.

Conjecture 3 (on 2-Poulet numbers):
For any 2-Poulet number \( P = d_1 \cdot d_2 \), where \( d_2 > d_1 \), the following statement is true: the number \( r = (d_2 - 1)/(d_1 - 1) \) is a rational number.

Verifying the conjecture 3:
(For the first seventy-five 2-Poulet numbers)

Note:
In the column I are listed the first seventy-five 2-Poulet numbers, in the column II are listed the cases when \( r = (d_2 - 1)/(d_1 - 1) \) is a natural number (put it in other way, the cases when \( d_2 = r\cdot d_1 - r + 1 \) and in the column III are listed the cases when \( r = (d_2 - 1)/(d_1 - 1) \) is a rational but not natural number.

\[
\begin{array}{ccc}
I. & II. & III. \\
1341 = 11 \cdot 31 & (d_2 = 3 \cdot d_1 - 2) & \\
1387 = 19 \cdot 73 & (d_2 = 4 \cdot d_1 - 3) & \\
2047 = 23 \cdot 89 & (d_2 = 4 \cdot d_1 - 3) & \\
\end{array}
\]
2701 = 37*73  
(d2 = 2*d1 - 1)

3277 = 29*113  
(d2 = 4*d1 - 3)

4033 = 37*109  
(d2 = 2*d1 - 1)

4369 = 17*257  
(d2 = 16*d1 - 15)

4681 = 31*151  
(d2 = 5*d1 - 4)

5461 = 43*127  
(d2 = 3*d1 - 2)

7957 = 73*109  
(d2 - 1)/(d1 - 1) = 3/2

8321 = 53*157  
(d2 = 3*d1 - 2)

10261 = 31*331  
(d2 = 11*d1 - 10)

13747 = 59*233  
(d2 = 4*d1 - 3)

14491 = 43*337  
(d2 = 8*d1 - 7)

15709 = 23*683  
(d2 = 31*d1 - 30)

18721 = 97*193  
(d2 = 5*d1 - 4)

19951 = 71*281  
(d2 = 4*d1 - 3)

23377 = 97*241  
(d2 - 1)/(d1 - 1) = 5/2

256999 = 127*337  
(d2 = 2*d1 - 1)

24799 = 127*337  
(d2 - 1)/(d1 - 1) = 9/2

249141 = 157*313  
(d2 = 2*d1 - 1)

25981 = 151*331  
(d2 = 2*d1 - 1)

260701 = 101*601  
(d2 = 6*d1 - 5)

27087 = 89*683  
(d2 - 1)/(d1 - 1) = 31/4

286077 = 59*1103  
(d2 = 19*d1 - 18)

296281 = 97*673  
(d2 = 7*d1 - 6)

308051 = 61*1321  
(d2 = 22*d1 - 21)

316121 = 103*307  
(d2 = 3*d1 - 2)

328549 = 53*1613  
(d2 = 31*d1 - 30)

338837 = 149*593  
(d2 = 4*d1 - 3)

3490751 = 151*331  
(d2 = 2*d1 - 1)

360701 = 101*601  
(d2 = 6*d1 - 5)

3712351 = 59*2089  
(d2 = 36*d1 - 35)

37129899 = 193*673  
(d2 - 1)/(d1 - 1) = 7/2

38130561 = 137*953  
(d2 = 7*d1 - 6)

39150851 = 251*601  
(d2 - 1)/(d1 - 1) = 12/5

40162193 = 241*673  
(d2 - 1)/(d1 - 1) = 14/5

41164737 = 257*641  
(d2 - 1)/(d1 - 1) = 5/2

42181901 = 101*1801  
(d2 = 18*d1 - 17)

43188057 = 89*2113  
(d2 = 24*d1 - 23)

44194221 = 167*1163  
(d2 = 7*d1 - 6)

45196093 = 157*1249  
(d2 = 8*d1 - 7)

46215749 = 79*2731  
(d2 = 35*d1 - 34)

47219781 = 271*811  
(d2 = 3*d1 - 2)

48220729 = 103*2143  
(d2 = 21*d1 - 20)

49226801 = 337*673  
(d2 = 2*d1 - 1)

50233017 = 43*5419  
(d2 = 129*d1 - 128)

51241001 = 401*601  
(d2 - 1)/(d1 - 1) = 3/2

52249841 = 433*577  
(d2 - 1)/(d1 - 1) = 4/3

53253241 = 157*1613  
(d2 - 1)/(d1 - 1) = 31/3

54256999 = 233*1103  
(d2 - 1)/(d1 - 1) = 19/4
Comment:

It can be seen that are already outlined few subsets of 2-Poulet numbers, such the following ones:

: 2-Poulet numbers $P = d_1 \times d_2$ for which $r = (d_2 - 1)/(d_1 - 1)$ is of the form $r = p^m/2^n$, where $p$ odd prime and $m$, $n$ positive integers; such numbers are: 7957, 23377, 35333, 60787, 129889, 164737, 241001, 256999, 318361, 481573, 129889, 164737, 241001, 256999, 318361, 481573 (...);

: 2-Poulet numbers $P = d_1 \times d_2$ for which $r = (d_2 - 1)/(d_1 - 1)$ is of the form $r = n/3$, where $n$ positive integer; such numbers are: 42799, 249841, 253241, 280601, 556169 (...);

: 2-Poulet numbers $P = d_1 \times d_2$ for which $r = (d_2 - 1)/(d_1 - 1)$ is of the form $r = n/5$, where $n$ positive integer; such numbers are: 150851, 162193, 452051 (...).