Are Galaxies Structured by Riccati Equation? The First Graph of Rational Bar

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Abstract A mother, a father, and their daughter were taking a picture. They were 5, 7, and 2 feet tall respectively. The parents stood in a row, and their daughter stood in front of her mother. My son saw this and ran quickly in front of the father before the picture was taken. I asked my son why. He answered that he was exactly 4 feet tall. I figured out his reasoning, and afterwards I have become an astrophysicist. A pattern is a distribution of differences. In the array pattern of the above-said four people, the height differences between adults and between kids are equal, and the height differences between females and between males are equal too. This simple pattern can be generalized into any array of numbers. Assume the differences of numbers in a row are equal to the corresponding differences in any other row. That is, there exist the common differences in all rows. Similarly assume the common differences in all columns. Then the pattern is called a rational structure. Assume the number at the bottom left corner is zero, C(0,0) = 0, and denote the series of numbers in the bottom row by U(i) and the series of numbers in the first column by V(j). I found the formula for the rational array: C = U(i) + V(j). This is called Skew Law. I generalized the rows and columns to be curved, and required that the curves cross each other at a right angle. This was exactly my idea of galaxy patterns. In this paper I show that the patterns are governed by the Riccati equation with constant coefficients; and the curves are governed by a type of algebraic equations. The cubic equation of the type gives a pattern which resembles the sharp bar of galaxy NGC 1073. Are all barred galaxies governed by the cubic and higher degrees of algebraic equations? The question will be resolved in the near future.

keywords: Rational Structure; Riccati Equation; Spiral Galaxies

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Contents

1 Introduction

Understanding the origin of natural structure is the great challenge to humans. For example, we haven't had an accepted theory for the origin of the fundamental biological unit, DNA. Most natural structures are governed by their environment so strongly that their structuring force is difficult to identify. There exists a kind of natural structure whose existence is relatively independent from their environment. These are galaxies. Spiral galaxies are planar distribution of stars. Those with a barred pattern are referred to as barred spirals and those without a barred pattern are termed ordinary spirals. The stellar distribution of an ordinary spiral galaxy is axi-symmetric with respect to the galaxy center, and its stellar density decreases exponentially radially. This distribution of density is the exponential disk. Astronomers have shown that the main structure of barred galaxies can also be described by an exponential law. When a fitted exponential disk is subtracted from a barred spiral galaxy what is the result? I propose that the resulting structure resembles human breasts [1-3]. I call it a double-breast structure. Barred spiral galaxies, however, have more than a pair of breasts. The bar of barred spiral galaxies is comprised of two or three pairs of breasts which are usually aligned. The addition of the two or three pairs of breasts to the major structure of the exponential disk becomes a bar-shaped pattern which crosses the galaxy center.

The only accepted theory applied to galaxies is Newton's universal gravity which in actuality is a theory of two bodies. When applied to three or more bodies, the theory gives chaotic distribution of matter. However, natural distribution of matter always displays regular patterns. Hence, Newton's theory cannot predict the natural structures of many bodies. Newton's theory applied to galaxy structure has been used previously to predict kinematical phenomena. These predictions are always wrong. A well known example is the problem of constant rotational curves. To maintain the status of Newton's theory, people created the terminology of dark matter [4] which has never been observed.

Understanding that galaxy structure may have a simple origin, we then want to determine what is the simplest structure in a plane? The answer is the one of "equal derivatives". For example, let's study a squared structure $f(x, y)$ in a plane. We define the row derivatives to be the partial derivatives $u(x, y_0) = f'_x(x, y_0)$ along the horizontal direction and the column derivatives to be the partial derivatives $v(x_0, y) = f'_y(x_0, y)$ along the vertical direction. The simplest pattern is realized if we assume that all row derivatives are the same function of x and all column derivatives are the same function of y: $u(x, y_0) = u(x)$, $v(x_0, y) = v(y)$. The integration of the row derivative defines a new function (denoted as $U(x)$) and the integration of the column derivative defines a new function $V(y)$: $U(x) = \int u(x)dx$, $V(y) = \int v(y)dy$. They completely determine the structure. For example, $U(x) + V(y)$ must be the structure itself in a difference of a constant, $f(x, y) = U(x)+V(y)+c$ where c is a constant. The quantities, $U(x), V(y)$, are called parallel quantities, and the sum $C = U(x) + V(y)$ is called the skew quantity. We generalize the rows and columns to be curved and they cross each other at a right angle. This is exactly my idea of galaxy structure [5].

A structure is a two or three dimensional distribution of similar matter. This paper deals with two dimensional structure only. A curve in the plane of the distribution is called a proportion curve or a Darwin curve if the matter density on one side of the curve is in constant ratio to the one on the other side of the curve. If there exists an orthogonal net of Darwin curves in the plane, the distribution of matter is called a rational structure. We found many evidences that galaxies are rational stellar distribution [2, 3]. We list a few examples. Firstly, galaxy components (disks and bars) can be fitted with the above-said rational structure [1]. Secondly, spiral arms can be fitted with the above-said proportion curves [6]. Thirdly, if galaxy structure must be rational and gravitational force-lines must be conservative then a new universal gravity is uniquely determined. The new gravity generalizes Newton's theory and explains constant rotation curves simply and elegantly [7].

Rational structure in two dimension

$$\rho(x, y) \tag{1}$$

means that not only there exists an orthogonal net of curves in the plane

$$\begin{cases} x = x(\lambda, \mu), \\ y = y(\lambda, \mu) \end{cases} \tag{2}$$

but also, along each curve, the matter density on one side of the curve is in constant ratio to the one on the other side. Because the density ratio is equivalent to the derivative to the logarithm of the density

$$f(x, y) = \ln \rho(x, y) \tag{3}$$

we from now on, are only concerned with the logarithmic density $f(x, y)$. We show that rational structures are governed by a Riccati equation with constant coefficients, and the orthogonal curves are governed by a type of algebraic equations. The cubic equation of the type gives a pattern which resembles the sharp bar of galaxy NGC 1073. Are all barred galaxies governed by this type of algebraic equations? The question will be resolved in the near future. Now we study the mathematical theory of rational structure and its graph in the following Sections.

2 Basic Theory of Rational Structure

2.1 Rational Structure Equation and Parallel Law

We know that, given the two partial derivatives

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y} \tag{4}$$

the structure $f(x, y)$ is determined provided that the Green's theorem is satisfied

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \tag{5}$$

Now we are interested in rational structure. Instead of calculating the partial derivatives (4), we calculate the directional derivatives along the tangent direction to the above curves

$$\frac{\partial f}{\partial l_\lambda}, \quad \frac{\partial f}{\partial l_\mu} \tag{6}$$

where l_λ is the linear length in the *physical* plane (x, y) and is along the row curves whose parameter is λ while l_μ is the linear length along the column curves whose parameter is μ. The two derivatives (6) are called the parallel derivatives. Given the two parallel derivatives, however, the structure $f(x, y)$ may not be determined. A similar Green's theorem, which is called the rational structure equation, must be satisfied

$$\frac{\partial}{\partial \mu}\left(P \frac{\partial f}{\partial l_\lambda}\right) - \frac{\partial}{\partial \lambda}\left(Q \frac{\partial f}{\partial l_\mu}\right) = 0 \tag{7}$$

where

$$\begin{aligned} P(\lambda, \mu) &= \sqrt{x_\lambda'^2 + y_\lambda'^2}, \\ Q(\lambda, \mu) &= \sqrt{x_\mu'^2 + y_\mu'^2} \end{aligned} \tag{8}$$

are the lengths of the tangential vectors (x'_λ, y'_λ) and (x'_μ, y'_μ), respectively, and are called the line magnitudes. Note that we use the simple notation $x'_\lambda = \frac{\partial x}{\partial \lambda}$. From now on, we always use the similar simple notations.

It is straightforward to show that the partial derivatives in the parameter space (λ, μ) are

$$f'_\lambda = P \frac{\partial f}{\partial l_\lambda}, \quad f'_\mu = Q \frac{\partial f}{\partial l_\mu} \tag{9}$$

Therefore, the rational structure equation (7) is nothing but the following simple equation

$$\frac{\partial^2 f}{\partial \mu \partial \lambda} = \frac{\partial^2 f}{\partial \lambda \partial \mu} \tag{10}$$

To simplify the expression of our equations, we introduce the important notations for our parallel derivatives

$$\begin{cases} \hat{u}(\lambda, \mu) = \frac{\partial f}{\partial l_\lambda}, \\ \hat{v}(\lambda, \mu) = \frac{\partial f}{\partial l_\mu} \end{cases} \tag{11}$$

The requirement of rational structure is that the parallel derivative $\hat{u}$ depends only on λ and the parallel derivative $\hat{v}$ depends only on μ

$$\hat{u} = \hat{u}(\lambda),\ \hat{v} = \hat{v}(\mu) \tag{12}$$

This necessary condition of rational structure is called the parallel law.

Now we prove the parallel law. Assume you walk along a row curve. The logarithmic ratio of the density on your left side to the immediate density on your right side is approximately the directional derivative to $f(x, y)$ along the column direction. That is, the logarithmic ratio is approximately the directional derivative $\hat{v}(\lambda, \mu)$. Because $\hat{v}(\lambda, \mu)$ is constant along the row curve (rational), $\hat{v}(\lambda, \mu)$ is independent of λ, $\hat{v} = \hat{v}(\mu)$. Similarly, we can prove that $\hat{u}(\lambda, \mu) = \hat{u}(\lambda)$. This concludes our proof.

Based on the parallel law, the Green's theorem (7) turns out to be much simpler

$$\hat{u}(\lambda)P'_\mu = \hat{v}(\mu)Q'_\lambda \tag{13}$$

which is our rational structure equation [1, 8]. The following equation is the necessary and sufficient condition for the curves to be orthogonal

$$x'_\lambda x'_\mu + y'_\lambda y'_\mu = 0 \tag{14}$$

In fact, rational structures depend only on the geometric curves, not the choice of coordinate parameters. Therefore, the general coordinate parameters $(\sigma,\ \tau)$ for the orthogonal net of curves are

$$\begin{cases} \lambda = g(\sigma), \quad \mu = h(\tau), \\ x = x(g(\sigma),\ h(\tau)), \\ y = y(g(\sigma),\ h(\tau)) \end{cases} \tag{15}$$

All these expressions give the same orthogonal net of Darwin curves and generate the same rational structure. In the following Section, we prove the existence of harmonic coordinate parameters, and all of our following discussion is based on the parameters.

2.2 Harmonic Coordinate System and Skew Law

A harmonic coordinate system is the choice of coordinate parameters such that the following equations hold

$$\begin{cases} x'_\lambda = y'_\mu, \\ x'_\mu = -y'_\lambda \end{cases} \tag{16}$$

They are the well known Cauchy-Riemann equations, and the following complex function in the parameter space (λ, μ)

$$z(\lambda, \mu) = x(\lambda, \mu) + \mathrm{i}\ y(\lambda, \mu) \tag{17}$$

must be analytic. The real part $x(\lambda, \mu)$ and the imaginary part $y(\lambda, \mu)$ are determined by each other, and both are harmonic functions,

$$\begin{aligned} x''_{\lambda\lambda} + x''_{\mu\mu} = 0, \\ y''_{\lambda\lambda} + y''_{\mu\mu} = 0 \end{aligned} \tag{18}$$

They are called the conjugates to each other (see some textbook on analytic complex functions).

Does there exist a harmonic pair of coordinate parameters for any net of curves? We prove that the necessary and sufficient condition for the existence of harmonic coordinates is that the net of curves is orthogonal. Application of the chain derivative rule to the composite functions (15) gives

$$\begin{array}{ll} x'_\sigma = x'_\lambda g'_\sigma, & x'_\tau = x'_\mu h'_\tau, \\ y'_\sigma = y'_\lambda g'_\sigma, & y'_\tau = y'_\mu h'_\tau \end{array} \tag{19}$$

The Cauchy-Riemann equations for the harmonic coordinate parameter (σ, τ) are

$$\begin{cases} x'_\sigma = y'_\tau, \\ x'_\tau = -y'_\sigma \end{cases} \tag{20}$$

That is,

$$\begin{cases} x'_\lambda g'_\sigma - y'_\mu h'_\tau = 0, \\ y'_\lambda g'_\sigma + x'_\mu h'_\tau = 0 \end{cases} \tag{21}$$

The above linear algebraic equation system has non-zero solution $\lambda = g(\sigma), \mu = h(\tau)$ if and only if the determinant of the equation system is zero. The determinant turns out to be the orthogonal condition (14). We have now proved an important and simple theorem concerning analytic complex functions.

Orthogonal Curves' Theorem: An analytic complex function defines an orthogonal net of curves, and an orthogonal net of curves corresponds to a family of analytic complex functions.

Because rational structure is always defined on some orthogonal net of curves, its harmonic coordinate parameters always exist. Therefore, we from now on, assume that the pair of parameters (λ, μ) is itself harmonic. An immediate result is the following

$$P(\lambda, \mu) = \sqrt{x'^2_\lambda + y'^2_\lambda} \equiv \sqrt{x'^2_\mu + y'^2_\mu} = Q(\lambda, \mu) \tag{22}$$

(see the formulas (8)). Accordingly the rational structure equation (13) becomes much simpler

$$\hat{u}(\lambda) P'_\mu = \hat{v}(\mu) P'_\lambda \tag{23}$$

The symbol $Q(\lambda, \mu)$ is no longer needed. Given $\hat{u}(\lambda)$ and $\hat{v}(\mu)$, we know that the above equation is a first order partial differential equation whose unknown is $P(\lambda, \mu)$. Its general solution can be obtained with the standard characteristic method. The characteristic equation is

$$\frac{d\lambda}{\hat{v}(\mu)} = \frac{d\mu}{-\hat{u}(\lambda)} \tag{24}$$

which is an ordinary differential equation. The general solution to the ordinary equation is

$$\hat{U}(\lambda) + \hat{V}(\mu) = \hat{C} \tag{25}$$

where $\hat{C}$ is an arbitrary constant, and $\hat{U}(\lambda)$ and $\hat{V}(\mu)$ are the indefinite integrals of the parallel derivatives $\hat{u}(\lambda)$ and $\hat{v}(\mu)$, respectively

$$\begin{cases} \hat{u}(\lambda) = \hat{U}'_\lambda(\lambda), \\ \hat{v}(\mu) = \hat{V}'_\mu(\mu) \end{cases} \tag{26}$$

The quantities $\hat{U}(\lambda)$ and $\hat{V}(\mu)$ are called the parallel integrals. From now on, the derivative $\hat{U}'_\lambda(\lambda)$ is simply denoted by $\hat{U}'(\lambda)$ or $\hat{U}'$. The notation applies to other quantities when confusion may be avoided. Now the general solution to the original rational structure equation (23) is

$$W(P, \hat{C}) = 0 \tag{27}$$

where $W(P, \hat{C})$ is an arbitrary function of the two variables P and $\hat{C}$, and P is replaced with the unknown $P(\lambda, \mu)$ and $\hat{C}$ is replaced with the left–hand side of the equation (25) where the arbitrary constant $\hat{C}$ must be isolated into the right–hand side. If we do not pay much attention to possible multi-valued functions, the general solution (27) is essentially the fact that P is a function of the single variable $\hat{C}$ (see (25))

$$P = P(\hat{C}) = P(\hat{U}(\lambda) + \hat{V}(\mu)) \tag{28}$$

The above is our second result derived with the harmonic parameters. Note that $P(\lambda, \mu)$ and $P(\hat{C})$ are two different functions but we use the same symbol P to denote them. Which is which can be recognized with its accompanying notations of the derivatives. For simplicity we follow this convention for P and for other quantities if applicable.

The above result of rational structure, (28), is called the skew law. Based on the formulas (25), we call $\hat{C}$ the skew variable, and call any function of the single-variable $\hat{C}$ a skew quantity. Therefore, the line magnitude of any rational structure is a skew quantity. In the following, we prove that the structure $f(x, y)$ itself is a skew quantity too.

The formulas (9) now become

$$\begin{aligned} f'_\lambda &= \hat{u}(\lambda) P(\lambda, \mu), \\ f'_\mu &= \hat{v}(\mu) P(\lambda, \mu) \end{aligned} \tag{29}$$

Therefore,

$$\hat{u}(\lambda) f'_\mu = \hat{v}(\mu) f'_\lambda \tag{30}$$

Given $\hat{u}(\lambda)$ and $\hat{v}(\mu)$, we know that the above first order partial differential equation is identical to the one (23). Therefore, we have

$$f = f(\hat{C}) = f(\hat{U}(\lambda) + \hat{V}(\mu)) \tag{31}$$

This proves that $f(x, y)$ is a skew quantity. Therefore,

$$\begin{aligned} f'_\lambda &= \hat{u}(\lambda) f'_{\hat{C}}(\hat{C}), \\ f'_\mu &= \hat{v}(\mu) f'_{\hat{C}}(\hat{C}) \end{aligned} \tag{32}$$

Comparison of the above equation with the equation (29) leads to

$$f'_{\hat{C}}(\hat{C}) = P(\hat{C}) \tag{33}$$

This is the important relation between the line magnitude and the density structure, and is used to draw the graph of rational structure.

For simplicity, we introduce new quantities which are parallel to the above hatted ones. That is, we introduce the corresponding un-hatted quantities:

$$C = \hat{C}/h, \quad u = \hat{u}/h, \quad v = \hat{v}/h, \quad U = \hat{U}/h, \quad V = \hat{V}/h, \quad \text{etc.} \tag{34}$$

where h is a common constant. The pairs of quantities differ by a common constant factor. But the hatted ones correspond to the physical variables. We will know that only negative h can help achieve a realistic galaxy structure.

Now we have proved the basic theorems of rational structure. Firstly, an orthogonal net of curves is equivalent to an analytic complex function. Secondly, the parallel law is a necessary condition for rational structure. Thirdly, the rational structure equation (23), i. e., Green's theorem, results in the skew law for the line magnitude. However, they are not the sufficient condition for rational structure. They are the necessary ones. For example, the net of curves (2) solved from the following equation

$$P(\lambda, \mu) = \sqrt{x_\lambda'^2 + x_\mu'^2} \tag{35}$$

may not be orthogonal. Therefore, we look for the necessary and sufficient condition for rational structure.

2.3 The Necessary and Sufficient Condition for Rational Structure

The net of curves (2) is orthogonal and its parameters are harmonic if and only if the Cauchy-Riemann equations (16) hold. The equations hold if and only if $x(\lambda, \mu)$ is a harmonic function. The function $x(\lambda, \mu)$ is harmonic if and only if the derivatives

$$\Phi(\lambda, \mu) = x_\mu' + \mathrm{i}\, x_\lambda' \tag{36}$$

satisfy Cauchy-Riemann equations. The above complex function satisfies Cauchy-Riemann equations if and only if the logarithm of its modulus

$$\ln P(\lambda, \mu) = \ln \sqrt{x_\lambda'^2 + x_\mu'^2} \tag{37}$$

is a harmonic function. Finally, applying harmonic operation to the above function gives the necessary and sufficient condition for the net of curves to be orthogonal and its parameters to be harmonic. We use the symbol L to denote the logarithm

$$L(\lambda, \mu) = \ln P(\lambda, \mu) = \ln P(C) = \ln P(U(\lambda) + V(\mu)) \tag{38}$$

Its derivatives are

$$\begin{aligned} L_{\lambda\lambda}'' &= \tfrac{P''P - P'^2}{P^2} u^2(\lambda) + \tfrac{P'}{P} u'(\lambda), \\ L_{\mu\mu}'' &= \tfrac{P''P - P'^2}{P^2} v^2(\mu) + \tfrac{P'}{P} v'(\mu) \end{aligned} \tag{39}$$

Finally, the harmonic equation, $L''_{\lambda\lambda} + L''_{\mu\mu} = 0$, leads to the following necessary and sufficient condition for the parameters (λ, μ) to be harmonic

$$\frac{P''}{P'} - \frac{P'}{P} = -\frac{u'(\lambda) + v'(\mu)}{u^2(\lambda) + v^2(\mu)} \tag{40}$$

That is,

$$\frac{(\ln P)''}{(\ln P)'} = -\frac{U''(\lambda) + V''(\mu)}{U'^2(\lambda) + V'^2(\mu)} \tag{41}$$

Finally, the above equation is the necessary and sufficient condition for the harmonic parameters (λ, μ). It is also the necessary and sufficient condition for rational structure if we always remember the necessary conditions, i. e., the laws of parallel and skew. The above equation is called the skew equation. However, it can be further simplified.

2.4 Riccati Equation and its Solutions

We know that skew equation is the necessary and sufficient condition for rational structure. Its solution is all the single-variable functions $P(C)$ whose derivatives must satisfy the equation. In fact, the derivative of $P(C)$ with respect to C is still a function of C. Therefore, the left-hand side of skew equation is still a function of C. Let us denote it by $R(C)$

$$R(C) = \frac{P''}{P'} - \frac{P'}{P} = -\frac{U''(\lambda) + V''(\mu)}{U'^2(\lambda) + V'^2(\mu)} \tag{42}$$

That is,

$$(U'^2(\lambda) + V'^2(\mu))R(C) = -U''(\lambda) - V''(\mu) \tag{43}$$

The two variables λ and μ on the right-hand side of the above equation are separated. Accordingly, the mixed partial derivatives to the left hand side of the above equation must be zero

$$\begin{aligned} 0 &= ((U'^2(\lambda) + V'^2(\mu))R(C))''_{\lambda\mu} \\ &= U'V' \left(2(U'' + V'')R'(C) + (U'^2 + V'^2)R''(C)\right) \end{aligned} \tag{44}$$

Therefore, the final factor in the above formula must be zero, and we have

$$\frac{R''(C)}{2R'(C)} = -\frac{U''(\lambda) + V''(\mu)}{U'^2(\lambda) + V'^2(\mu)} = R(C) \tag{45}$$

Finally, we have the simple equation which rational structures must satisfy

$$R''(C) - 2R(C)R'(C) = 0 \tag{46}$$

Its first integration is exactly the Riccati equation

$$R'(C) - R^2(C) = \pm a^2 \tag{47}$$

where a is a positive constant. The above ordinary differential equation must be satisfied by any rational structure. It is the Riccati equation with constant coefficients. Table 1 is the list of all solutions to the equation where d is the integral constant for the first order equation.

Table 1: Solutions to Riccati Equation $R'(C) - R^2(C) = \pm a^2$

constant term	$a = 0$ /name	$+a^2$ /name	$-a^2$ /name
constant solution	$R(C) = 0$ /Riccati-0	no solution	$R(C) = a$ /Riccati-a
nonconstant solution	$-\frac{1}{C}$ /Riccati-Pw	$a\tan(aC + d)$ /Riccati-Tw	$-a\tanh(aC + d)$ /Riccati-Hw

The solution $-\frac{1}{C}$ in the Table corresponds to the zero coefficient, $a = 0$. Therefore, it is called the parabolic solution and denoted by Riccati-Pw. The last two solutions are denoted by Riccati-Tw and Riccati-Hw respectively, because their resulting orthogonal curve equations involve trigonometric functions and hyperbolic functions, respectively.

This paper presents a preliminary study on the Riccati-Pw solution only,

$$R(C) = -\frac{1}{C} \tag{48}$$

3 A Preliminary Study on the Parabolic Solution

3.1 Skew Density

The two equations (42) and (48) combined lead to the following

$$\frac{P''(C)}{P'(C)} - \frac{P'(C)}{P(C)} = -\frac{1}{C} \tag{49}$$

whose solution $P(C)$ is

$$P(C) = m|C|^l \tag{50}$$

where $m(> 0), l$ are arbitrary constants. Finally, the formula (33) determines the rational structure f

$$\begin{aligned} f(x, y) &= f(\hat{C}) = \int P(C)d\hat{C} = h\int P(C)dC \\ &= hm\int |C|^l dC \end{aligned} \tag{51}$$

where we see that the negative h can help achieve a decreasing stellar density ρ away from the center if $l + 1$ is positive.

3.2 Parallel Integrals

The logarithmic density distribution (51) is the function of the single variable C, not the function of the coordinates (x, y). Therefore, we can not draw its graph. To find (x, y), firstly we need to calculate the parallel integrals $U(\lambda), V(\mu)$. Substitution of the formula (48) into the left-hand side of the equation (42) leads to

$$-\frac{1}{U+V} = -\frac{U''(\lambda)+V''(\mu)}{U'^2(\lambda)+V'^2(\mu)} \tag{52}$$

The above equation is

$$U(\lambda)V''(\mu) + V(\mu)U''(\lambda) = U'^2(\lambda) + V'^2(\mu) - U(\lambda)U''(\lambda) - V(\mu)V''(\mu) \tag{53}$$

The two variables λ and μ on the right-hand side of the above equation are separated. Accordingly, the mixed partial derivatives to the left hand side must be zero

$$U'(\lambda)V'''(\mu) + V'(\mu)U'''(\lambda) = 0$$

The two variables λ and μ in the above equation are separated either. Accordingly, we have

$$\begin{cases} U''(\lambda) - w^2U(\lambda) = c_1, \\ V''(\mu) + w^2V(\mu) = c_2 \end{cases} \tag{54}$$

where $w(>0), c_1, c_2$ are constants. The quantity w is exactly the letter included in the name Riccati-Pw. The only solution to both the equation (53) and the equation (54) is

$$\begin{array}{l} U(\lambda) = A\cosh(w\lambda+\epsilon), \\ u(\lambda) = U'(\lambda) = Aw\sinh(w\lambda+\epsilon), \\ U''(\lambda) = Aw^2\cosh(w\lambda+\epsilon), \\ V(\mu) = A\cos(w\mu+\delta), \\ v(\mu) = V'(\mu) = -Aw\sin(w\mu+\delta), \\ V''(\mu) = -Aw^2\cos(w\mu+\delta) \end{array} \tag{55}$$

where $A(>0), \epsilon, \delta$ are constants.

It appears that we have solved our rational structure equations. However, our final goal is the determination of rational structure $f(x, y)$ in the physical space (x, y). But the results we have achieved are defined in the parameter space (λ, μ). The relationship between the two spaces are exactly the net of orthogonal curves (2) which is determined by $P(C)$ (see the formula (35)). Therefore, in the following Sections we try to solve $x(\lambda, \mu)$ and $y(\lambda, \mu)$ and their inverse based on the formula (35).

3.3 Orthogonal Curve Equations and their Solution

Now we look for $x(\lambda, \mu)$ and $y(\lambda, \mu)$, i.e., the harmonic coordinate curves (2) based on the formula (35). Because $C = A\cosh(w\lambda+\epsilon) + A\cos(w\mu+\delta)$ is never negative, we have

$$P(\lambda, \mu) = m|C|^l = m|U+V|^l = mC^l \tag{56}$$

The conjugate $\alpha(\lambda, \mu)$ to $L = \ln(P)$, satisfies the following Cauchy-Riemann equations

$$\begin{array}{l} \alpha'_\mu = L'_\lambda = (\ln P)'_\lambda = l\frac{U'(\lambda)}{C}, \\ \alpha'_\lambda = -L'_\mu = -l\frac{V'(\mu)}{C} \end{array} \tag{57}$$

Therefore,

$$\alpha'_\lambda U'(\lambda) = -\alpha'_\mu V'(\mu) \tag{58}$$

The solution to the corresponding characteristic equation is

$$\int \frac{d\lambda}{U'(\lambda)} - \int \frac{d\mu}{V'(\mu)} = c_3 \tag{59}$$

where c_3 is another integral constant. Finally, the solution to the characteristic equation is

$$\tanh(\xi)\tan(\eta) = c \tag{60}$$

where c is a constant and

$$\xi = \frac{w\lambda + \epsilon}{2}, \quad \eta = \frac{w\mu + \delta}{2} \tag{61}$$

Therefore, $\alpha(\lambda, \mu)$ must be a composite function of the product $\Upsilon = \tanh(\xi)\tan(\eta)$

$$\alpha(\lambda, \mu) = \alpha(\Upsilon) \tag{62}$$

Substituting the above into an equation in (57), we have

$$\alpha'_\lambda(\lambda, \mu) = \alpha'_\Upsilon \Upsilon'_\lambda = \frac{w}{2}\alpha'_\Upsilon \operatorname{sech}^2\xi \tan\eta = -l\frac{V'(\mu)}{C} = l\frac{Aw\sin(2\eta)}{A\cosh(2\xi) + A\cos(2\eta)} \tag{63}$$

That is,

$$\alpha'_\Upsilon = 2l\frac{2\cosh^2\xi\cos^2\eta)}{\cosh(2\xi) + \cos(2\eta)} \tag{64}$$

Here we present a very important mathematical identity whose verification is left for the readers,

$$\frac{2\cosh^2\xi\cos^2\eta}{\cosh(2\xi) + \cos(2\eta)} = \frac{1}{1 + \tanh^2\xi\tan^2\eta} \tag{65}$$

Finally, we obtained the conjugate α to L,

$$\alpha(\lambda, \mu) = 2l\arctan\Upsilon + \alpha_0 = 2l\arctan(\tanh\xi\tan\eta) + \alpha_0 \tag{66}$$

where α_0 is an integral constant. The constant rotates our structure in the (x, y) plane, therefore, it is ignored.

Finally, we have the orthogonal curve equations derived from the formula (35)

$$\begin{aligned} x'_\mu &= P\cos\alpha = mC^l\cos(2l\arctan(\tanh\xi\tan\eta)), \\ x'_\lambda &= P\sin\alpha = mC^l\sin(2l\arctan(\tanh\xi\tan\eta)) \end{aligned} \tag{67}$$

If we choose

$$l = \frac{n}{2} \tag{68}$$

then we have the following result

$$\begin{aligned} x'_\mu &= m(2A)^{n/2}\left(\Gamma^n - \tbinom{l}{2}\Gamma^{n-2}\Lambda^2 + \tbinom{l}{4}\Gamma^{n-4}\Lambda^4 - \cdots\right), \\ x'_\lambda &= m(2A)^{n/2}\left(\tbinom{l}{1}\Gamma^{n-1}\Lambda - \tbinom{l}{3}\Gamma^{n-3}\Lambda^3 + \cdots\right) \end{aligned} \tag{69}$$

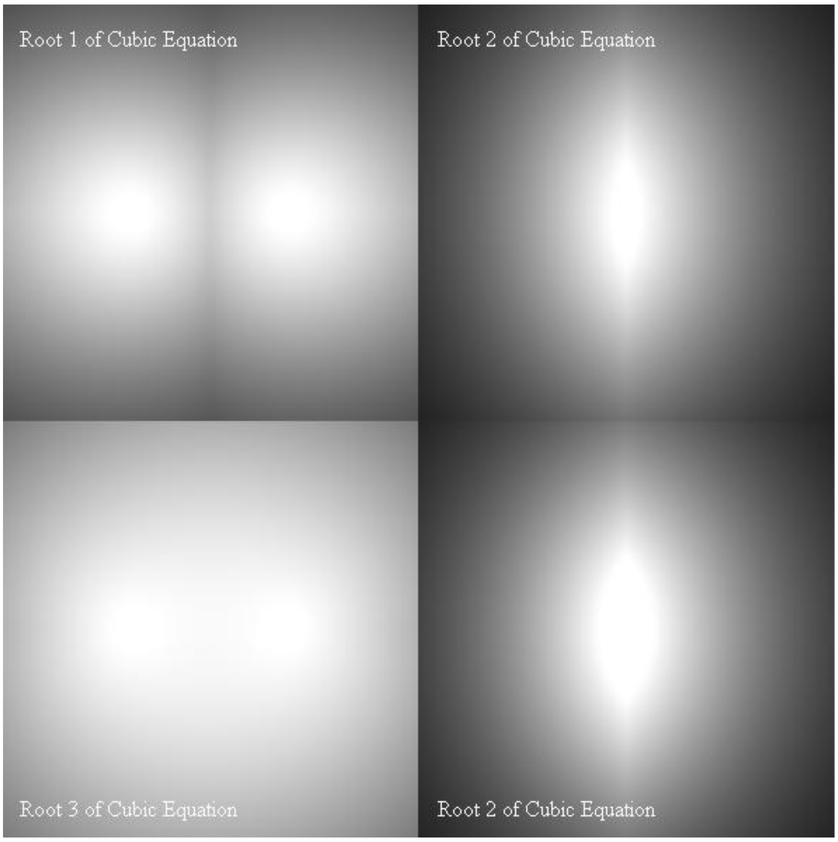

Figure 1: Rational structures which result from the three roots of rational cubic equation. The two graphs on the right side are the same rational structure in different resolutions. The structure resembles the sharp bar of galaxy NGC 1073.

where $\binom{l}{k}$ are Newton's binomial coefficients and

$$\Gamma = \cosh\xi\cos\eta, \quad \Lambda = \sinh\xi\sin\eta \tag{70}$$

In the above formulas and in the following We assume that

$$\cos\eta \geq 0 \tag{71}$$

Introducing the complex variable

$$\zeta = \xi + \mathrm{i}\,\eta \tag{72}$$

we have

$$x'_\mu + \mathrm{i}\,x'_\lambda = m\sqrt{2A}^n \cosh^n\zeta \tag{73}$$

This solution is denoted by Riccati-$Pw\frac{n}{2}$. However, this paper deals with $n = 3$ only. In the following Section, we solve the above equation and calculate $f(x, y)$.

4 Graph of Riccati-$Pw\frac{3}{2}$ Solution

4.1 Orthogonal Curves

$$\begin{aligned} x'_\mu &= P\cos\alpha = mC^{\frac{3}{2}}\cos(3\arctan(\tanh\xi\tan\eta)), \\ x'_\lambda &= P\sin\alpha = mC^{\frac{3}{2}}\sin(3\arctan(\tanh\xi\tan\eta)) \end{aligned} \tag{74}$$

According to the formula (73), we have

$$x'_\mu + \mathrm{i}\, x'_\lambda = m\sqrt{2A}^3 \cosh^3\zeta = \frac{m\sqrt{A}^3}{\sqrt{2}}(3\cosh\zeta + \cosh(3\zeta)) \tag{75}$$

Therefore,

$$x'_\eta + \mathrm{i}\, x'_\xi = x'_\eta - \mathrm{i}\,(-y)'_\eta = \frac{d}{d\zeta}(-y + \mathrm{i}\, x) = \frac{m\sqrt{2}\sqrt{A}^3}{w}(3\cosh\zeta + \cosh(3\zeta)) \tag{76}$$

Its integration is

$$-y + \mathrm{i}\, x = \frac{m\sqrt{2}\sqrt{A}^3}{w}(3\sinh\zeta + \frac{1}{3}\sinh(3\zeta)) = H(3\sinh\zeta + \sinh^3(\zeta)) \tag{77}$$

where H is a constant

$$H = \frac{m2\sqrt{2A}^3}{3w} \tag{78}$$

To draw the graph of our rational structure, we need the inverse complex function to the formula (77)

$$\begin{cases} \xi = \xi(x, y), \\ \eta = \eta(x, y) \end{cases} \tag{79}$$

That is, we need solve the following algebraic equation with a complex coefficient q

$$\nu^3 + p\nu + q = 0 \tag{80}$$

where

$$\begin{aligned} &\nu = \sinh\zeta, \\ &p = 3, \\ &q = \hat{y} - \mathrm{i}\,\hat{x} = y/H - \mathrm{i}\, x/H \end{aligned} \tag{81}$$

This equation is called a rational algebraic equation. In our case, it is a cubic equation.

Now we solve the cubic equation. It is straightforward to show that

$$(\beta + \gamma)^3 - 3\beta\gamma(\beta + \gamma) - (\beta^3 + \gamma^3) = 0 \tag{82}$$

which holds for any pair of complex numbers β, γ. Therefore,

$$\nu = \beta + \gamma \tag{83}$$

must be a solution to the cubic equation if

$$-3\beta\gamma = 3 = p \tag{84}$$

and

$$-(\beta^3 + \gamma^3) = \hat{y} - \mathrm{i}\,\hat{x} = q \tag{85}$$

Simultaneously, β^3, γ^3 must be the two roots of the following quadratic equation

$$\kappa^2 + q\kappa - p^3/27 = \kappa^2 + q\kappa - 1 = 0 \tag{86}$$

because the following identity holds

$$(\kappa-\beta^3)(\kappa-\gamma^3)=\kappa^2-(\beta^3+\gamma^3)\kappa+\beta^3\gamma^3 \tag{87}$$

Now we look for the two roots of the quadratic equation (86). Its determinant is

$$\Delta=q^2+2^2=\delta_1+\mathrm{i}\,\delta_2=4+\hat{y}^2-\hat{x}^2+\mathrm{i}\,(-2\hat{x}\hat{y}) \tag{88}$$

The two roots are

$$\begin{array}{l}\kappa_1=\beta^3=\phi_1+\mathrm{i}\,\phi_2=\frac{1}{2}\left(-\hat{y}+\sqrt{|\Delta|}\cos(\theta/2)+\mathrm{i}\,(\hat{x}+\sqrt{|\Delta|}\sin(\theta/2))\right),\\ \kappa_2=\gamma^3=\psi_1+\mathrm{i}\,\psi_2=\frac{1}{2}\left(-\hat{y}-\sqrt{|\Delta|}\cos(\theta/2)+\mathrm{i}\,(\hat{x}-\sqrt{|\Delta|}\sin(\theta/2))\right)\end{array} \tag{89}$$

where θ is a specific amplitude of the complex number Δ while $|\Delta|$ is its modulus (in this paper, arg is always a specific amplitude)

$$\begin{array}{l}\theta=\arg\Delta=\arccos\left(\delta_1/\sqrt{\delta_1^2+\delta_2^2}\right),\\ |\Delta|=\sqrt{(4-\hat{r}^2)^2+16\hat{y}^2},\\ \hat{r}=\sqrt{\hat{x}^2+\hat{y}^2}\end{array} \tag{90}$$

By now we have found the quantities β^3,γ^3. Their cubic roots are

$$\begin{array}{ll}\beta_m=(\phi_1^2+\phi_2^2)^{1/6}e^{\mathrm{i}(\arg\kappa_1+2m\pi)/3}, & m=1,2,3\\ \gamma_n=(\psi_1^2+\psi_2^2)^{1/6}e^{\mathrm{i}(\arg\kappa_2+2n\pi)/3}, & n=1,2,3\end{array} \tag{91}$$

The sum $\nu=\beta_m+\gamma_n$ for an arbitrary pair of m and n may not be the solution to the original cubic equation because the equation (84) may not be satisfied. However, the pair (β_m,γ_n) which satisfies (84) must exist. Suppose the pair is β,γ. Then the three roots of the original cubic equation are

$$\begin{array}{l}\nu_1=\beta+\gamma,\\ \nu_2=\varepsilon\beta+\bar{\varepsilon}\gamma,\\ \nu_3=\bar{\varepsilon}\beta+\varepsilon\gamma\end{array} \tag{92}$$

where

$$\varepsilon=e^{2\pi\mathrm{i}/3}=-1/2+\mathrm{i}\,\sqrt{3}/2,\ \bar{\varepsilon}=\varepsilon^2=e^{-2\pi\mathrm{i}/3}=-1/2-\mathrm{i}\,\sqrt{3}/2 \tag{93}$$

Now we obtained the important quantity ν (see the formulas (92)). We can further calculate $\zeta=\sinh^{-1}\nu$ (see the formulas (81)), and obtain the inverse function (79). However, this is not necessary. In this paper we study the graph of rational structure only. Therefore, our major concern is the logarithmic density $f(x,y)$. It is the function of the single variable C only (see (51)). That is, it is a skew quantity. Fortunately we have a very simple formula for C

$$\begin{aligned}C&=A(\cosh(2\xi)+\cos(2\eta))=2A|\cosh\zeta|=2A\left|\sqrt{1+\sinh^2\zeta}\right|\\&=2A\left|\sqrt{1+\nu^2}\right|\end{aligned} \tag{94}$$

Its proof is straightforward and ignored. Therefore,

$$f(x,y)=hm\int C^l dC=\frac{hm}{l+1}C^{l+1}=\frac{2hm}{5}C^{5/2} \tag{95}$$

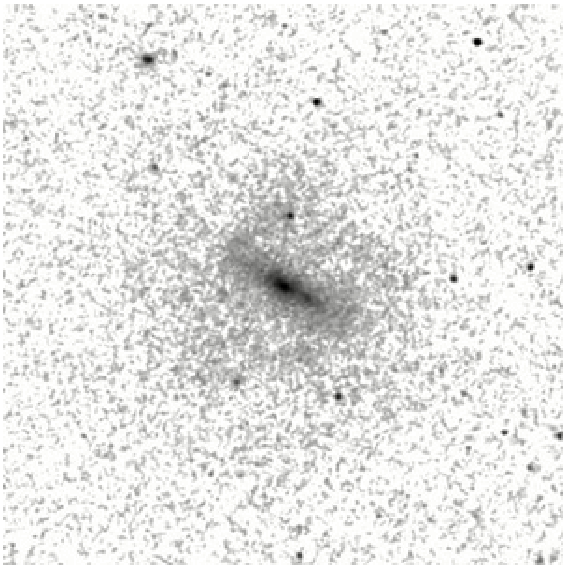

Figure 2: Longer wavelength image of NGC 1073 (image credit [9]).

Our structure involves the constant parameters, h, m, w, A. For example, a larger absolute value of h corresponds to a sharper structure.

The graphs of the three rational structures which correspond to the three roots of the cubic equation respectively, are displayed in Fig. 1. The structure which results from the second root presents a bar pattern. It resembles the sharp bar of galaxy NGC 1073 (see Fig. 2, image credit [9]). The first important question is: Do the structures approach the exponential disk when $r = \sqrt{x^2 + y^2} \to \infty$?

4.2 The Limiting Structure of Riccati-$Pw\frac{3}{2}$

Taking the complex number q to be in the first quadrant in the (x, y) plane (see the formula (85)) and assuming

$$|q| = |\hat{y} - \mathrm{i}\,\hat{x}| = \sqrt{\hat{x}^2 + \hat{y}^2} \to \infty \tag{96}$$

we have

$$\begin{aligned}
&\hat{x} \le 0, \quad , \hat{y} \ge 0, \quad 0 \le \theta \le \pi, \\
&+\sqrt{\Delta} \to q, \\
&\kappa_1 \to 0, \quad \beta_m \to 0, \quad m = 1, 2, 3 \\
&\kappa_2 \to q, \quad |\gamma_n| \to |q|^{1/3} = \hat{r}^{1/3}, \quad n = 1, 2, 3 \\
&|\nu| \to |\gamma| \to \hat{r}^{1/3}, \\
&C = 2A\left|\sqrt{1+\nu^2}\right| \to 2A\left|\sqrt{\nu^2}\right| = 2A|\nu| \to 2A\hat{r}^{1/3}, \\
&f \sim C^{5/2} \to \hat{r}^{5/6} \sim \hat{r}, \\
&\rho \sim \exp(-\hat{r})
\end{aligned} \tag{97}$$

That is, our rational structures do approach the exponential disk.

5 Conclusion and Future Work

The discovery in 2013 of the skew law (see the formula (25) and the reference [10]), is the greatest breakthrough in rational structure study. The mathematical theory discussed in the current paper is based on the law. Our Riccati equation results from the law, and all rational structures must satisfy the equation. The equation is an ordinary differential equation and we found all its solutions in our case. Substituting each solution to the necessary and sufficient condition for rational structure (i. e., the skew equation (41)), leads to the equation systems governing the parallel integrals. The solutions of parallel integrals always involve a constant w due to the method of separated variables. This constant always appears in the names of our solutions. For example, Riccati-0w is the constant solution to the Riccati equation with the zero coefficient $a = 0$ and nonzero $w(\neq 0)$. It is interesting that Riccati-00 is the familiar rational structure of exponential disk. Astronomical observation shows that galaxy structure extends "infinitely" in its disk plane and its stellar density becomes zero in the infinite area. These are called the galaxy principles. A preliminary study shows that Riccati-0w, Riccati-a0, and Riccati-aw do not meet the principles. The preliminary study also indicates that the density distribution of Riccati-Tw and Riccati-Hw may not decrease radially in an exponential law. Instead, it may decrease in a power law. Coincidentally, the stellar density of elliptical galaxies obeys a power law. Therefore, the solutions of Riccati-Tw and Riccati-Hw deserve further investigation.

The structure of Riccati-P$w\frac{1}{2}$ is the Heaven Breasts. Its density decreases in an exponential law, $\rho(x, y) \sim \exp(-r^3)$. However, it does not approach the exponential disk $\exp(-r)$. Astronomical observation shows that galaxies always have a bright center. Therefore, Heaven Breasts can not exist independently. I added the Heaven Breasts structure $\rho_b \sim \exp(-r^3)$ to the exponential disk $\rho_d \sim \exp(-r)$ to simulate barred galaxy structure,

$$\rho \approx \rho_d + \rho_{b1} + \rho_{b2} \tag{98}$$

It fits to galaxy images very well. However, rational structure must satisfy the nonlinear Riccati equation. Therefore, the above formula is not mathematically a rational structure. However, it is a good approximation.

This paper presents a detailed study on Riccati-$Pw\frac{3}{2}$. It gives three different rational structures corresponding to the three roots of rational cubic equation. Two of the solutions present the similar structure to Heaven Breasts. The third solution displays a bar pattern. It resembles the sharp bar of galaxy NGC 1073. Therefore, we need solve rational algebraic equations of higher degrees and display the structure of Riccati-$Pw\frac{n}{2}$. Especially we need solve the rational quartic equation to see its four patterns. Are all barred galaxies governed by the rational algebraic equations? The question will be resolved in the near future.

6 Acknowledgement

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7 Appendix

The following is the c++ computer program file whose data output $gim[m][n]$ is used in our graph (Figure 1):

```
#include<stdio.h>
#include<math.h>
#include<iomanip.h>
#include<fstream.h>
const int Mm=100,Nm=100;
int rs=2;
long double pi=3.14159265358979323846,hpi=pi*0.5,tpi=pi*2,qpi=pi*0.25;
long double rh0=8000,hh=-0.4,mm=1,AA=1,www=1,gim[102][102]=0;
int main()
int m,n,hM=Mm/2,hN=Nm/2;
long double MX,MY,DI,DJ;
long double cc,X,Y,xb,yb,da,db,argD,rb,rom,phi1,phi2,psi1,psi2,argka1,argka2,
   betaR,betaI,gammaR,gammaI,nuR,nuI;
MX=10;MY=MX;DI=2*MX/Mm;DJ=2*MY/Nm;
for(m=hM+1;m¡=Mm;m++){
   X=(m-hM-0.5)*DI;
   xb=X*3*www/( 2*mm*exp(1.5*log(2*AA)) );
   for(n=1;n¡=hN;n++){
    Y=(n-hN-0.5)*DJ;
    yb=Y*3*www/( 2*mm*exp(1.5*log(2*AA)) );
    da=4+yb*yb-xb*xb; db=-2*xb*yb;
    argD=acos(da/sqrt(da*da+db*db));
    rb=sqrt(xb*xb+yb*yb);
    rom=sqrt( (4-rb*rb)*(4-rb*rb)+16*yb*yb );
    phi1=0.5*( -yb+sqrt(rom)*cos(0.5*argD) );
    phi2=0.5*( xb+sqrt(rom)*sin(0.5*argD) );
    psi1=0.5*( -yb-sqrt(rom)*cos(0.5*argD) );
    psi2=0.5*( xb-sqrt(rom)*sin(0.5*argD) );
    argka1=acos(phi1/sqrt(phi1*phi1+phi2*phi2));
    argka2=acos(psi1/sqrt(psi1*psi1+psi2*psi2));
    if(rs==1){
     betaR=exp(log(phi1*phi1+phi2*phi2)/6.)*cos(argka1/3.+tpi/3.);
     betaI=exp(log(phi1*phi1+phi2*phi2)/6.)*sin(argka1/3.+tpi/3.);
     gammaR=exp(log(psi1*psi1+psi2*psi2)/6.)*cos(argka2/3.);
     gammaI=exp(log(psi1*psi1+psi2*psi2)/6.)*sin(argka2/3.);
    }else if(rs==2){
     betaR=exp(log(phi1*phi1+phi2*phi2)/6.)*cos(argka1/3.+2*tpi/3.);
     betaI=exp(log(phi1*phi1+phi2*phi2)/6.)*sin(argka1/3.+2*tpi/3.);
     gammaR=exp(log(psi1*psi1+psi2*psi2)/6.)*cos(argka2/3.-tpi/3.);
     gammaI=exp(log(psi1*psi1+psi2*psi2)/6.)*sin(argka2/3.-tpi/3.);
```

```
  }else{
    betaR=exp(log(phi1*phi1+phi2*phi2)/6.)*cos(argka1/3.);
    betaI=exp(log(phi1*phi1+phi2*phi2)/6.)*sin(argka1/3.);
    gammaR=exp(log(psi1*psi1+psi2*psi2)/6.)*cos(argka2/3.+tpi/3.);
    gammaI=exp(log(psi1*psi1+psi2*psi2)/6.)*sin(argka2/3.+tpi/3.);
  }
  nuR=betaR+gammaR;   nuI=betaI+gammaI;
  cc=AA*2*sqrt( (1+nuR*nuR-nuI*nuI)*(1+nuR*nuR-nuI*nuI)+4*nuR*nuI*nuR*nuI );
  cc=hh*mm*exp(   2.5*log(cc)   )/2.5;
  gim[m][n]=rh0*exp(cc);
  gim[hM-(m-hM)+1][n]=gim[m][n];
  gim[m][hM-(n-hM)+1]=gim[m][n];
  gim[hM-(m-hM)+1][hM-(n-hM)+1]=gim[m][n];
 }//n
}//m
return 0;
}
```

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