Application of the Ptolemy theorem (3)

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ABSTRACT: Sixty years ago, Rollett set a problem which was basing on the famous problem of the Arbelos, this cause many responses from others, and many solutions were received from readers, which suggest this problem was of sufficient intractability as to be deserving of dinine inspiration. Another problem we study in this paper is an analogue of Ptolemy’s theorem in the hyperbolic plane, that is, what is the relations between the mutual distances of n points in the hyperbolic plane. We also study the generalized Ptolemy theorem, which has Minkowskian geometry property.

INTRODUCTION

In general, incenter I belongs, together with three other centers \( G, S, S' \) (the gergonne point and the inner, outer soddy centers) to another important line, and known as the soddy line. In this paper, a number of points on the soddy line are identified as centers of perspective for pairs of triangles that have a common axis, the gergonne line which is orthogonal to the soddy line.

Relations between the mutual distances of three points are obtained which are necessary and sufficient to insure that those points determine a line, circle, horocycle, or equidistant curves.

We also present some new results that illustrate the power of complex numbers in planar geometry. The reversed triangle inequality in Minkowskian geometry is the unexpected key to the extension. This connection between the two geometries shows up again when we consider Ptolemy’s theorem with a Minkowskian metric.

1. The Apollonius problem and the vector methods

Figure 1 (see [1]):

Repeated use the equation obtained by Satterly (in [1]) to generate subsequent values:

\[
\begin{align*}
\ell_1 &= \frac{Rab}{Ra + b^2}, \quad \ell_2 = \frac{Rab}{Ra + 4b^2}, \quad \ell_n = \frac{Rab}{Ra + n^2b^2} \\
\ell_n &= \frac{R (n^2b^2 - a^2)}{Ra + n^2b^2}, \quad y_n = \frac{2Rab}{Ra + n^2b^2} = \frac{2nc}{n} \\
\frac{(x - b)^2}{A^2} + \frac{y}{B^2} &= 1, \quad A = \frac{a + R}{2}, \quad B = \sqrt{Ra}
\end{align*}
\]

(1)
1.1 The Apollonius circle and the orthocenter of a triangle

Here, we connect \( MN \) and \( c_n \) to construct a triangle (in figure 1), and then we analyze this triangle by the following theorem.

Theorem 1.1.1: Every point not on the circumscribed circle to a triangle has a single definite isogonal conjugate. The relation between the two is symmetrical.

Theorem 1.1.2: Given two non-intersecting circles which possess the property that a ring of \( n \) circles may be constructed all tangent to them and tangent to one another making \( m \) complete circuits, and if two circles of the ring touch the original ones at points on one circle orthogonal to these two, then the original circles are members of a ring of \( n \), circles, making \( m \), complete circuits, all tangent to the first ring, where:

\[
\frac{m}{n} + \frac{m_1}{n_1} = \frac{1}{2}
\]

Figure 1.1.1 (see [3]):

Let us consider the pedal circles of two isogonally conjugate points, then the circumscribed circles of their pedal triangles show:

\[
\angle A_3 A_1 P = \alpha_i, \quad \angle A_2 A_1 P = \alpha_i
\]

\[
\angle A_3 A_2 P = \alpha_j, \quad \angle A_1 A_2 P = \alpha_j
\]

\[
\angle A_1 A_3 P = \alpha_k, \quad \angle A_2 A_3 P = \alpha_k
\]

While,

\[
\frac{\sin \alpha_i}{\sin \alpha_j} \cdot \frac{\sin \alpha_j}{\sin \alpha_k} = -1
\]
Equivalently,

$$\frac{\rightarrow A_j P a_i}{\rightarrow A_j P a_k} = \cos \alpha_j = \frac{\rightarrow A_j P a_i}{\rightarrow A_j P a_k}$$

(3)

Next, we introduce the following geometric model to study figure 1.1.1:

Figure 1.1.2:

Let us continue to study the relations in figure 1.1.2, and we assume that:

$$\rightarrow X A = p, \rightarrow X B = q, \rightarrow X C = r$$

$$\rightarrow Z A = \frac{1}{2}(p - q - r), \rightarrow Z B = \frac{1}{2}(q - p - r), \rightarrow Z C = \frac{1}{2}(r - p - q)$$

(4)

Which lead that:

$$p \cdot q = q \cdot r = r \cdot p = m$$

$$\rightarrow Z A^2 = Z B^2 = Z C^2 = \frac{1}{4}(p^2 + q^2 + r^2) - \frac{m^2}{2} = t^2$$

(5)

It follows from observation that:

$$\rightarrow Z B - Z A = A_1 A_2 = q - p$$

$$\rightarrow Z C - Z B = A_2 A_3 = r - q$$

$$\rightarrow Z A - Z C = A_3 A_1 = p - r$$

(6)

Substitute the results above to theorem 1.1.1,

$$\cos \alpha_j = \frac{(p - r)(p - q - r)}{2t(\frac{Ra}{2} + b)}$$

$$\cos \alpha_j = \frac{(p - q)(p - q - r)}{2t(\frac{Ra}{2} + a)}$$

$$\cos \alpha_j = \frac{(q - r)(r - p - q)}{2t(a + b)}$$

$$\cos \alpha_j = \frac{(p - q)(p - q - r)}{2t(\frac{Ra}{2} + a)}$$
\[
\cos \alpha_k = \frac{(r - p)(p - q - r)}{2t(Rab + b)} \quad \text{and} \quad \cos \alpha_k = \frac{(r - q)(p - r - q)}{2t(a + b)}
\]

(7)

Since,

\[
\cos \alpha_i' = \sin \alpha_j'
\]

(8)

Also note that the distance from the orthocenter to the vertices keep unchanged, and the length is unchanged too.

Therefore, by rotation property, we obtain:

\[
\cos \alpha_i' \cos \alpha_j' \cos \alpha_k' = \sin \alpha_i' \sin \alpha_j' \sin \alpha_k' 
\]

(9)

In addition,

\[
\sin \alpha_i = \frac{PPa_2}{t} = \sqrt{t^2 - APa_2^2} \quad \text{for} \quad \alpha_i' \quad \text{and} \quad \alpha_j', \quad \alpha_k'
\]

(10)

\[
\rightarrow \rightarrow \rightarrow A Pa_2 = \frac{\cos \alpha_i}{A Pa_3} \quad \text{and} \quad \cos \alpha_i 
\]

(11)

The following results are obtained from (11) and figure 1,

\[
\rightarrow \rightarrow 2 \quad \frac{\cos \alpha_i}{A Pa_2} = \left(\frac{\cos \alpha_i}{A Pa_3}\right)^2 = \left(\frac{\cos \alpha_i}{Ra + n^2 b^2}\right)^2 \left(\frac{Rab}{Ra + n^2 b^2} + b^2 - h^2\right)
\]

(12)

\[
h = 2\Delta / (Ra + n^2 b^2) = 2\Delta / (c_n + a)
\]

(13)

From (13), we now try to handle the height of the triangle, Figure 1.1.3 (see [3]):

Immediately calculations show that:

\[
h = 2\Delta / (c_n + a) = \frac{2nc_n(a + b)}{c_n + a}
\]

(14)
Then we make an assumption to continue our study, such that:

\[ r < q < p \]  \hspace{1cm} (15)

From (15), it is clear that:

\[ \frac{\cos \alpha_i}{\cos \alpha_j} = \left( \frac{p-r}{p-q} \right) \left( \frac{c_n + a}{c_n + b} \right)^2 \leq \left( \frac{c_n + a}{c_n + b} \right)^2 \]  \hspace{1cm} (16)

\[ \frac{\cos \alpha_j}{\cos \alpha_k} = \left( \frac{q-r}{q-p} \right) \left( \frac{c_n + a}{a+b} \right)^2 \leq \left( \frac{c_n + a}{a+b} \right)^2 \]  \hspace{1cm} (17)

\[ \frac{\cos \alpha_k}{\cos \alpha_i} = \left( \frac{r-p}{r-q} \right) \left( \frac{c_n + b}{c_n + b} \right)^2 \leq \left( \frac{c_n + b}{c_n + b} \right)^2 \]  \hspace{1cm} (18)

Using (10) again, to get:

\[ \sin \alpha_i = \frac{\sqrt{2 - APa} \left( \frac{c_n + a}{c_n + b} \right)^2}{t} \leq \frac{\sqrt{t - \left( \frac{c_n + a}{c_n + b} \right)^2 \left( \frac{c_n + b}{c_n + b} \right)^2 - h^2}}{t} \]  \hspace{1cm} (19)

\[ \sin \alpha_j = \frac{\sqrt{2 - APa} \left( \frac{a+b}{c_n + b} \right)^2}{t} \leq \frac{\sqrt{t - \left( \frac{a+b}{c_n + b} \right)^2 \left( \frac{c_n + b}{c_n + b} \right)^2 - h^2}}{t} \]  \hspace{1cm} (20)

\[ \sin \alpha_k = \frac{\sqrt{2 - APa} \left( \frac{c_n + b}{c_n + b} \right)^2}{t} \leq \frac{\sqrt{t - \left( \frac{c_n + b}{c_n + b} \right)^2 \left( \frac{c_n + b}{c_n + b} \right)^2 - h^2}}{t} \]  \hspace{1cm} (21)

It follows from theorem 1.1.1,

\[ \frac{\sin \alpha_i}{\sin \alpha_j} \frac{\sin \alpha_j}{\sin \alpha_k} \frac{\sin \alpha_k}{\sin \alpha_i} = -1 \]  \hspace{1cm} (22)

Now it is enough to finish our proof as follow:

\[ \frac{(p-q)(p-q-r)}{2(c_n+a)} \left( \frac{p-q}{2(c_n+a)} \right) \left( \frac{r-q}{2(a+b)} \right) \]

\[ \leq \frac{2 - \left( \frac{c_n + a}{c_n + b} \right)^2 \left( \frac{c_n + b}{c_n + b} \right)^2 - h^2}{t} \cdot \frac{2 - \left( \frac{a+b}{c_n + b} \right)^2 \left( \frac{c_n + b}{c_n + b} \right)^2 - h^2}{t} \cdot \frac{2 - \left( \frac{c_n + b}{c_n + b} \right)^2 \left( \frac{c_n + b}{c_n + b} \right)^2 - h^2}{t} \]  \hspace{1cm} (23)

At last, we use the sidelines of a triangle to substitute the vectors in (23),

\[ \frac{a^4 b^2}{8 ac^2} = \frac{(p-q)(p-q-r)}{2(c_n+a)} \left( \frac{(p-q)}{2(c_n+a)} \right) \left( \frac{r-q}{2(a+b)} \right) \]
\[
\begin{align*}
&\frac{2}{r} - \left(\frac{c + a}{c + b}\right)^2 \left(\frac{c + a}{c + b}\right)^2 , \\
&\frac{2}{r} - \left(\frac{c + a}{a + b}\right)^2 \left(\frac{c + a}{a + b}\right)^2 , \\
&\frac{2}{r} - \left(\frac{c + a}{c + b}\right)^2 \left(\frac{c + a}{c + b}\right)^2 , \\
&\frac{2}{r} - \left(\frac{c + a}{c + b}\right)^2 \left(\frac{c + a}{c + b}\right)^2
\end{align*}
\]

\[
\frac{2}{r} = \left(\frac{c + a}{b}\right)^2 (b^2 - h^2) . \quad \frac{2}{r} = \left(\frac{c + a}{a}\right)^2 (b^2 - h^2) . \quad \frac{2}{r} = \left(\frac{c + a}{c + b}\right)^2 (b^2 - h^2)
\]

1.2 The radius of the circumcircle circle and the incircle

Figure 1.2.1 (see [2]):

Theorem 1.2: If two circles be related that a triangle inscribed in the one is circumscribed to the other, then the former is the inverse in the latter of the nine-point circle of the triangle whose vertices are the points of contact.

It follows from figure 1.2.1 that,

\[
\frac{1}{r} = \frac{2}{r} + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)
\]

(1)

Next, we introduce the following geometric model.

Figure 1.2.2:

Here, we let \( BP \perp AP \) in figure 1.2.2. Then,

\[
\begin{align*}
\overrightarrow{BP} \cdot \overrightarrow{CQ} &= \overrightarrow{AP} \cdot \overrightarrow{CB} - r^2 + \overrightarrow{AB} \cdot \overrightarrow{AC} \\
\overrightarrow{AB} \cdot \overrightarrow{AC} &= -(R - a + x_n)(R - b - x_n) + y_n^2
\end{align*}
\]

(2) (3)

By formula (1) (2) (3) in figure 1, we have:
\[ \overrightarrow{AP} \cdot \overrightarrow{CB} = r(a+b)(\sin B \sqrt{(a+c)^2 - r^2} + \cos B \frac{r}{a+c}) \] (4)

\[ \overrightarrow{BP} \cdot \overrightarrow{CQ} = BP \cdot (-\overrightarrow{AP} - \overrightarrow{AC}) = \]
\[ - (\sqrt{(a+c)^2 - r^2} \cdot \sin A \frac{r}{a+c} + \cos A \sqrt{(a+c)^2 - r^2}) \] (5)

Nextly, we study the property of the incircle in theorem 1.2 (see [3]):

Figure 1.2.3:

The nine point circle is circumscribed to a similar triangle, so that its radius is one half of the circumscribed circle, respectively:

\[ \overrightarrow{O'B} = \frac{R^2}{r-d}, \quad \overrightarrow{O'C} = \frac{R^2}{r+d} \] (6)

\[ C'B = R = \frac{R^2}{r+d} + \frac{R^2}{r-d} = \frac{R^2}{O'B} + \frac{R^2}{O'C} \] (7)

Recall the property of the Apollonius problem,

\[ \overrightarrow{O'B} = (2Rr + d^2)(r+d) / R^2 \]

\[ \overrightarrow{O'C} = (2Rr + d^2)(r-d) / R^2 \] (8)

Where,

\[ \frac{1}{R} = \frac{1}{r+d} + \frac{1}{r-d}, \quad r^2 = 2Rr + d^2, \quad d = \sqrt{r(r-2R)} \] (9)

Now, it is clearly that:

\[ \overrightarrow{O'B} + \overrightarrow{O'C} \geq \max\{a+b, a+c, b+c\} \] (10)

Which lead that:

\[ 2r(2Rr + r(r-2R)) / R^2 \geq a+c \] (11)

Therefore, we can relate figure 1.2.2 and figure 1.2.3 to get:
\[
\sqrt{(a+c_n)^2 - r^2} \leq r \sqrt{\frac{4}{R^4}} (2Rr + r(r-2R))^2 - 1 = r \sqrt{l^2 - 1} < \frac{2r^2}{R^2} \tag{12}
\]

Now, we can substitute (12) into (4) and (5),

\[-(R-a+x_n)(R-b-y_n) + \frac{2}{n} - r^2 = BP \cdot CQ - AP \cdot CB \]

\[-(a+c_n)^2 - r + (a+c_n)^2 - r - (b+c_n)(\sin A - r + \cos A \sqrt{(a+c_n)^2 - r^2}) \tag{13}
\]

Hence, our proof is completed:

\[-(a+b)(\sin B \cos B + \cos B \frac{r}{a+c_n}) \]

\[\geq -(r^2 - 1 + r \sqrt{l^2 - 1} - (b + c_n)(\sin A \cos A + \cos A \sqrt{l^2 - 1})) \tag{14}\]

Consequently,

\[-r(a+b)(\sin B \cos B + \cos B \frac{1}{a+c_n}) \]

\[\geq -(2r^2 l^2 - 1 + 2r^2 \frac{b(\sin A + \cos A 2r^2)}{R^2} - ra(\sin B \cos B + \cos B \frac{r}{c}) \tag{15}\]

In which, we use the sidelines of the triangle to replace the radius of the Apollonius circle (see figure 1).

### 1.3 The Euler-Gergonne-Soddy triangle of a triangle

Figure 1.3.1 (see [2]):

![Figure 1.3.1](image)

Theorem 1.3: The orthocenter and the centroid are centers of similitude for the nine point circle and circumscribed circles, the ratios of similitude being 1:2 and -1:2, respectively.

There is another circle much less well known than the nine point circle but possessing a number of analogous properties. Let the inscribed circle touch a sideline while the escribed circle corresponding to this side.
Figure 1.3.2 (see [2]):

Please note that here we use the trilinear coordinate.

Figure 1.3.1 and 1.3.2 show:

\[ DX = dIX, \quad EY = eIY, \quad FZ = fIZ \]

\[ X:Y:Z = (1, 1+2e, 1+2f):(1+2d, 1, 1+2f):(1+2d, 1, 1+2e, 1) \]

\[ P:Q:R = (1+2d, 1+e, 1+f):(1+d, 1+2e, 1+f):(1+d, 1+e, 1+2f) \]  \(1\)

Nextly, we use the Ptolemy theorem to the quadrilateral $SQRX$ (in figure 1.3.2):

\[ IX \cdot QR \leq \sigma (XQ + XR) \]  \(3\)

Where,

\[ XQ = c \cos B - \frac{2s}{a} \left( \frac{a}{2} (s-b) \right) \]

\[ XR = a \cos C - \frac{2s}{b} \left( \frac{b}{2} (s-c) \right) \]  \(4\)
We now compute (3) and (4),

\[ QR / d \leq \sigma \left( (c \cos B - \frac{2s}{a} \left( \frac{a}{2} \right) - (s-b)) + (a \cos C - \frac{2s}{b} \left( \frac{b}{2} \right) - (s-c)) \right) \]  

(5)

Which lead that :

\[ QR^2 = e^2 + (a - \sqrt{b^2 - (1 + e)^2} - \sqrt{c^2 - (1 + 2e)^2})^2 \]  

(6)

The property of soddy line and soddy circle gives:

\[ \frac{1}{\sigma} = \frac{ab - R_c^2}{R_a R_b R_c} = \frac{bc - R_a^2}{R_a R_b R_c} = \frac{ac - R_b^2}{R_a R_b R_c} \]  

(7)

Substitute (6) (7) into (5),

\[ QR^2 = e^2 + (a - \sqrt{ac - \left( \frac{1}{\sigma} \right) R_a R_b R_c - (1 + e)^2} - \sqrt{ab - \left( \frac{1}{\sigma} \right) R_a R_b R_c - (1 + 2e)^2})^2 \]

\[ \leq [d \sigma \left( (c \cos B - \frac{2s}{a} \left( \frac{a}{2} \right) - (s-b)) + (a \cos C - \frac{2s}{b} \left( \frac{b}{2} \right) - (s-c)) \right)]^2 \]

\[ = [d \sigma \left( (\cos B + a \cos C - \left( \frac{s(b-c)}{a} + \frac{s(c-a)}{b} \right) \right)]^2 \]

(8)

Now we can make an assumption to the ratios in (1),

\[ ad = \frac{be + cf}{2} = be = cf \]  

(9)

Consequently,

\[ e^2 + (a - \sqrt{ac - \left( \frac{1}{\sigma} \right) R_a R_b R_c - (1 + e)^2} - \sqrt{ab - \left( \frac{1}{\sigma} \right) R_a R_b R_c - (1 + 2e)^2})^2 \]

\[ = e^2 + a^2 + ac - 2 \left( \frac{1}{\sigma} \right) R_a R_b R_c - (1 + e)^2 - \sqrt{ab - \left( \frac{1}{\sigma} \right) R_a R_b R_c - (1 + 2e)^2} + ab - (1 + 2e)^2 - \]

\[ 2a \left( \sqrt{ac - \left( \frac{1}{\sigma} \right) R_a R_b R_c - (1 + e)^2} + \sqrt{ab - \left( \frac{1}{\sigma} \right) R_a R_b R_c - (1 + 2e)^2} \right) + \]

\[ 2 \sqrt{ac - \left( \frac{1}{\sigma} \right) R_a R_b R_c - (1 + e)^2} + \sqrt{ab - \left( \frac{1}{\sigma} \right) R_a R_b R_c - (1 + 2e)^2} + \]

\[ \geq e^2 + a^2 - 2a \left( \sqrt{ac - \left( \frac{1}{\sigma} \right) R_a R_b R_c - (1 + e)^2} + \sqrt{ab - \left( \frac{1}{\sigma} \right) R_a R_b R_c - (1 + 2e)^2} \right) + \]

\[ 4 \sqrt{ac - \left( \frac{1}{\sigma} \right) R_a R_b R_c - (1 + e)^2} + \sqrt{ab - (1 + 2e)^2} + 4 \left( \sqrt{ac - (1 + e)^2} + \sqrt{ab - (1 + 2e)^2} \right) \]

\[ \geq e^2 + a^2 - 2a \left( \sqrt{ac - (1 + e)^2} + \sqrt{ab - (1 + 2e)^2} \right) + 4 \left( \sqrt{ac - (1 + e)^2} + \sqrt{ab - (1 + 2e)^2} \right)^2 \]  

(10)
If we select that:

\[\sqrt{ac - \left(\frac{1}{\sigma} - \frac{2}{r}\right)}R_a R_b R_c - (1+e)^2 = \sqrt{ab - \left(\frac{1}{\sigma} - \frac{2}{r}\right)}R_a R_b R_c - (1+2e)^2\]  \(11\)

\[C \frac{9a^2}{b^2 r^2} \leq (c \cos B + a \cos C - \left(\frac{s(b-c)}{a} + \frac{s(c-a)}{b}\right))^2\]  \(12\)

In which,

\[\sqrt{ac - \left(\frac{1}{\sigma} - \frac{2}{r}\right)}R_a R_b R_c - (1+e)^2 + \sqrt{ab - \left(\frac{1}{\sigma} - \frac{2}{r}\right)}R_a R_b R_c - (1+2e)^2) \leq a\]  \(13\)

Basing on the fact that the intouch triangles are similar, we have:

\[C \frac{9a^2}{b^2 r^2} \leq\]

\[C \frac{9a^2}{b^2 r^2} (e^2 + a^2 - 2a(\sqrt{ac - (1+e)^2} + \sqrt{ab - (1+2e)^2}) + 4 \sqrt{ac - (1+e)^2} \sqrt{ab - (1+2e)^2})\]

\[\leq (c \cos B + a \cos C - \left(\frac{s(b-c)}{a} + \frac{s(c-a)}{b}\right))^2\]  \(14\)

When \(n \to \infty\), the equality holds.

### 1.4 The Brocard Angle and the Area of a Triangle

Figure 1.4.1 (see [3]):

![Image](image)

Theorem 1.4.1: The two Brocard points are isogonal conjugate of one another.

Theorem 1.4.2: The triangle into which the given triangle is divided by connecting its vertices with the positive Brocard point are equal to those obtained by connecting them with the negative one.

Theorem 1.4.1 gives:

\[\angle \Omega A_1 A_2 = \angle \Omega A_2 A_3 = \angle \Omega A_3 A_1 = w, \quad \angle \Omega A_2 A_1 = \angle \Omega A_1 A_3 = \angle \Omega A_3 A_2 = w\]  \(1\)

\[\Omega A_2 : a_3 = \sin w : \sin \angle A_2, \quad \Omega A_2 : a_1 = \sin (\angle A_3 - w) : \sin \angle A_3\]  \(2\)
\[
\sin \angle A_3 / \sin \angle A_1 = \sin(\angle A_3 - w) \sin \angle A_2 / \sin w \sin \angle A_3
\] (3)

\[
\text{ctnw} = \sum \text{ctn} \angle A_i
\] (4)

Figure 1.4.2 (see [3]):

Theorem 1.4.2 imply that:

\[
csc^2 w = \csc^2 \angle A_1 + \csc^2 \angle A_2 + \csc^2 \angle A_3 + 2 \sum \text{ctn} \angle A_i \text{ctn} \angle A_j
\]

\[
\frac{\text{ctn} \angle A_3 = \frac{1 - \text{ctn} \angle A_1 \text{ctn} \angle A_2}{\text{ctn} \angle A_1 + \text{ctn} \angle A_2}
\] (5)

Let us consider the property of Brocard angle:

\[
csc^2 w = \sum \csc^2 \angle A_i
\]

\[
\sin^2 w = \frac{\Pi \sin \angle A_i}{\sum \sin^2 \angle A_i \sin^2 \angle A_j} = \frac{4\Delta^2}{\sum a_i^2 a_j^2}
\] (6)

Then, the area can be written as:

\[
16 \Delta^2 = 16 \Pi (s - a_i) = 2 \sum a_i^2 a_j^2 = 4 \sum a_i^4
\] (7)

\[
\cos^2 w = \frac{(\sum a_i^2)^2}{\sum a_i^2 a_j^2}, \quad \text{ctn}^2 w = \frac{\sum \sin^2 \angle A_i}{2\Pi \sin \angle A_i} = \frac{\sum a_i^2}{4\Delta}
\] (8)

Figure 1.4.3:
Here, we use the geometric model in figure 1.4.3 to handle theorem 1.4.1 and 1.4.2. Figure 1.4.3 imply:

$$
\sum |AA_i|^2 = \sum (OA_i^2 - 2 OA_i \cdot OA + OA^2) \tag{9}
$$

Recall the property of the Brocard angle:

Figure 1.4.4:

Here we use \(a_i\) to represent \(OA_i\).

The area of the triangle is:

$$
S_{\Delta A_1 A_2 A_3} = \frac{1}{2} \left| \overrightarrow{A_1 A_2} \times \overrightarrow{A_1 A_3} \right| = \frac{1}{2} (a_1 \times a_2 + a_3 \times a_2 + a_1 \times a_3) \tag{10}
$$

Now we introduce the Brocard angle to (10).

$$
S_{\Delta A_1 A_2 A_3} = \frac{1}{2} (a_1 c + a_2 a + a_3 b) \frac{\Delta}{\sum a_i^2} = \frac{1}{2} (a_1 c + a_2 a + a_3 b) \frac{\Delta \cos w}{\sum a_i^2} \tag{11}
$$

We compute the vectors as follow:

$$
\sum a_i^2 = \sum |AA_i|^2 = (a_3 - a_2)^2 + (a_3 - a_2)^2 = \sum (OA_i^2 - 2 OA_i \cdot OA + OA^2) + (a_3 - a_2)^2
$$

$$
= (a_3 - a_2)^2 + OA_2^2 - 2 OA_1 \cdot OA_1 + OA_1^2 + OA_3^2 - 2 OA_3 \cdot OA_1 + OA_1^2
$$

$$
= -2a_3 a_2 \cos B + 2a_2^2 - 2a_1 a_2 \cos A + 2a_1^2 + 2a_3^2 - 2a_1 a_3 \cos C \tag{12}
$$

It can be seen easily that:

$$
\frac{1}{2} (a_1 \times a_2 + a_3 \times a_2 + a_1 \times a_3) = \frac{1}{2} (a_1 a_2 \sin A + a_3 a_2 \sin B + a_1 a_3 \sin C) \tag{13}
$$
Recall the proof of theorem 1.4.2,

$$\csc^2 w = \csc^2 \angle A_1 + \csc^2 \angle A_2 + \csc^2 \angle A_3 + 2 \sum \csc \angle A_i \csc \angle A_j$$  \hspace{1cm} (14)

$$\csc^2 \angle A_1 + \csc^2 \angle A_2 + \csc^2 \angle A_3 + 2 \sum \csc \angle A_i \csc \angle A_j \geq 4 + 2 \sum \csc \angle A_i \csc \angle A_j = 6$$  \hspace{1cm} (15)

Immediately calculations show:

$$1 + \frac{\sum a_i^2}{4 \Delta} =$$

$$\csc \angle A_1 + \csc \angle A_2 + \csc \angle A_3 + 2 \sum \csc \angle A_i \csc \angle A_j \geq 4 + 2 \sum \csc \angle A_i \csc \angle A_j = 6$$  \hspace{1cm} (16)

Which imply:

$$(a_1 c + a_2 a + a_3 b) \frac{\Delta \cos w}{\sum a_i^2} \leq \frac{1}{20} (a_1 c + a_2 a + a_3 b) \cos w$$  \hspace{1cm} (17)

$$(a_1 a_2 \sin A + a_3 a_2 \sin B + a_1 a_3 \sin C) \leq \frac{1}{10} (a_1 c + a_2 a + a_3 b) \cos w$$  \hspace{1cm} (18)

Substitute the area into (17) and (18), we have:

$$\cos w = \frac{(\sum a_i^2)}{\sqrt{\sum a_i^2 a_j}} = \sqrt{2(\sum a_i^2) / \sqrt{16 \Delta^2 + \sum a_i^4}} \geq \sqrt{2(\sum a_i^2) / \sqrt{(\sum a_i^2)^2 / 25 + \sum a_i^4}} = 5 \sqrt{2\Delta}$$  \hspace{1cm} (19)

At last, we use the extremes to study the inequality below:

$$(\sum a_i^2)^2 = \sum a_i^4 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2 \leq \sum a_i^4 + C$$  \hspace{1cm} (20)

We consider the following case:

$$\sqrt{2x} / \sqrt{x^2 / 25 + x^2 / 25 - C} = \sqrt{2x} / \sqrt{26x^2 / 25 - C}$$  \hspace{1cm} (21)

When $C = 0$ or $C = x^2$, we can get the extreme value, and our last conclusion follows:

$$C = x^2 = 2 \sum a_i^2 a_j = 16 \Delta^2 + \sum a_i^4$$  \hspace{1cm} (22)

$$x = -2a^2a_2 \cos B + 2a^2_1 \cos A + 2a^2_3 \cos C \geq \frac{20 \Delta}{\sqrt{26}}$$  \hspace{1cm} (23)

In this section we use the vector method and the Ptolemy theorem to study the Apollonius problem, and we obtain four results:

1.1 by applying the property of the Apollonius circle and the orthocenter, we get a formula about the sideline of a triangle.

1.2 using the vector method to get a relation between the circumradius and inradius.

1.3 using the Ptolemy theorem to study the Apollonius circle and the ratios for its soddy circles, and we use the squeeze method to relate the ratios and the triangle.

1.4 use the vector method and the extreme method to study the Brocard angle and area.
2. The Ptolemy theorem in Euclidian spherical geometry and matrix analysis

Theorem 2.1: If \( p_1, p_2, p_3, p_4 \) are four points, then the determinant:

\[
\gamma(p_1, p_2, p_3, p_4) = \left| \sin^2 \left( \frac{\rho_i}{2} \right) \right| \leq 0
\]

And \( \gamma(p_1, p_2, p_3, p_4) = 0 \) if the four points lie on a great circle.

Theorem 2.2: If \( p_1, p_2, p_3, p_4 \) are a quadruple of points, then the three products of the sines of half the “opposite” distance satisfy the triangle inequality.

Theorem 2.3: Let \( p_n, p_{n+1}, \ldots, p_{n+2} \) be \( n+2 \) non-cospherical points of \( E_n \), with no \( (n+1) \)-tuple in an \( E_{n-1} \), then the sum of the reciprocals of the power of each point with respect to the sphere circumscribing the remaining \( n+1 \) points is zero.

Here, we study the following type matrix (see [4][5][6]):

\[
D(p_1, p_2, p_3, p_4) = \begin{vmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & p_1 p_2^2 & p_1 p_3^2 & p_1 p_4^2 \\
1 & 1 & p_1 p_2^2 + 1 & p_1 p_3^2 + 1 & p_1 p_4^2 + 1 \\
1 & 1 & p_1 p_3^2 + 1 & p_2 p_3^2 + 1 & p_2 p_4^2 + 1 \\
1 & 1 & p_1 p_4^2 + 1 & p_2 p_4^2 + 1 & p_3 p_4^2 + 1
\end{vmatrix}
\]

\[
D(p_1, p_2, p_3, p_4) = \begin{vmatrix}
p_1 p_2^2 + 1 & p_1 p_3^2 + 1 & p_1 p_4^2 + 1 \\
p_2 p_3^2 + 1 & p_2 p_4^2 + 1 \\
p_2 p_4^2 + 1 & p_3 p_4^2 + 1 \\
p_3 p_4^2 + 1 & 1 \\
p_4 p_4^2 + 1 & p_3 p_4^2 + 1 & 1
\end{vmatrix}
\]

\[
D(p_1, p_2, p_3, p_4) = \begin{vmatrix}
p_1 p_2^2 + 1 & p_1 p_3^2 + 1 & p_1 p_4^2 + 1 \\
p_2 p_3^2 + 1 & p_2 p_4^2 + 1 \\
p_3 p_4^2 + 1 & 1 \\
p_4 p_4^2 + 1 & p_3 p_4^2 + 1 & 1
\end{vmatrix}
\]

\[
\ldots
\]

\[
(1)
\]
Then, we decompose the matrix (1) into four smaller matrices:

\[
\begin{pmatrix}
1 & p_1 p_2 + 1 & p_1 p_3 + 1 & p_1 p_4 + 1 \\
1 & 1 & p_2 p_3 + 1 & p_2 p_4 + 1 \\
p_2 p_3 + 1 & 1 & p_3 p_4 + 1 \\
p_2 p_4 + 1 & p_3 p_4 + 1 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & p_1 p_2 + 1 & p_1 p_3 + 1 & p_1 p_4 + 1 \\
p_1 p_2 + 1 & 1 & p_2 p_3 + 1 & p_2 p_4 + 1 \\
p_1 p_3 + 1 & p_2 p_3 + 1 & 1 & p_3 p_4 + 1 \\
p_1 p_4 + 1 & p_2 p_4 + 1 & p_3 p_4 + 1 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & p_1 p_2 + 1 & p_1 p_3 + 1 & p_1 p_4 + 1 \\
p_1 p_2 + 1 & 1 & p_2 p_3 + 1 & p_2 p_4 + 1 \\
p_1 p_3 + 1 & p_2 p_3 + 1 & 1 & p_3 p_4 + 1 \\
p_1 p_4 + 1 & p_2 p_4 + 1 & p_3 p_4 + 1 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & a + 1 & b + 1 & c + 1 \\
a + 1 & a + 1 & b + 1 & c + 1 \\
a & a + 1 & b + 1 & c + 1 \\
a & a + 1 & b + 1 & c + 1
\end{pmatrix}
\]

(2)

It is easy to verify that:

\[
\begin{pmatrix}
0 & a + 1 & b + 1 & c + 1 \\
a - 1 & 1 & d + 1 & e + 1 \\
b - 1 & d + 1 & f + 1 \\
c - 1 & e + 1 & f + 1 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & a + 1 & b + 1 & c + 1 \\
a & a + 1 & b + 1 & c + 1 \\
0 & 1 & d + 1 & e + 1 \\
e & e + 1 & f + 1 & 1
\end{pmatrix}
\]

(3)

By observing the matrix (3), we find:

\[
\begin{pmatrix}
a & a + 1 & b + 1 & c + 1 \\
0 & 1 & d + 1 & e + 1 \\
d & d + 1 & f + 1 \\
e & e + 1 & f + 1 & 1
\end{pmatrix}
= \begin{pmatrix}
a & a + 1 & b + 1 & c + 1 \\
0 & 1 & d + 1 & e + 1 \\
d & d + 1 & f + 1 \\
e & e + 1 & f + 1 & 1
\end{pmatrix}
\]

(4)

Which lead that:

\[
\begin{pmatrix}
a & a + 1 & b + 1 & c + 1 \\
1 & 1 & d + 1 & e + 1 \\
d + 1 & d + 1 & f + 1 \\
e + 1 & e + 1 & f + 1 & 1
\end{pmatrix}
= 0
\]

(5)

Similarly, we use the same method again to the other three matrices:
Now, it is sufficient to compute:

\[
\begin{align*}
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|}
| a+1 & a+1 & d+1 & e+1 & b+1 & b+1 & d+1 & f+1 & c+1 & c+1 & e+1 & f+1 | \\
| 1 & 1 & b+1 & c+1 & = 0 , & 1 & 1 & a+1 & c+1 & = 0 , & 1 & 1 & a+1 & b+1 & = 0 \\
| b+1 & b+1 & 1 & f+1 & a+1 & a+1 & 1 & e+1 & a+1 & a+1 & 1 & d+1 \\
| c+1 & c+1 & f+1 & 1 & c+1 & c+1 & e+1 & 1 & b+1 & b+1 & d+1 & 1 | \\
\end{array}
\end{align*}
\]

We can see it that, when \( a = b = c \), the last three terms can be expressed as:

\[ a(d^2 - (e - d)(e - f) + (f - e)(f - d) - de + (d - e)(d - f) - ef ) \]  

Therefore, if \( a = b = c = d = e \), we have:

\[ \Delta = 2a(f^2 - 2af) = 0 \]  

Theorem 2.2 gives:

\[ \gamma(p,q,r,s) = A \cdot B \cdot C \cdot D \leq 0 \]  

In which,

\[ A = -[\sin \alpha \sin \delta + \sin \varepsilon \sin \theta + \sin \alpha \sin \beta] \]  
\[ B = [\sin \alpha \sin \delta + \sin \varepsilon \sin \theta - \sin \alpha \sin \beta] \]  
\[ C = [\sin \alpha \sin \delta - \sin \varepsilon \sin \theta + \sin \alpha \sin \beta] \]  
\[ D = [-\sin \alpha \sin \delta + \sin \varepsilon \sin \theta + \sin \alpha \sin \beta] \]
Now we substitute (8) into (9) to obtain:

\[ A \cdot B \cdot C \cdot D = -\sin \left( \frac{\sqrt{a}}{2r} \right) (2 \sin \frac{\sqrt{a}}{2r} + \sin \frac{\sqrt{f}}{2r} - 2 \sin \frac{\sqrt{a} + \sqrt{f}}{2r} + 2 \sin \frac{\sqrt{a} - \sqrt{f}}{2r} \leq 0 \] (10)

Also note that:

\[ 2 \sin \frac{\sqrt{a}}{2r} - \sin \frac{\sqrt{f}}{2r} = 2 \sin \frac{\sqrt{a}}{2r} - \sin \frac{\sqrt{2a}}{2r} \geq 0 \] (11)

Then, we try to analyze the equation below:

\[ (2 \sin x - \sin \sqrt{2x})' = 2 \cos x - \sqrt{2} \cos \sqrt{2x} \] (12)

We discover that:

When, \( \frac{\sqrt{a}}{2r} \in \left( 2k\pi + \frac{\pi}{6}, 2k\pi + \frac{5\pi}{6} \right) \)

We have:

\[ 2 \sin \frac{\sqrt{a}}{2r} - \sin \frac{\sqrt{2a}}{2r} \geq 0 \] (13)

So the following assumption holds:

\[ \frac{\sqrt{a}}{2(2k\pi + l_1)} \leq r \leq \frac{\sqrt{a}}{2(2k\pi + l_2)} \] (14)

And the range of the angle is:

\[ l \in \left( 2k\pi - \frac{\pi}{6}, 2k\pi + \frac{\pi}{6} \right) \cup \left( 2k\pi + \frac{5\pi}{6}, 2k\pi + \frac{7\pi}{6} \right) \] (15)

In which,

\[ 2\sum_{l=1}^{n} \frac{(-1)^n 2^n + 1}{(2n+1)!} = \sum_{l=1}^{n} \frac{(-1)^n (\sqrt{2})^{2n+1}}{(2n+1)!} \] (16)

We unfold the series in (16) to get:

\[ (2 - \sqrt{2})x + ((\sqrt{2})^3 - 2)x^3 / 3! + ((\sqrt{2})^5 - 2)x^5 / 7! + \ldots \]

\[ = ((\sqrt{2})^5 - 2)x^5 / 5! + ((\sqrt{2})^9 - 2)x^9 / 9! + \ldots \] (17)

We also compute the series in (17),

\[ (2 - \sqrt{2})(2k\pi + \frac{5\pi}{6}) + ((\sqrt{2})^3 - 2)(2k\pi + \frac{5\pi}{6})^3 / 3! + ((\sqrt{2})^7 - 2)(2k\pi + \frac{5\pi}{6})^3 / 7! + \ldots \]

\[ \leq ((\sqrt{2})^5 - 2)(2k\pi + \frac{7\pi}{6})^5 / 5! + ((\sqrt{2})^9 - 2)(2k\pi + \frac{7\pi}{6})^9 / 9! + \ldots \] (18)

Respectively,

\[ (2 - \sqrt{2})(2k\pi + \frac{\pi}{6}) + ((\sqrt{2})^3 - 2)(2k\pi + \frac{\pi}{6})^3 / 3! + ((\sqrt{2})^7 - 2)(2k\pi + \frac{\pi}{6})^3 / 7! + \ldots \]
\[ \geq ((\sqrt{2})^5 - 2)(2k\pi - \frac{\pi}{6})^5 / 5! + ((\sqrt{2})^9 - 2)(2k\pi - \frac{\pi}{6})^9 / 9! + \ldots \]  

(19)

At last, we use theorem 2.3 to calculate,

\[ (op_{n+1}^2 - r^2) \over D(p_0 \ldots p_n) + 2(op_{n+1}^2 - r^2) \over r(p_0 \ldots p_n) + op_{n+1} C(p_0 \ldots p_{n+1}) = 0 \]  

(20)

Theorem 2.3 gives:

\[ C(p_0 \ldots p_n) = \gamma^2 (p_0 \ldots p_n p_{n+1}) / D(p_0 \ldots p_n) \]  

(21)

\[ D(op_{n+1}^2 - r^2) + 2 \gamma (op_{n+1}^2 - r^2) + \gamma^2 / D = 0 \]  

(22)

It is enough to assume that:

\[ op_{n+1}^2 - r^2 = -\frac{\gamma}{D} \]  

(23)

\[ \Sigma 1/(op_{n+1}^2 - r^2) = 0 \]  

(24)

Our last result can be found now,

\[ \frac{1}{op_1^2 - r^2} + \frac{1}{op_2^2 - r^2} + \ldots + \frac{1}{op_n^2 - r^2} + \frac{1}{op_{n+1}^2 - r^2} = 0 \]  

(25)

Recall (14) and (15),

\[ \frac{1}{op_1^2} + \frac{1}{op_2^2} + \ldots + \frac{1}{op_n^2} + \frac{1}{op_{n+1}^2} \leq 0 \]  

\[ \Rightarrow \frac{1}{op_1^2 - \frac{\alpha}{4l^2}} + \frac{1}{op_2^2 - \frac{\alpha}{4l^2}} + \ldots + \frac{1}{op_n^2 - \frac{\alpha}{4l^2}} + \frac{1}{op_{n+1}^2 - \frac{\alpha}{4l^2}} \leq 0 \]  

(26)

Therefore,

\[ \frac{n}{r^2 (1 - \frac{r^2}{t^2})} + \frac{1}{op_{n+1}^2 - \frac{r^2}{t^2}} \leq 0 \]  

(27)

\[ op_{n+1}^2 \geq \frac{r^2}{t^2} (1 + \frac{r^2}{n}) - \frac{r^2}{n} \]  

(28)

Here, we study the Euclid spherical geometry, and we use the Ptolemy theorem as a tool to study the property of the matrix, even to find a new relation about the points on the spherical. Our procedure is: we assume that there are four points on the spherical and there exist six lines which connect the four points, five of them are equal, then the left line on the spherical is \( \sqrt{2} \) times as long as the other five lines. By this property, we get a new formula for spherical geometry,
such that: \[ n_{op + 1} \geq r_1^2 \left( 1 + \frac{r_2^2}{n} \right) - \frac{r_2^2}{n} \]

3. **Ptolemy inequality and the Minkowskian geometry** (see [7][8][9])

Theorem 3.1: The generalized Ptolemy theorem (see formula (1) (2), below)

Theorem 3.2: Let \( P_1, P_2, \ldots, P_n \) denote points on a circle of unit radius. For each k-shuffle \( s = (s_1, s_2, \ldots, s_n) \) with its complement \( s^- = (t_1, t_2, \ldots, t_{n-k}) \), define:

\[ d(S) = \prod P_s P_t \]

Then, \( \sum \frac{1}{d(s)} \geq 1 \)

Figure 3:

Here, we handle the cyclic polygon with the Ptolemy theorem:

\[ c_1 |A_1 P|^n - c_2 |A_2 P|^n - \cdots + c_n |A_n P|^n \geq 0 \] (1)

\( n \) is odd, the equality hold if and only if \( P \) lie on the arc.

\( n \) is even \( (d = 0, 2, 4, \ldots, n-2) \), then:

\[ c_1 |A_1 P|^d - c_2 |A_2 P|^d + \cdots + c_n |A_n P|^d = 0 \] (2)

Here, we introduce a special polygon to study theorem 3.1

(1) Every angle of the polygon is equal

(2) The sidelines of the polygon are: \( 1^2, 2^2, \ldots, N^2 \)

(3) Then, \( \sum_{s=0}^{N} n_x e^{isa} = 0 \)

(4) Here we can view the complex number as mass, and we give an unit circle to place
these \( n \) mass \( 1^2, 2^2, \ldots, N^2 \) on the circle in some sequence, and we also make a regulation that the centroid lies on the center of the circle.

It follows from theorem 3.1 that:

\[
W(P) = w_1 A_1 P_1^d - w_2 A_2 P_2^d + w_2 A_2 P_2^d - \ldots (-1)^{n-1} w_n A_n P_n^d
\]  

(3)

When \( P \) is on the arc, then

\[
W(P) = 0
\]  

(4)

Since \( n \) is an even number, we can make two points a pair, such that:

\[
(1^2, 2^2), (3^2, 4^2), \ldots, ((n-1)^2, n^2)
\]  

(5)

Then, we decompose \( N/2 = p_1 a_1 p_2 a_2 \ldots p_n a_n \) (\( p \) is a prime number)

First suppose that:

\[
N/2l = p \cdot q
\]  

(6)

Where,

\[
p, q \text{ are primes, } \gamma = 2\pi/p, \beta = 2\pi/q
\]  

(7)

Then, the mass of every pair is:

\[
\sum_{k=1}^{l} (4k-1), \sum_{k=l+1}^{2l} (4k-1), \sum_{k=2l+1}^{3l} (4k-1), \ldots
\]  

(8)

The result after we divide them into groups is (every group has \( p \) points)

\[
\begin{align*}
&\sum_{k=1}^{pl} (4k-1) + \sum_{k=1}^{pl} (4k-1) e^{ik\beta} + \ldots \sum_{k=1}^{pl} (4k-1) e^{ik\beta} + \ldots \\
&\sum_{k=l+1}^{2l+1} (4k-1) + \sum_{k=l+1}^{2l+1} (4k-1) e^{ik\beta} + \ldots \sum_{k=2l+1}^{3l+1} (4k-1) e^{ik\beta} + \ldots \\
&\sum_{k=2l+1}^{3l+1} (4k-1) + \sum_{k=2l+1}^{3l+1} (4k-1) e^{ik\beta} + \ldots \sum_{k=2l+1}^{3l+1} (4k-1) e^{ik\beta} + \ldots
\end{align*}
\]  

(9)

Then we add the points in the same group together (\( p \) points a group):

\[
\eta = \sum_{k=1}^{l} (4k-1) + \sum_{k=l+1}^{2l} (4k-1) e^{i\gamma} + \sum_{k=2l+1}^{3l} (4k-1) e^{2i\gamma} + \ldots
\]  

(10)

Which is equivalent with \( P \) is on the arc of the circle, it means:

\[
\eta(1 + e^\beta i + \ldots + e^{(q-1)\beta i}) = 0
\]  

(11)
If the radius of the circumcircle is finite, we have:

\[ w_1 \sin^d (w_1 - \phi) - w_2 \sin^d (w_2 - \phi) + \ldots + (-1)^n w_n \sin^d (w_n - \phi) = 0 \]  \hspace{1cm} (12)

Consequently,

\[ \sum_{r=1}^{N} (-1)^{r-1} w_r e^{i(d-2)\phi} \sum_{i=1}^{r} e^{i(d-2)\phi} + \ldots + \sum_{d} e^{-i\phi} e^{i\phi} \]  \hspace{1cm} (13)

In other words,

\[ \sum_{r=1}^{N} (-1)^{-r-1} w_r e^{i\phi} \sum_{r} e^{i\phi} = 0 \]  \hspace{1cm} (14)

If the radius of the circumcircle is infinite, we have:

\[ \sum_{r=1}^{N} (-1)^{-r-1} v_r e^{i\phi} \sum_{r} e^{i\phi} = 0 \]  \hspace{1cm} (15)

Where, \( r_r = (-1)^{d} w_r \)

Then, the following equivalent condition appears:

\[ \eta (1 + e^{\phi \beta} + \ldots + e^{(q - 1)\beta}) = 0 \Leftrightarrow (13)(14) \]  \hspace{1cm} (16)

Next, we construct a regular polygon inscribed the circle and apply the Ptolemy theorem to study theorem 3.1.

Firstly, we arrange the vertices to make the points satisfy \( n \) is even:

\( A_1 P \), \( A_3 P \), \( A_5 P \), \ldots \,... \,
\( A_2 P \), \( A_4 P \), \( A_6 P \), \ldots \,...

It follows that:

\[ w_1 \left| A_1 P \right|^d + w_3 \left| A_3 P \right|^d + \ldots + w_n \left| A_n P \right|^d = w_2 \left| A_2 P \right|^d + w_4 \left| A_4 P \right|^d + \ldots \]  \hspace{1cm} (17)

Using the Ptolemy theorem to the regular polygon we defined above,

\[ \frac{PA_1 + PA_2 + \ldots + PA_N}{PA_1 + PA_N} = \frac{1}{1 - \cos \pi / n} \]  \hspace{1cm} (18)

Therefore, we can make a transition to relate the cyclic polygon and the regular inscribed polygon.

\[ w_1 \left| A_1 P \right|^d + w_3 \left| A_3 P \right|^d + \ldots + w_n \left| A_n P \right|^d > w_2 \left| A_2 P \right|^d + w_4 \left| A_4 P \right|^d + \ldots \]  \hspace{1cm} (20)

By the correspondence above, use the sequence inequality to get:
\[ w_1 \left| A_1 P \right|^d + w_3 \left| A_3 P \right|^d + \ldots + w_n \left| A_n P \right|^d \geq \pi / 2 \left( \left| A_1 P \right|^d + \ldots + \left| A_n P \right|^d \right) \quad (21) \]

Oppositely,
\[ w_2 \left| A_2 P \right|^d + w_4 \left| A_4 P \right|^d + \ldots \geq \pi / 2 \left( \left| A_2 P \right|^d + \ldots + \left| A_n P \right|^d \right) \quad (22) \]

It follows from (18) and (19) that:
\[ w_1 \left| A_1 P \right|^d + w_3 \left| A_3 P \right|^d + \ldots + w_n \left| A_n P \right|^d + w_2 \left| A_2 P \right|^d + w_4 \left| A_4 P \right|^d + \ldots \geq \pi / 2 \left( \left| A_1 P \right|^d + \ldots + \left| A_n P \right|^d \right) = \pi nr^2 \quad (23) \]

The general case is:
\[ w_1 \left| A_1 P \right|^d + w_3 \left| A_3 P \right|^d + \ldots + w_n \left| A_n P \right|^d > \pi nr^{t-1} r^{2s} \sum_{j=0}^{r-1} \frac{C s^j}{2s^j} / 2 \]
\[ = C \pi nr^{2s} \geq w_2 \left| A_2 P \right|^d + w_4 \left| A_4 P \right|^d + \ldots \quad (24) \]

From above, we can search the value we need by the existence of (24),
\[ w_1 \left| A_1 P \right|^d + w_3 \left| A_3 P \right|^d + \ldots + w_n \left| A_n P \right|^d = w_2 \left| A_2 P \right|^d + w_4 \left| A_4 P \right|^d + \ldots \]
\[ = C_1 \pi nr^d \quad (25) \]

Next, we make a regulation by the average of the sum \( C \pi nr^{2s} \):
\[ w_1 \left| A_1 P \right|^d . w_3 \left| A_3 P \right|^d . . . . w_{n/2-1} \left| A_{n/2-1} P \right|^d \quad \leq C_2 \pi r^d \]
\[ w_{n/2+1} \left| A_{n/2+1} P \right|^d . w_{n/2+2} \left| A_{n/2+2} P \right|^d . . . . \quad \geq C_2 \pi r^d \quad (26) \]

Similarly, we can get the inequality for \( w_2 \left| A_2 P \right|^d . w_4 \left| A_2 P \right|^d . . . . \)

By the equivalent property we get above, we can make the mass two two a pair to obtain:
\[ \eta = w_1 \sin^d (w_1 - \phi) - w_2 \sin^d (w_2 - \phi) + \ldots \]
\[ \eta e^{Bi} = w_{2lp+1} \sin^d (w_{2lp+1} - \phi) - w_{2lp+2} \sin^d (w_{2lp+2} - \phi) + \ldots \]
\( \eta = \frac{1}{2l} \sum_{k=1}^{l} (4k-1) + \frac{2l}{2l+1} \sum_{k=1}^{l} (4k-1)e^{i\gamma} + \frac{3l}{2l+1} \sum_{k=1}^{l} (4k-1)e^{2i\gamma} + \ldots \) (p formulas altogether)

\[
= \sum_{r=1}^{2lp} (-1)^{r-1} w_r \left[ s_0 e^{i(d-2)w_r} \cdot e^{-i(d-2)\phi} + s_1 e^{i(d-2)w_r} \cdot e^{-i(d-2)\phi} + \ldots + s_d e^{-idw_r} \cdot e^{id\phi} \right]
\]

(2lp values altogether)

We add the values in the same group together,

\[
\sum_{k=1}^{l} \sum_{r=1}^{4l} (-1)^{r-1} w_r \left[ s_0 e^{i(d-2)w_r} \cdot e^{-i(d-2)\phi} + s_1 e^{i(d-2)w_r} \cdot e^{-i(d-2)\phi} + \ldots + s_d e^{-idw_r} \cdot e^{id\phi} \right]
\]

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We add the values in the same group together,

\[
\sum_{k=1}^{l} \sum_{r=1}^{4l} (-1)^{r-1} w_r \left[ s_0 e^{i(d-2)w_r} \cdot e^{-i(d-2)\phi} + s_1 e^{i(d-2)w_r} \cdot e^{-i(d-2)\phi} + \ldots + s_d e^{-idw_r} \cdot e^{id\phi} \right]
\]

\[
\sum_{k=1}^{l} \sum_{r=1}^{4l} (-1)^{r-1} w_r \left[ s_0 e^{i(d-2)w_r} \cdot e^{-i(d-2)\phi} + s_1 e^{i(d-2)w_r} \cdot e^{-i(d-2)\phi} + \ldots + s_d e^{-idw_r} \cdot e^{id\phi} \right]
\]

We now compute the parameter defined by the average :

\[
\sum_{k=1}^{l} (4k-1) \leq 2l \pi r^d = C_1 \pi r^d
\]

\[
\sum_{k=1}^{l} (4k-1)e^{i\gamma} \geq -C_1 \pi r^d
\]

\[
\sum_{k=1}^{l} (4k-1)e^{ip\gamma} \leq C_1 \pi r^d
\]

Then we introduce the well known formulas of the minkowskian geometry (in [8]):

\[
\frac{|A_1 A_2|^2}{|PA_1|^2 \cdot |PA_2|^2} = -(2e^{i\rho})^{-2} \frac{|A_1 A_2|^2}{(q - \cos a)(q - \cos b)}
\]

\[
|PA_1| \cdot |PA_2| = i(2e^{i\rho})^{-1} \sqrt{(q - \cos a)(q - \cos b)}
\]
\[ 4l - \sum_{k=1}^{l} (4k-1) = \sum_{k=1}^{l} (1 - w^j_j - s^m_m) = \sum_{r=1}^{4l} (1 - w^j_j e^{ikw_r}) \geq 4l - C_3 l \pi r^d \quad (33) \]

Use theorem 3.2 to get:

\[ e^{i2\pi / n \cdot (s^j_j - s^m_m)} = 1 + 2 \sin([2\pi / n \cdot (s^j_j - s^m_m)] / 2) \exp[i(\pi + 2\pi / n \cdot (s^j_j - s^m_m)) / 2] \]

\[ = 1 + 2 \sin(\pi / n \cdot (s^j_j - s^m_m)) \sin(\pi / n \cdot (s^j_j - s^m_m)) - i \cos(\pi / n \cdot (s^j_j - s^m_m)) \]

\[ = 1 + 2 \sin a(a - i \cos a) \quad (34) \]

Where,

\[ w^j_j = e^{i2\pi / n \cdot (s^j_j - s^m_m)} \quad (35) \]

\[ w_r = e^{ie^{ikw_r}} \quad (36) \]

Recall (30), we have:

\[ \sum_{r=1}^{4l} 2 \sin a(a - i \cos a) \geq 4l - C_3 l \pi r^d \quad (37) \]

In which, \( a \in (0, \pi) \)

It is enough to assume that:

\[ e^{i\pi / n \cdot (s^j_j - s^m_m)} = w_r \quad (38) \]

\[ e^{i\pi / n \cdot (s^j_j - s^m_m)} = e^{ikw_r} \quad (39) \]

Substitute (38) and (39) into (37), we obtain:

\[ \sum_{r=1}^{4l} 2 \sin(kw_r)(\sin(kw_r) - i \cos(kw_r)) \geq 4l - C_3 l \pi r^d \quad (40) \]

The property of minkowskian geometry gives,

\[
\begin{align*}
\left| PA_1 \right| \cdot \left| PA_2 \right| &= e^{ik(w_1 + w_2)} = \cos k(w_1 + w_2) + i \sin k(w_1 + w_2) \\
\left| PA_3 \right| \cdot \left| PA_4 \right| &= e^{ik(w_3 + w_4)} = \cos k(w_3 + w_4) + i \sin k(w_3 + w_4)
\end{align*}
\]

\[ \cdots \]

Now, we obtain the relation for the polar angles from [8]:

\[ \sin k(w_1 + w_2) = (2 e^{i\rho})^{-1} \sqrt{(q - \cos a)(q - \cos b)} = 1 \quad (42) \]

\[ (2 e^{i\tau}) = \sqrt{(q - \cos a)(q - \cos b)} \quad (43) \]

Which lead that:

\[ \]
\[
\sum_{k=1}^{l} (4k-1) \cdot \sum_{k=1}^{pl} (4k-1)e^{(p-1)i\gamma} \leq C_3 l^2 \pi^2 r^2 d \\
\sum_{k=1}^{k=1} (4k-1) \cdot \sum_{k=1}^{k=1} (4k-1)e^{(p-1)i\gamma} \geq C_3 l^2 \pi^2 r^2 d
\]  
\[(44)\]

We finish our proof as follow:

\[
1 - e^{(p-1)i\gamma} \leq 1 - \frac{C_3 l^2 \pi^2 r^2 d}{\sum_{k=1}^{l} (4k-1) \cdot \sum_{k=1}^{pl} (4k-1)}
\]

\[
1 - e^{(p-1)i\gamma} \geq 1 - \frac{C_3 l^2 \pi^2 r^2 d}{\sum_{k=1}^{2l} (4k-1) \cdot \sum_{k=1}^{(p-1)l} (4k-1)}
\]

\[(45)\]

\[
\Pi \left(1 - \frac{C_3 l^2 \pi^2 r^2 d}{\sum_{k=1}^{l} (4k-1) \cdot \sum_{k=1}^{pl} (4k-1)}\right) > \Pi \left(1 - \frac{C_3 l^2 \pi^2 r^2 d}{\sum_{k=1}^{2l} (4k-1) \cdot \sum_{k=1}^{(p-1)l} (4k-1)}\right) > 0
\]

\[(46)\]

\[
\sum_{r=1}^{4l} 2 \sin(ikw_r)(\sin(ikw_r) - i \cos(ikw_r)) \geq 4l - C_3 l\pi r^d
\]

\[(47)\]

\[
\Pi \left(1 - \frac{C_3 l^2 \pi^2 r^2 d}{\sum_{k=1}^{2l} (4k-1) \cdot \sum_{k=1}^{(p-1)l} (4k-1)}\right) \leq \Pi \left(1 - \frac{4l - C_3 l\pi r^d}{\sum_{k=1}^{4l} 2 \sin(ikw_r)(\sin(ikw_r) - i \cos(ikw_r))^2}\right)
\]

\[(48)\]

\[
(\text{r is odd})
\]

\[
\Pi \left(1 - \frac{C_3 l^2 \pi^2 r^2 d}{\sum_{k=1}^{l} (4k-1) \cdot \sum_{k=1}^{pl} (4k-1)}\right) \leq \Pi \left(1 - \frac{4l - C_3 l\pi r^d}{\sum_{k=1}^{l} (4k-1) \cdot \sum_{k=1}^{pl} (4k-1)}\right)
\]

\[(49)\]

\[
(\text{r is even})
\]

Here, we study the symmetry in Minkowskian geometry, by applying the Ptolemy theorem. And we also use the techniques in combinatorics, and properly introduce some properties of the Minkowskian geometry. At last, we discover that: the polar angle \( ikw_r \) in Minkowskian geometry and even the length of the tangent line to the polar angle, is dependent on the ratio below:
$$\eta = \sum_{k=1}^{l} (4k-1) + \sum_{k=l+1}^{2l} (4k-1) e^{i}\gamma + \sum_{k=2l+1}^{3l} (4k-1) e^{2i}\gamma + \ldots$$  \hspace{1cm} (50)

Reference:
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