MEASURING COMPLEXITY BY USING REDUCTION TO
SOLVE P VS NP AND NC & PH

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1. Abstract

This article prove that NC and PH is proper (especially P is not NP) by using reduction difference. We can prove that NC is proper by using ALO is not NC. This means L is not P. We can prove P is not NP by using reduction difference between L and P. And we can also prove that PH is proper by using P is not NP.

2. NC is proper

We use circuit problem as follows;

Definition 1. We will use the term “AC”, “NC” as each complexity decision problems classes. “FAC” as function problems class of AC. These complexity classes also use uniform circuits family set that compute target complexity classes problems. “f ◦ g” as composite circuit that output of g are input of f. In this case, we also use complexity classes to show target circuit. For example, A ◦ BB when A is circuits family and BB is circuits family set mean that a ◦ b | a ∈ A, b ∈ B ∈ BB. “R(A)” as subset of reversible NC that include A. Reversible mean that \( (R(A) ◦ (R(A))^{-1}) (x) = x \). Circuits family uniformity is that these circuits can compute FAC⁰.

Theorem 2. NL \( \leq_{AC^0} NC^2 \)

Proof. Mentioned [1] Theorem 10.40, all NC² are closed by FL reduction. This reduction is validity of \( (c_1, c_2) \) transition function. Transition function change \( O(1) \) memory and keep another memory. Therefore this validity can compute AC⁰ and we can replace FL to FAC⁰.

\( \square \)

Theorem 3. AC¹ has Universal Circuits Family that can emulate all AC¹ circuits family. That is, every AC¹ has AC⁰ – Complete under FAC⁰.

Proof. To prove this theorem by making universal circuit family \( A' \in AC^0 \) that emulate circuit family \( \{C_j\} \in AC^1 \) by using “depth circuit tableau”. Universal circuit \( U_j \in A' \) have partial circuit \( u_{k,d} \) that emulate all \( C_j \) gates \( g_k \in \) (include input value) and partial circuit \( v_{p-q,d} \) that emulate all wires \( w_{p-q} \) from \( g_q \) output to \( g_q \) input in every depth \( d \). \( U_j \) use three value \( \{\top, \bot, \emptyset\} \). \( \emptyset \) is special value that all \( g_q \) ignore this value. All gate in a depth \( d \) is \( u_{d} \), all wires that input connected \( k \) in a depth \( d \) is \( v_{k-d} \), output connected \( k \) in a depth \( d \) is \( v_{-k,d} \).

\( v_{p-q,d} \) input connected each \( w_{p,d} \) output and \( w_{p-q} \). \( v_{p-q,d} \) Output connected each \( w_{q,d+1} \) input. If \( w_{p-q} \) does not exist, \( v_{p-q,d} \) output \( \emptyset \). Else if \( w_{p-q} \) have negative then \( v_{p-q,d} \) output \( u_{k,d} \) negative value. Else \( v_{p-q,d} \) output \( u_{k,d} \) positive value.
u_{k,d} input connected each v_{k,d-1} output and g_k. u_{k,d} output connected each v_k output. If g_k is one of C, input value, u_{k,d} output the input value. Else (g_k is And / Or gate) u_{k,d} output the gate value that compute from all v_{k,d-1} output values. In this computation, u_{k,d} ignore all 0. If all value are 0, u_{k,d} output 0.

This U_j that consists of u, v emulate C_j. We can make every u, v in FAC^0 because C_j is uniform circuit 1. Therefore, A' in AC^i and this theorem was shown.

\[ \text{Theorem 4. } NC^i = NC^{i+1} \implies NC^i \text{ – Complete } = AC^i \text{ – Complete } = NC^{i+1} \text{ – Complete.} \]

\[ \text{Proof. } \text{If } NC^i = NC^{i+1}, \text{ all } NC^i \text{ – Complete, } AC^i \text{ – Complete, } NC^{i+1} \text{ – Complete can reduce each other and } NC^i \text{ – Complete, } AC^i \text{ – Complete, } NC^{i+1} \text{ – Complete in } NC^i. \text{ Therefore, this theorem was shown.} \]

\[ \text{Theorem 5. } nc \subset ne \circ NC^1 \mid ne \subset NC^i \]

\[ \text{Proof. To prove it using reduction to absurdity. We assume that } nc = ne \circ NC^1 \mid ne \subset NC. \text{ It is trivial that } nc = NC^1 = AC^i = NC^{i+1} = AC^{i+1} = \ldots. \]

\[ \text{Because } nc = ne \circ NC^1 \text{ and mentioned above 4, } R(\text{FAC}^0 \text{ – Complete}) = \text{FAC}^0 \text{ – Complete. Therefore} \]

\[ nc = ne \circ NC^1 \implies \forall A, B \in R(\text{FAC}^0 \text{ – Complete}) \exists C \in \text{FAC}^0 (A \circ B = A \circ C) \]

\[ A \text{ is reversible circuits family. Therefore } A \text{ have } A^{-1}. \]

\[ nc = ne \circ NC^1 \]

\[ \implies \forall A, B \in R(\text{FAC}^0 \text{ – Complete}) \exists C \in \text{FAC}^0 (A^{-1} \circ A \circ B = B^{-1} \circ A \circ C) \]

\[ \implies \forall B \in R(\text{FAC}^0 \text{ – Complete}) \exists C \in \text{FAC}^0 (B = C) \]

This means FAC^0 = FAC^i. But this contradict AC^0 \subset NC^1 \subset AC^i.

Therefore, this theorem was shown than reduction to absurdity.

\[ \text{Corollary 6. } NC^i \subset NC^{i+1} \]

\[ \text{Theorem 7. } AC^i \subset AC^{i+1} \]

\[ \text{Proof. If } AC^i = AC^{i+1} \text{ then } AC^i = NC^{i+1} = AC^{i+1} = NC^{i+2} = AC^{i+2} \text{ and contradict mentioned above 5 } NC^i \subset NC^{i+1}. \text{ Therefore, this theorem was shown than reduction to absurdity.} \]

\[ \text{Theorem 8. } NC = AC \subset P \]

\[ \text{Proof. To prove it using reduction to absurdity. We assume that } NC = P. \text{ It is trivial that we can reduce some } A \in P \text{ – Complete to } B \in NC. \text{ But } B \text{ is also in } NC^i. \text{ Therefore, this mean that } NC^i = NC^i \text{ and contradict mentioned above 5 } NC^i \subset NC^{i+1}. \text{ Therefore, this theorem was shown than reduction to absurdity.} \]

\[ \text{Corollary 9. } L \subseteq P \]

### 3. PH is proper

**Definition 10.** We will use the term “L”, “P”, “P – Complete”, “NP”, “NP – Complete”, “FL”, “FP” as each complexity classes. These complexity classes also use Turing Machine (TM) set that compute target complexity classes problems. We will use the term “Δ_k”, “Σ_k”, “Π_k” as each Polynomial hierarchy classes. “f \circ g” as composite problem that output of g are input of f. “R(A)” as “reversible TM” that equal A. Reversible mean that \( (R(A) \circ (R(A))^{-1}) (x) = x. \)
Theorem 11. \( R(\Sigma_k) \subset \Sigma_k, \ R(\Pi_k) \subset \Pi_k. \)

Proof. We can reduce \( \Sigma_k \) and \( \Pi_k \) to another \( \Sigma_k \) and \( \Pi_k \) that have tree graph of computation history. (If all configuration keep input, computation history become tree graph.) These \( \Sigma_k, \Pi_k \) are \( R(\Sigma_k), \ R(\Pi_k) \) because each computation history of each output only reach one input. Therefore \( \left( R(A) \circ (R(A))^{-1} \right)(x) = x \). We can compute these reduction in \( FP \). Therefore, this theorem was shown. \( \square \)

Theorem 12. \( R(\Sigma_k - \text{Complete}) \subset \Sigma_k - \text{Complete} \)

Proof. Mentioned above11, it takes atmost \( O(n) \) times and spaces to reduce \( \Sigma_k \)
into \( R(\Sigma_k) \). Therefore this theorem was shown. \( \square \)

Theorem 13. \( P \subseteq NP \)

Proof. To prove it using reduction to absurdity. We assume that \( P = NP \).
As we all know that if \( P = NP \) then all \( NP \) can reduce \( P - \text{Complete} \) under \( FL \). And all \( NP \circ FP \subset NP \). Therefore
\[
P = NP \rightarrow \forall A \in NP - \text{Complete} \forall B \in FP \exists C \in FL (A \circ B = A \circ C)
\]

Mentioned above11, \( R(NP - \text{Complete}) \subset NP - \text{Complete} \). Therefore
\[
P = NP \rightarrow \forall D \in R(NP - \text{Complete}) \forall B \in FP \exists C \in FL (D \circ B = D \circ C)
\]
\( D \) is reversible function. Therefore \( D \) have \( D^{-1} \).
\[
P = NP \rightarrow \forall D \in R(P - \text{Complete}) \forall B \in FP \exists C \in FL (B = C)
\]
This means \( FP = FL \). But this contradict \( FL \nsubseteq F P \) mentioned above5. Therefore, this theorem was shown than reduction to absurdity. \( \square \)

Theorem 14. \( \sigma_k \subseteq \sigma_k \circ \Sigma_1 \mid \sigma_k \subset \Sigma_k \)

Proof. To prove it using reduction to absurdity. We assume that \( \sigma_k = \sigma_k \circ \Sigma_1 \).
Mentioned [2] Theorem 6.26, we can reduce all \( \sigma_k \) to \( \Sigma_k - \text{Complete} \) under \( FP \).

Because mentioned above 12, \( R(\Sigma_k) \subset \Sigma_k \). Therefore
\[
\sigma_k = \sigma_k \circ \Sigma_1 \rightarrow \exists A \in R(\Sigma_k - \text{Complete}) \forall B \in \Sigma_1 \exists C \in FP (A \circ B = A \circ C)
\]
\( A \) is reversible function. Therefore \( A \) have \( A^{-1} \).
\[
\sigma_k = \sigma_k \circ \Sigma_1 \rightarrow \exists A \in R(\Sigma_k - \text{Complete}) \forall B \in \Sigma_1 \exists C \in FP (A^{-1} \circ A \circ B = A^{-1} \circ A \circ C)
\]
This means \( \Sigma_1 = FP \). But this contradict mentioned above13. Therefore, this theorem was shown than reduction to absurdity. \( \square \)

Corollary 15. \( \Pi_k \subseteq \Pi_{k+1}, \Sigma_k \not\subseteq \Sigma_{k+1} \)

Theorem 16. \( \Delta_k \subseteq \Sigma_k, \Sigma_k \neq \Pi_k \)

\[
\Sigma_k = \Pi_k \rightarrow \Sigma_k = \Pi_k = PH
\]
\[
\Delta_k \subseteq \Sigma_k \rightarrow \Delta_k = \Sigma_k = \Pi_k = PH
\]
This contrapositive is,
\[
(\Sigma_k \not\subseteq PH) \lor (\Pi_k \not\subseteq PH) \rightarrow \Sigma_k \neq \Pi_k
\]
\[
(\Delta_k \not\subseteq PH) \lor (\Sigma_k \not\subseteq PH) \lor (\Pi_k \not\subseteq PH) \rightarrow \Delta_k \neq \Sigma_k
\]
From mentioned above 14,
\[
\Sigma_k \subseteq \Pi_{k+1} \subset PH
\]
Therefore, $\Delta_k \neq \Sigma_k, \Sigma_k \neq \Pi_k$. 


$\Sigma_k \subseteq \Sigma_{k+1}, \Pi_k \subseteq \Pi_{k+1}, \forall k \geq 1 (\Delta_k \subset (\Sigma_k \cap \Pi_k) \subset (\Sigma_k \cup \Pi_k) \subset \Delta_{k+1})$

Therefore, $\Delta_k \not\subseteq \Sigma_k, \Sigma_k \not\subseteq \Pi_k.$  

\[\square\]

**Theorem 17.** $\Pi_k \not\subseteq \Sigma_k, \Sigma_k \not\subseteq \Pi_k$

**Proof.** To prove it using reduction to absurdity. We assume that $\Pi_k \subset \Sigma_k$. This means that all $\Sigma_k = \Pi_k$ is also $\Sigma_k$.

$\Pi_k \subset \Sigma_k \rightarrow \forall A \in \Sigma_k \left( \overline{A} \in \Pi_k \subset \Sigma_k \right)$

Mentioned [2] Theorem 6.21, all $\Sigma_k$ are closed under polynomial time conjunctive reduction. We can emulate these reduction by using $\Pi_k$. That is, $\exists B \in \Sigma_k \forall C \in \Sigma_k \exists D \in \Pi_1 (B \circ D = C)$

Therefore, $\Pi_k \subset \Sigma_k$

$\rightarrow \exists B \in \Sigma_k \forall C \in \Sigma_k \exists D \in \Pi_1 \forall A \in \Sigma_k (B \circ D = C) \land (\overline{A} \in \Pi_k \subset \Sigma_k)$

$\rightarrow \exists B \in \Sigma_k \forall C \in \Sigma_k \exists D \in \Pi_1 (B \circ D = C) \land (\overline{B} \in \Sigma_k)$

$\rightarrow \exists B \in \Sigma_k \forall C \in \Sigma_k \exists D \in \Pi_1 (B \circ D = C) \land (B \in \Pi_k)$

Therefore $\Sigma_k \subset \Pi_k$ because $B \circ D \in \Pi_k$. But this means $\Sigma_k = \Pi_k$ and contradict $\Sigma_k \neq \Pi_k$ mentioned above. Therefore $\Pi_k \not\subseteq \Sigma_k$.

We can prove $\Sigma_k \not\subseteq \Pi_k$ like this.

Therefore, this theorem was shown than reduction to absurdity.  

\[\square\]

**Theorem 18.** $\Delta_k \subseteq \Pi_k$

**Proof.** To prove it using reduction to absurdity. We assume that $\Delta_k = \Pi_k$.


$\Sigma_k \subset \Sigma_{k+1}, \Pi_k \subset \Pi_{k+1}, \forall k \geq 1 (\Delta_k \subset (\Sigma_k \cap \Pi_k) \subset (\Sigma_k \cup \Pi_k) \subset \Delta_{k+1})$

Therefore

$\Delta_k = \Pi_k$

$\rightarrow \Delta_k \subset (\Sigma_k \cap \Pi_k) \subset \Sigma_k \subset (\Sigma_k \cup \Pi_k) \subset \Delta_{k+1}$

$\Pi_k \subset \Sigma_k$

But this result contradict mentioned above. Therefore, this theorem was shown than reduction to absurdity.  

\[\square\]

**Theorem 19.** $\Sigma_k \not\subseteq \Delta_{k+1}, \Pi_k \not\subseteq \Delta_{k+1}$

**Proof.** To prove it using reduction to absurdity. We assume that $\Sigma_k = \Delta_{k+1}$.


$\forall k \geq 1 (\Delta_k \subset (\Sigma_k \cap \Pi_k) \subset (\Sigma_k \cup \Pi_k) \subset \Delta_{k+1})$

Therefore

$\Sigma_k = \Delta_{k+1}$

$\rightarrow \Delta_k \subset (\Sigma_k \cap \Pi_k) \subset \Pi_k \subset (\Sigma_k \cup \Pi_k) \subset \Sigma_k = \Delta_{k+1}$

$\Pi_k \subset \Sigma_k$

But this result contradict mentioned above. Therefore $\Sigma_k \not\subseteq \Delta_{k+1}$.

We can prove $\Pi_k \not\subseteq \Delta_{k+1}$ like this.

Therefore, this theorem was shown than reduction to absurdity.  

\[\square\]

**Theorem 20.** $PH \not\subseteq \text{PSPACE}$

**Proof.** To prove it using reduction to absurdity. We assume that $PH = \text{PSPACE}$. It is trivial that we can reduce some $A \in \text{PSPACE} - \text{Complete}$ to $B \in PH$. But
B is also in $\Delta_k$. Therefore, this mean that $\Delta_k = \Delta_{k+1}$ and contradict mentioned above $1819 \; \Delta_k \subsetneq \Delta_{k+1}$. Therefore, this theorem was shown than reduction to absurdity. \hfill \Box

REFERENCES

