Abstract. In this report, we study differential forms on a manifold $M$. We first give the definition of differential forms. Then the exterior derivative, Lie derivative, and integrations of differential forms are discussed. Finally we will look at a special family of differential forms, called harmonic forms. This report is a preparation for de Rham cohomology and Hodge theorem that will be studied in the second report.

Let $M$ be an $n$-dimensional differentiable manifold. The tangent bundle of $M$ is denoted as $TM$ and the cotangent bundle of $M$ is denoted as $T^*M$. Throughout this report, all manifolds, vector fields, functions, etc. are assumed to be smooth (of class $C^\infty$) and all manifolds are assumed path-connected and with boundary, if there is no extra claim.

1 Construction of differential forms

In this section, we construct differential forms on manifolds. We first introduce the exterior algebra of finite dimensional vector spaces.

Definition 1 (page 217 of [2], page 63 of [4]). Let $R$ be a commutative ring with multiplicative identity. The exterior algebra over $R$ is the free $R$-module $\Lambda R[\alpha_1, \alpha_2, \cdots]$ with basis the finite products $\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k}, i_1 < \cdots < i_k$, with associative and distributive multiplication defined by the rules $\alpha_i \wedge \alpha_j = -\alpha_j \wedge \alpha_i$ for $i \neq j$, and $\alpha_i \wedge \alpha_i = 0$. The product $\wedge$ is called wedge product. The empty product of $\alpha_i$’s is allowed and provides an identity element $1 \in \Lambda R[\alpha_1, \alpha_2, \cdots]$. In particular, if $\{\alpha_1, \cdots, \alpha_n\}$ is a basis of an
\[ \text{n-dimensional vector space } V \text{ over } \mathbb{R}, \text{ then } \Lambda_{\mathbb{R}}[\alpha_1, \cdots, \alpha_n] \text{ is denoted as } \Lambda^*V \text{ for simplicity.} \]

**Definition 2** (page 64 of [4]). Let \( V \) be an \( n \)-dimensional vector space over \( \mathbb{R} \). A multilinear map

\[ \omega : \underbrace{V \times \cdots \times V}_{k} \longrightarrow \mathbb{R} \]

is called an alternating form of degree \( k \) on \( V \), if for any \( X_i \in V, i = 1, \cdots, n, \)

\[ \omega(X_{\sigma(1)}, \cdots, X_{\sigma(n)}) = \text{sgn} \sigma \cdot \omega(X_1, \cdots, X_n). \]

The set of all alternating forms of degree \( k \) on \( V \), denoted as \( A^k(V) \), is a vector space with respect to the natural sum and the multiplication of alternating forms by real numbers. If we let \( A^0(V) = \mathbb{R} \), then the collection of all alternating forms on \( V \) is \( A^*(V) = \bigoplus_{k=0}^{n} A^k(V) \).

Let \( V^* \) be the dual space of \( V \). Then there is an isomorphism \( \iota : \Lambda^*V^* \longrightarrow A^*(V) \) such that for \( \omega = \alpha_1 \wedge \cdots \wedge \alpha_k \), where \( \alpha_1, \cdots, \alpha_k \in V^* \) and \( k = 1, \cdots, n, \)

\[ \iota(\omega)(X_1, \cdots, X_k) = \frac{1}{k!} \det(\alpha_i(X_j)), \]

and extend linearly on \( \Lambda^*V^* \). Hence we can identify \( \Lambda^*V^* \) and \( A^*(V) \) (page 65 of [4]). For \( \omega \in A^k(V) \) and \( \eta \in A^l(V) \), we consider the wedge product \( \omega \wedge \eta \) as an element of \( A^{k+l}(V) \). Then (page 65 of [4])

\[ \omega \wedge \eta(X_1, \cdots, X_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn} \sigma \cdot \omega(X_{\sigma(1)}, \cdots, X_{\sigma(k)})\eta(X_{\sigma(k+1)}, \cdots, X_{\sigma(k+l)}). \]

Now we begin to construct differential forms on manifolds.

For an \( n \)-dimensional manifold \( M \), we say that \( \omega \) is a differential form of degree \( k \), or a \( k \)-form, on \( M \) if it assigns \( \omega_p \in A^k(T_pM) = \Lambda^k(T^*_pM) \) to each point \( p \in M \) and \( \omega_p \) is of class \( C^\infty \) with respect to \( p \). Precisely, if we denote \( \Gamma(TM) \) (resp. \( \Gamma(T^*M) \)) as the collection of all smooth cross-sections of \( TM \) (resp. \( T^*M \)), which forms a \( C^\infty(M) \)-module, then a \( k \)-form \( \omega \) is a \( C^\infty(M) \)-multilinear map

\[ \omega : \Gamma(TM) \times \cdots \times \Gamma(TM) \longrightarrow C^\infty(M). \]
Let $U$ be a coordinate neighborhood of $M$ and $x_1, \ldots, x_n$ the local coordinates in $U$. Then for any $p \in U$, $\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_n}|_p$ is a basis of $T_p M$ and $dx_1|_p, \ldots, dx_n|_p$ is the dual basis of $T^*_p M$ with $dx_i(\frac{\partial}{\partial x_j}) = \delta_{ij}$. For a $k$-form $\omega$ on $M$, $\omega$ has the local expression in $U$

$$\omega_p = \sum_{i_1 < \cdots < i_k} f_{i_1 \cdots i_k}(p)(dx_{i_1}|_p) \wedge \cdots \wedge (dx_{i_k}|_p)$$

for any $p \in U$, where $f_{i_1 \cdots i_k}$ is a $C^\infty$-function of $p$.

Let $W$ be another coordinate neighborhood of $M$ such that $U \cap W \neq \emptyset$. Let $y_1, \ldots, y_n$ be the local coordinate in $W$. Then for each $p \in U \cap W$, if we denote the Jacobian of $x_1, \ldots, x_i$ with respect to $y_j, \ldots, y_k$ as

$$D(x_1, \ldots, x_i) \Big|_{D(y_j, \ldots, y_k)}$$

we have

$$\omega = \sum_{i_1 < \cdots < i_k} f_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

$$= \sum_{i_1 < \cdots < i_k} f_{i_1 \cdots i_k} \left( \sum_{j_1=1}^n \frac{\partial x_{i_1}}{\partial y_{j_1}} dy_{j_1} \right) \wedge \cdots \wedge \left( \sum_{j_k=1}^n \frac{\partial x_{i_k}}{\partial y_{j_k}} dy_{j_k} \right)$$

$$= \sum_{j_1 < \cdots < j_k} f_{i_1 \cdots i_k} x_{i_1}(y_{j_1}, \ldots, y_{j_k}), \ldots, x_{i_k}(y_{j_1}, \ldots, y_{j_k}) \frac{D(x_1, \ldots, x_i)}{D(y_1, \ldots, y_k)} dx_{i_1} \wedge \cdots \wedge dy_{j_k}.$$ 

Let $\Lambda^k(M)$ be the vector bundle

$$\Lambda^k(M) = \{(p, \omega_p) \mid p \in M, \omega_p \in A^k(T_p M)\}.$$ 

Then $\omega$ is a $k$-form if and only if it is a smooth cross-section of $\Lambda^k(M)$, denoted as $\omega \in \Omega^k(M) := \Gamma(\Lambda^k(M))$. Let $\Lambda^*(M) = \bigoplus_{k=0}^n \Lambda^k(M)$ and $\Omega^*(M) = \bigoplus_{k=0}^n \Omega^k(M)$. Then $\Omega^*(M)$ is the collection of all differential forms on $M$, forming a $C^\infty(M)$-module. The wedge product (exterior product) of two differential forms $\omega \in \Omega^k(M), \eta \in \Omega^l(M)$, is denoted as $\omega \wedge \eta$, and is defined as the wedge product $(\omega \wedge \eta)_p = \omega_p \wedge \eta_p$ at each point $p \in M$. $\omega \wedge \eta$ is an element in $\Omega^{k+l}(M)$. It can be verified that $\omega \wedge \eta = (-1)^{kl}\eta \wedge \omega$. 

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2 Pull-back of differential forms

Let $M_1, M_2$ be an $n_1$-dimensional and $n_2$-dimensional differentiable manifold respectively. Let $f : M_1 \to M_2$ be a smooth map. Let $\omega \in \Omega^k(M_2)$. Then we can define a $k$-form on $M_1$, called the pull-back of $\omega$, as follows.

Let $p \in M_1$ and $f(p) = q \in M_2$. Then the tangent map of $f$ at $p$ is a linear map $df|_p : T_p M_1 \to T_q M_2$. $df|_p$ induces its dual map $f^*|_p : T^*_q M_2 \to T^*_p M_1$, i.e., the map defined by $f^*|_p(\alpha)(X) = \alpha(df|_p(X))$ for all $X \in T_p M_1$ and $\alpha \in T^*_q M_2$. Furthermore, $f^*|_p$ induces a linear map $f^*|_p : \Lambda^k(T^*_q M_2) \to \Lambda^k(T^*_p M_1)$ for each non-negative integer $k$. Hence we can let $(f^*\omega)(X_1, \cdots, X_k) = \omega(df(X_1), \cdots, df(X_k))$ for all $X_1, \cdots, X_k \in \Gamma(TM_1)$. Then it can be proved that $f^*\omega \in \Omega^k(M_1)$ is a smooth cross-section of $\Lambda^k(M_1)$.

**Proposition 1** (page 72 of [4]). Let $f : M_1 \to M_2$. Then $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$ for any $\omega \in \Omega^k(M_2)$ and $\eta \in \Omega^l(M_2)$ and any non-negative integers $k,l$.

**Remark 2.** Although differential forms on $M_2$ can be pulled back to be differential forms on $M_1$, vector fields on $M_1$ cannot be mapped to be vector fields on $M_2$ in general. Suppose $n_1 = n_2$. If for $p \in M_1$, $df|_p$ is non-degenerate at $T_p M_1$, then by the implicit function theorem, there exists open neighborhoods $U, V$ of $p, q$ respectively such that $f|_U : U \to V$ is a homeomorphism and at each $x \in U$, $df|_x$ is non-degenerate. The condition that $df|_p$ is non-degenerate for all $p \in M_1$ does not imply that $df(X)$ is a vector field on $M_2$ for vector field $X \in \Gamma(TM_1)$. The exponential map from $\mathbb{R}$ to the unit circle in $\mathbb{C}$ is a counterexample.

3 Exterior derivative of differential forms

In this section we discuss the exterior derivative of differential forms.

**Theorem 3** (page 75 of [7]). For an $n$-dimensional manifold $M$, there exists a unique map $d : \Omega^*(M) \to \Omega^*(M)$, such that $d(\Omega^k(M)) \subseteq \Omega^{k+1}(M)$ for all $k \in \mathbb{N}$ and satisfy the following properties:
(1). for any $\omega_1, \omega_2 \in \Omega^*(M)$, $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$;
(2). for any $\omega \in \Omega^r(M)$, $\omega_2 \in \Omega^*(M)$, $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^r \omega_1 \wedge d\omega_2$;
(3). for $f \in \Omega^0(M) = C^\infty(M)$, $d\omega$ is the differentiation of $f$, and $d(df) = 0$.

The unique map $d$ is called the exterior derivative on $\Omega^*(M)$.

For a $k$-form $\omega \in \Omega^k(M)$ on $M$, if in a local coordinate $(x_1, \cdots, x_n)$,
\[ \omega = \sum_{i_1 < \cdots < i_k} f_{i_1 \cdots i_k} dx_1 \wedge \cdots \wedge dx_k, \]
then its exterior differentiation is given by
\[ d\omega = \sum_{j=1}^n \sum_{i_1 < \cdots < i_k} \frac{\partial f_{i_1 \cdots i_k}}{\partial x_j} dx_j \wedge dx_1 \wedge \cdots \wedge dx_k. \]

**Proposition 4** (page 72 of [4]). Let $f : M_1 \longrightarrow M_2$. Then $d(f^*\omega) = f^*(d\omega)$ for any $\omega \in \Omega^k(M_2)$.

**Theorem 5** (page 71 of [4]). Let $\omega \in \Omega^k(M)$ for $k = 0, \cdots, n$. Then for any vector fields $X_1, \cdots, X_{k+1} \in \Gamma(TM)$, we have
\[ d\omega(X_1, \cdots, X_{k+1}) = \frac{1}{k+1} \sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1, \cdots, \hat{X}_i, \cdots, X_{k+1})) \]
\[ + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{k+1}). \]

Here the symbol $\hat{X}_i$ means $X_i$ is omitted. In particular, for the case $k = 1$ we have
\[ d\omega(X, Y) = \frac{1}{2} \{X \omega(Y) - Y \omega(X) - \omega([X, Y]) \}. \]

### 4 Lie derivative of differential forms

Let $X \in \Gamma(TM)$. Then we define a linear map
\[ i(X) : \Omega^k(M) \longrightarrow \Omega^{k-1}(M), \]
\[ (i(X)\omega)(X_1, \cdots, X_{k-1}) = k\omega(X, X_1, \cdots, X_{k-1}) \]
for any $\omega \in \Omega^k(M)$ and $X_1, \ldots, X_{k-1} \in \Gamma(TM)$. Note that if $k = 0$, then we define $i(X) = 0$. The $(k-1)$-form $i(X)\omega$ is called the interior product of $\omega$ by $X$. By definition, it can be verified that $i(X)$ is linear with respect to functions, i.e., $i(X)(f \omega_1 + g \omega_2) = f(i(X)\omega_1) + g(i(X)\omega_2)$ for any $f, g \in C^\infty(M)$. Hence $i(X)$ is a $C^\infty(M)$-module homomorphism. Moreover, it can be calculated that for any $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$,

$$i(X)(\omega \wedge \eta) = (i(X)\omega) \wedge \eta + (-1)^k \omega \wedge i(X)\eta.$$ 

Given $X \in \Gamma(TM)$, the Lie derivative of differential forms is defined to be the following linear operator

$$L_X : \Omega^k(M) \rightarrow \Omega^k(M),$$

$$(L_X \omega)(X_1, \ldots, X_k) = X\omega(X_1, \ldots, X_k) - \sum_{i=1}^k \omega(X_1, \ldots, [X, X_i], \ldots, X_k).$$

**Proposition 6** (page 74-75 of [4]). The Lie derivative satisfies the following equations

$$L_X = i(X) \circ d + d \circ i(X),$$

$$L_X \circ i(Y) - i(Y) \circ L_X = i([X, Y]),$$

$$L_X \circ L_Y - L_Y \circ L_X = L_{[X,Y]},$$

$$L_X(\omega \wedge \eta) = L_X \omega \wedge \eta + \omega \wedge L_X \eta,$$

$$L_X d\omega = dL_X \omega$$

for any $X, Y \in \Gamma(TM)$ and $\omega \in \Omega^k(M), \eta \in \Omega^l(M)$.

**Remark 7.** Let $X \in \Gamma(TM)$. A curve $\gamma : (a, b) \rightarrow M$ is called an integral curve of $X$ if the vector $\gamma'(t) \in T_{\gamma(t)}$ coincides with the value of $X$ at $\gamma(t)$, for each $t \in (a, b)$.

By the existence and uniqueness of the solution of ODEs, for any $p \in M$, there exists an integral curve passing through $p$; and if two integral curves pass through $p$, then they are connected as a single integral curve. Moreover, for any $p \in M$, there exists an open neighborhood $V$ of $p$ and $\epsilon(p) > 0$ such
that for any \( q \in V \), there exists an integral curve \( \gamma_q : (-\epsilon(p), \epsilon(p)) \rightarrow M \) with \( \gamma_q(0) = q \).

For each \( p \in M \), let \( (a(p), b(p)) \) be the maximal open interval such that the integral curve of \( X \) through \( p \) is defined on \( (a(p), b(p)) \). Let \( W = \{(t, p) \in \mathbb{R} \times M \mid t \in (a(p), b(p))\} \). Then \( \Phi : W \rightarrow M, \Phi(t, p) := \gamma_p(t) \) is a \( C^\infty \)-map. We denote \( \phi_t(p) = \Phi(t, p) \). Then for arbitrary \( t \), \( \phi_t : M_t = \{p \in M \mid t \in (a(p), b(p))\} \rightarrow M \) is a diffeomorphism onto an open submanifold of \( M \) and \( \phi_t \circ \phi_s(p) = \phi_{t+s}(p) \) as long as both sides are defined (page 42 of [4]). The set of all diffeomorphisms \( \{\phi_t \mid t \in \mathbb{R}\} \) is called the one parameter group of local transformations generated by \( X \) (although the set does not always form a group).

**Proposition 8** (page 78 of [4]). Let \( X \in \Gamma(TM) \) and \( \{\phi_t\} \) the on parameter group of local transformations generated by \( X \). Then for each integer \( k \) and \( \omega \in \Omega^k(M) \) we have

\[
L_X \omega = \lim_{t \to 0} \frac{\phi_t^* \omega - \omega}{t}.
\]

### 5 Integration of differential forms

In this section, we shall discuss the integration of differential forms on manifolds. In order to do this we first introduce the orientability of manifolds.

**Definition 3** (page 72 of [3]). An \( n \)-dimensional manifold \( M \) is called orientable if there exists \( \omega \in \Omega^n(M) \) such that \( \omega_p \neq 0 \) for any \( p \in M \). Such an \( \omega \) is called an orientation form on \( M \). Two orientation forms \( \omega_1, \omega_2 \) are equivalent if \( \omega_2 = f \omega_1 \) for some \( f \in C^\infty(M) \) with \( f(p) > 0 \) for all \( p \in M \).

**Remark 9.** For each \( p \in M \), \( A^n(T_p M) \) is an 1-dimensional vector space. Precisely, if \( (x_1, \cdots, x_n) \) is a local coordinate of a neighborhood around \( p \), then \( A^n(T_p M) = \text{Span}_\mathbb{R}(dx_1|_p) \wedge \cdots \wedge (dx_n|_p) \). Thus, if \( \omega_1, \omega_2 \) are two orientation forms, then \( \omega_1 = f \omega_2 \) for some \( f \) with \( f(p) \neq 0 \) for all \( p \in M \). Hence \( f \) is always positive or always negative. Hence either \( \omega_1 \) or \( -\omega_1 \) is equivalent to \( \omega_2 \). Therefore, an orientable manifold \( M \) has exactly 2-orientations.
Proposition 10 (page 85-86 of [7]). A manifold $M$ is orientable if and only if $M$ has an atlas $\{(U_\alpha, (x_1^\alpha, \cdots, x_n^\alpha))\}$ such that for any $U_\alpha \cap U_\beta \neq \emptyset$, the Jacobian of coordinate transformation $\frac{D(x_1^\alpha, \cdots, x_n^\alpha)}{D(x_1^\beta, \cdots, x_n^\beta)} > 0$.

Definition 4 (page 72 of [3]). Let $M_1, M_2$ be $n$-dimensional manifolds and $\phi : M_1 \rightarrow M_2$ a diffeomorphism. Suppose $M_1, M_2$ are oriented by their orientation forms $\omega_1, \omega_2$ respectively. We say $\phi$ is orientation preserving if $\phi^* \omega_2$ determines the same orientation of $M_1$ as $\omega_1$.

Remark 11. Generally, in the case $M_1, M_2$ are not required of the same dimension, we suppose $\phi : M_1 \rightarrow M_2$ with its Jacobian $J_\phi(p) \neq 0$ for all $p \in M_1$. If $M_2$ is oriented by its orientation form $\omega_2$, then $M_1$ is oriented by $\phi^* \omega_2$.

Now we start to define integrations of differential forms. To do this, we first define integrations of differential forms in a compact subset of $\mathbb{R}^n$.

Let $(x_1, \cdots, x_n)$ be the canonical coordinate of $\mathbb{R}^n$. Let $\omega = f(x)dx_1 \wedge \cdots \wedge dx_n$ be an $n$-form in $\mathbb{R}^n$ such that $\text{supp}\omega := \text{supp}f = \{x \in \mathbb{R}^n \mid f(x) \neq 0\}$ is compact. Then we can define

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} f d\mu_n.$$  

Here $d\mu_n$ is the Lebesgue measure of $\mathbb{R}^n$. The following proposition ensures that the integration is well-defined.

Proposition 12 (page 83 of [3]). Let $V, W$ be open subsets in $\mathbb{R}^n$ and $\phi : V \rightarrow W$ a diffeomorphism. If we denote $D_x \phi$ as the Jacobian of $\phi$ at $x \in V$, then for any $f$ integrable on $W$,

$$\int_{\mathbb{R}^n} f d\mu_n = \int_V (f \circ \phi)|D_x \phi| d\mu_n.$$  

As a consequence, if $D_x \phi$ is of constant sign $\delta = \pm 1$ for all $x \in V$, then for each $n$-form on $\mathbb{R}^n$ with $\text{supp}\omega \subseteq W$,

$$\int_V \phi^* \omega = \delta \int_W \omega.$$  

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Theorem 13 (Partition of unity, page 88 of [7]). Let $\Sigma = \{U_\alpha\}$ be an open cover of an $n$-dimensional manifold $M$. Then there exists a family of smooth functions $\{g_\alpha\}$ satisfying:

1. $0 \leq g_\alpha \leq 1$, $\text{supp} g_\alpha$ is compact and there exists $U_\alpha \in \Sigma$ such that $\text{supp} g_\alpha \subseteq U_\alpha$;
2. for each $p \in M$, there exists an open neighborhood $U$ of $p$ such that $\{g_\alpha \mid U \cap \text{supp} g_\alpha \neq \emptyset\}$ is finite;
3. $\sum_\alpha g_\alpha = 1$ at each point of $M$.

Let an $n$-dimensional manifold $M$ be oriented. Let $\Omega^n_c(M)$ be the set of $n$-forms with compact support on $M$. We will define an integral

$$\int_M : \Omega^n_c(M) \rightarrow \mathbb{R}.$$ 

Let $\{U_\alpha\}$ be an atlas on $M$ such that the orientation of each $U_\alpha$ is determined by the orientation of $M$. We choose $\{g_\alpha\}$ subordinate to $\{U_\alpha\}$ by Theorem 13. Then for any $\omega \in \Omega^n_c(M)$, define the integral of $\omega$ on $M$ by

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} g_\alpha \omega = \sum_\alpha \int_{\mathbb{R}^n} \phi_\alpha^*(g_\alpha \omega).$$

Here $\phi_\alpha$ is from an open subset of $\mathbb{R}^n$ to $U_\alpha$ and is a diffeomorphism with positive Jacobian at each point. By Theorem 13, the sum is finite at each $p \in M$. By Proposition 12, the integral of $\omega$ on $M$ does not depend on the choice of $\{U_\alpha\}$. Hence the integral is well-defined.

Remark 14. Let $M_1, M_2$ be $n$-dimensional oriented manifolds. Let $f : M_1 \rightarrow M_2$ be an orientation-preserving diffeomorphism. Then for any $\omega \in \Omega^n_c(M_2)$ we have (page 85 of [3])

$$\int_{M_2} \omega = \int_{M_1} f^* \omega.$$ 

Generally, let $f : M_1 \rightarrow M_2$ be a $n$-sheeted covering map preserving orientation. Then by similar argument using partition of unity, we obtain

$$n \int_{M_2} \omega = \int_{M_1} f^* \omega.$$
Given an orientation form on an $n$-dimensional oriented manifold $M$, it can induce a measure on $M$.

**Lemma 15** (page 40-41 of [5]). Let $X$ be a locally compact Hausdorff space. Let $T$ be a positive linear functional on $C_c(X)$ (the space of continuous functions with compact support). Then there exists a $\sigma$-algebra $\Sigma$ in $X$ which contains all Borel sets in $X$. And there exists a unique positive measure $\mu$ on $\Sigma$ which represents $T$ in the sense that $Tf = \int_X f \, d\mu$ for every $f \in C_c(X)$, which has the additional following properties

(a). $\mu(K) < \infty$ for every compact set $K \subseteq X$;
(b). for every $E \in \Sigma$ we have $\mu(E) = \inf\{\mu(V) \mid E \subseteq V, V \text{ open}\}$;
(c). the relation $\mu(E) = \sup\{\mu(K) \mid K \subseteq E, K \text{ compact}\}$ holds for every open set $E$ and for every $E \in \Sigma$ with $\mu(E) < \infty$;
(d). if $E \in \Sigma$ with $\mu(E) = 0$ and $A \subseteq E$, then $A \in \Sigma$.

Let $\omega \in \Omega^n(M)$ be an orientation form of $M$ which gives the orientation of $M$. Then $\omega$ is nowhere zero. Hence for any $f \in C_c(M)$, $\text{supp}\, f = \text{supp}(f \omega)$ is compact. Let

$$T_\omega : C_c(M) \rightarrow \mathbb{R},$$

$$f \mapsto \int_M f \omega.$$

Then $T_\omega$ is a positive linear functional, i.e., $T_\omega$ is linear over $\mathbb{R}$ and $T_\omega f \geq 0$ for all $f \geq 0$. By Lemma 15, $T_\omega$ determines a positive measure $\mu_\omega$ on $M$ which satisfies

$$\int_M f \, d\mu_\omega = \int_M f \omega, \quad f \in C_c(M).$$

Therefore, for any $\mu_\omega$-measurable function $g$ on $M$ with compact support, we can define the integrals of $g^+$ and $g^-$ by the same procedure as in the case of Lebesgue measure of $\mathbb{R}^n$. If the two integrals are both finite then $g$ is integrable on $M$.

We state the Stokes’ Theorem to end this section.

Let $M$ be an $n$-dimensional orientable manifold with its boundary $\partial M$. Then an orientation of $M$ induces an orientation of $\partial M$ (page 74 of [6]). Hence $\partial M$ is also orientable.

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Theorem 16 (Stokes Theorem, page 107 of [4]). Let $M$ be an oriented $n$-dimensional manifold. Assume $\partial M$ is equipped with an orientation induced from the orientation of $M$. Let $\omega$ be an $(n - 1)$-form on $M$ with compact support. Then

$$\int_M d\omega = \int_{\partial M} \omega.$$ 

Now let $M$ be an $n$-dimensional manifold. A $C^\infty$-map $\sigma : \Delta^k \to M$ from the standard $k$-simplex $\Delta^k$ to $M$ is called a $C^\infty$ singular $k$-simplex of $M$. The free abelian group generated by all $C^\infty$ singular $k$-simplices of $M$ is denoted by $S^\infty_k(M)$ and an element of it is called a $C^\infty$ singular $k$-chain of $M$. For an arbitrary $c \in S^\infty_k(M)$, we see that $\partial c \in S^\infty_{k-1}(M)$. For $\omega \in \Omega^k(M)$, the pull-back $\sigma^*\omega$ is defined (except for the $(k - 1)$-skeleton which is a subset of measure zero) on $\Delta^k$. Let

$$\int_{\sigma} \omega := \int_{\Delta^k} \sigma^* \omega.$$ 

For a general chain $c \in S^\infty_k(M)$, if $c = \sum a_i \sigma_i$, then we extend the definition linearly over $\mathbb{Z}$

$$\int_{\sigma} \omega := \sum i a_i \int_{\sigma_i} \omega.$$ 

Theorem 17 (Stokes Theorem on chains, page 109 of [4]). For a $C^\infty$ singular $k$-chain $c \in S^\infty_k(M)$ of a manifold $M$ and a $(k - 1)$-form $\omega$ on $M$, we have

$$\int_c d\omega = \int_{\partial c} \omega.$$ 

6 Harmonic forms

In this section we discuss about harmonic forms.

Let $M$ be an orientable manifold. We equip a Riemannian metric $g$ on $M$. Then for any $X \in \Gamma(TM)$, we have a $C^\infty(M)$-linear map

$$g(X, \cdot) : \Gamma(TM) \to C^\infty(M),$$

$$Y \mapsto g(X, Y).$$
Hence \( g(X, \cdot) \in \Omega^1(M) \). Let

\[
\hat{g} : \Gamma(TM) \longrightarrow \Omega^1(M),
X \longmapsto g(X, \cdot).
\]

Then \( \hat{g} \) can be proved to be an isomorphism of \( C^\infty(M) \)-modules (page 148 of [4]). Hence for any \( \omega, \eta \in \Omega^1(M) \) we can define their pointwise product by

\[
\langle \omega, \eta \rangle := g(\hat{g}^{-1}\omega, \hat{g}^{-1}\eta).
\]

In general, for \( k = 1, \cdots, n \), we can define the pointwise product on \( \Omega^k(M) \) as the following \( C^\infty(M) \)-bilinear map

\[
\langle \cdot, \cdot \rangle : \Omega^k(M) \times \Omega^k(M) \longrightarrow C^\infty(M),
(\alpha_1 \wedge \cdots \wedge \alpha_k, \beta_1 \wedge \cdots \wedge \beta_k) \longmapsto \operatorname{det}(\langle \alpha_i, \beta_j \rangle).
\]

Here \( \alpha_1, \cdots, \alpha_k, \beta_1, \cdots, \beta_k \) are all 1-forms. We also define the pointwise product between two differential forms of different degrees to be 0.

We will introduce some operators on \( \Omega^* M \). We first introduce the Hodge star operator.

Let \( p \in M \) and \( v_1, \cdots, v_n \) an orthonormal basis of \( T_p M \). Let \( v_1^*, \cdots, v_n^* \) be the dual basis of \( T^*_p M \). Then the Hodge star operator at the point \( p \) is a linear operator defined as

\[
\ast : A^k(T_p M) \longrightarrow A^{n-k}(T_p M),
v_{i_1}^* \wedge \cdots \wedge v_{i_k}^* \longmapsto \operatorname{sgn}(I, J)v_{j_1}^* \wedge \cdots \wedge v_{j_{n-k}}^*.
\]

Here \( k = 0, 1, \cdots, n \), \( i_1 < \cdots < i_k \) is a subset of \( \{1, \cdots, n\} \) and \( j_1 < \cdots < j_{n-k} \) is its complement. And \( \operatorname{sgn}(I, J) \) is the sign of the permutation \( i_1, \cdots, i_k, j_1, \cdots, j_{n-k} \).

Now we extend \( \ast \) globally on \( \Omega^*(M) \). Since \( M \) is orientable, there exists an atlas \( \{(U; x_1, \cdots, x_n) \mid U \text{ is a chart in this atlas}\} \) such that the Jacobians of coordinate transformations are all positive. By Schimidt orthogonalization process, for each \( U \) we can get a locally defined orthonormal frame
$e_1, \ldots, e_n \in \Gamma(TU)$ such that at each $p \in U$, $e_1|_p, \ldots, e_n|_p \in T_pU$ is an orthonormal basis. Let $\theta_1, \ldots, \theta_n \in \Gamma(T^*U)$ be the dual frame of $e_1, \ldots, e_n$. If

$$\omega = \sum_{i_1<\cdots<i_k} f_{i_1\cdots i_k} \theta_{i_1} \wedge \cdots \wedge \theta_{i_k}$$

then we let

$$*\omega = \sum_{i_1<\cdots<i_k} \text{sgn}(I,J)f_{i_1\cdots i_k} \theta_{j_1} \wedge \cdots \wedge \theta_{j_{n-k}}.$$ 

Therefore, we have the $C^\infty(M)$-linear map

$$* : \Omega^*(M) \rightarrow \Omega^*(M),$$

$$\Omega^k(M) \rightarrow \Omega^{n-k}(M), \quad k = 0, 1, \ldots, n.$$ 

This is called the Hodge star operator. In particular, $*1 \in \Omega^n(M)$ is called the volume form of $M$, denoted by $v_M$.

**Proposition 18** (page 151 of [4]). The Hodge star operator satisfies the following properties for $\omega, \eta \in \Omega^k(M)$:

1. $**\omega = (-1)^{k(n-k)}\omega$.
2. $\omega \wedge *\eta = \eta \wedge *\omega = \langle \omega, \eta \rangle v_M$.
3. $\langle *\omega, *\eta \rangle = \langle \omega, \eta \rangle$.

Now we turn to the adjoint operator $\delta$ of the exterior derivative $d$ for an oriented manifold without boundary.

**Definition 5.** Let $M$ be an oriented manifold without boundary. Let $\delta : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$, $k = 1, \ldots, n$, be the operator determined by the following commutative diagram

$$
\begin{array}{ccc}
\Omega^k(M) & \overset{*}{\rightarrow} & \Omega^{n-k}(M) \\
\downarrow \delta & & \downarrow d \\
\Omega^{k-1}(M) & \overset{(-1)^k*}{\rightarrow} & \Omega^{n-k+1}(M)
\end{array}
$$
Then \( \delta \) is called the adjoint operator of \( d \). Since \( * \) is isomorphism, \( \delta \) is unique. Thus the commutative diagram is equivalent to

\[
\delta = (-1)^k *^{-1} d * = (-1)^{n(k+1)+1} * d * .
\]

**Remark 19.** Additionally, we further suppose \( M \) is compact. Let \( \omega, \omega' \in \Omega^k(M) \). Then we can define their inner product

\[
(\omega, \omega') = \int_M \langle \omega, \omega' \rangle v_M.
\]

Then by the assumption that \( M \) is compact, it can be verified directly that \( (\cdot, \cdot) \) is a positive-definite symmetric bilinear form. Hence \( (\cdot, \cdot) \) is really an inner product on \( \Omega^k(M) \).

Now let \( \omega \in \Omega^k(M) \) and \( \eta \in \Omega^{k+1}(M) \). Then

\[
d\omega \wedge *\eta = d(\omega \wedge *\eta) - (-1)^k \omega \wedge d(*\eta) = d(\omega \wedge *\eta) + \omega \wedge (*\delta \eta).
\]

Integrating on each sides over \( M \) and apply Stokes Theorem, since we assumed \( M \) has no boundary,

\[
(d\omega, \eta) = \int_M (d\omega, \eta) v_M
= \int_M d\omega \wedge *\eta
= \int_M d(\omega \wedge *\eta) + \int_M \omega \wedge (*\delta \eta)
= \int_M \omega \wedge (*\delta \eta)
= (\omega, \delta \eta).
\]

We see that in this case, \( \delta \) is really the adjoint operator of \( d \) relative to the inner product \( (\cdot, \cdot) \). Since \( d \circ d = 0 \), from the diagram we see that \( \delta \circ \delta = 0 \).

Now we can define harmonic forms.

**Definition 6.** Let \( M \) be an oriented manifold without boundary. Then the operator defined by

\[
\Delta = d\delta + \delta d : \Omega^k(M) \to \Omega^k(M)
\]
is called the Laplacian. Here $k = 0, 1, \cdots, n$. Note that on $\Omega^0(M)$, $\delta = 0$ and on $\Omega^n(M)$, $d = 0$. A differential form $\omega \in \Omega^*(M)$ is called harmonic if $\Delta \omega = 0$. In particular, if $f \in \Omega^0(M)$ is harmonic, then $f$ is called a harmonic function.

**Remark 20** (page 155-157 of [4]). According to this definition, for a $k$-form

$$\omega = f dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

in $\mathbb{R}^n$, we have

$$\Delta \omega = -\sum_{s=1}^{n} \frac{\partial^2 f}{\partial x_s^2} dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

**Remark 21.** Suppose $M$ is compact. Then for $\omega, \eta \in \Omega^*(M)$,

$$(\Delta \omega, \eta) = ((d\delta + \delta d) \omega, \eta)$$

$$= (d\delta \omega, \eta) + (\delta d \omega, \eta)$$

$$= (\omega, d\delta \eta) + (\omega, \delta d \eta)$$

$$= (\omega, \Delta \eta).$$

Hence $\Delta$ is self-adjoint. Moreover,

$$(\Delta \omega, \omega) = (d\delta \omega, \omega) + (\delta d \omega, \omega) = (\delta \omega, \delta \omega) + (d\omega, d\omega).$$

Hence if $\Delta \omega = 0$ then $d\omega = \delta \omega = 0$. Clearly $d\omega = \delta \omega = 0$ implies $\Delta \omega = 0$. Hence $\Delta \omega = 0$ if and only if $d\omega = \delta \omega = 0$.

**Proposition 22.** Let $M$ be a compact oriented $n$-dimensional (Riemannian) manifold without boundary. Then a harmonic function on $M$ is constant. And a harmonic $n$-form $\omega$ on $M$ is a constant multiple of $v_M$.

**Proof.** Let $f \in C^\infty(M)$ such that $\Delta f = 0$. Then $df = 0$. Hence $f$ is constant on $M$. Let $\omega \in \Omega^n(M)$. Then $\omega = fv_M$ for some $f \in C^\infty(M)$. If $\Delta \omega = 0$, then $\Delta f = \Delta (f * v_M) = \Delta * (fv_M) = \Delta * \omega = * \Delta \omega = 0$. Hence $\omega$ is a constant multiple of $v_M$. $\square$

What will happen if $M$ is with boundary and not assumed to be compact? I may study in future.
References


[7] 陈省身，陈维桓，微分几何讲义，北京大学出版社，2001。
Abstract. In this report, we study topological structures of manifolds using differential forms. We first state the de Rham cohomology Theorem and introduce Čech cohomology as a tool. Then we discuss about Hodge Theorem. Finally, we study some applications of these theorems.

Let $M$ be an $n$-dimensional differentiable manifold. Throughout this report, all manifolds, vector fields, functions, etc. are assumed to be smooth (of class $C^\infty$) and all manifolds are assumed path-connected and with boundary, if there is no extra claim. The notations and results in reading report I are assumed.

1 De Rham cohomology

In this section we will state the de Rham Theorem of cohomology of manifolds.

Recall that if we denote $\Omega^k(M)$ as the set of all $k$-forms on the manifold $M$, then a linear map

$$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

called the exterior differential operator, is defined. Since $d \circ d = 0$, we obtain a cochain complex, called the de Rham complex

$$\cdots \longrightarrow \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0.$$
The $k$-dimensional cohomology group of this cochain complex is called the $k$-dimensional de Rham cohomology group of $M$, denoted as $H^{k}_{DR}(M)$, $k = 0, 1, \cdots, n$. Moreover, we call the direct sum

$$H^{*}_{DR}(M) = \bigoplus_{k=0}^{n} H^{k}_{DR}(M)$$

the de Rham cohomology group of $M$ (page 112 of [3]).

Let $\omega \in \Omega^{k}(M)$. Then $\omega$ is called a closed form if $d\omega = 0$, and an exact form if there exists $\eta \in \Omega^{k-1}(M)$ such that $\omega = d\eta$. We denote the set of all closed $k$-forms by $Z^{k}(M)$ and the set of all exact $k$-forms by $B^{k}(M)$. Then

$$B^{k}(M) = \text{Im}\{d : \Omega^{k-1}(M) \rightarrow \Omega^{k}(M)\},$$

$$Z^{k}(M) = \text{Ker}\{d : \Omega^{k}(M) \rightarrow \Omega^{k+1}(M)\},$$

$$B^{k}(M) \subseteq Z^{k}(M),$$

$$H^{k}_{DR}(M) = \frac{Z^{k}(M)}{B^{k}(M)}, \quad k = 0, 1, \cdots, n.$$

We write elements of $H^{k}_{DR}(M)$ in the form $[\omega] := \omega + B^{k}(M)$ for $\omega \in Z^{k}(M)$. Let $[\omega'] = [\omega] \in H^{k}_{DR}(M), [\eta'] = [\eta] \in H^{l}_{DR}(M)$. Then $\omega' = \omega + d\xi$, $\eta' = \eta + d\tau$ for some $\xi, \in \Omega^{k-1}(M), \tau \in \Omega^{l-1}(M)$. Hence

$$\omega' \wedge \eta' = (\omega + d\xi) \wedge (\eta + d\tau)$$

$$= \omega \wedge \eta + \omega \wedge d\tau + d\xi \wedge \eta + d\xi \wedge d\tau$$

$$= \omega \wedge \eta + d((-1)^{k}\omega \wedge \tau + \xi \wedge \eta + \xi \wedge \eta \wedge \tau).$$

Hence $\omega' \wedge \eta' \in [\omega \wedge \eta]$ and vice versa. Hence $[\omega' \wedge \eta'] = [\omega \wedge \eta]$. Therefore, the wedge product of differential forms induces a product of de Rham cohomology

$$\wedge : H^{k}_{DR}(M) \times H^{l}_{DR}(M) \rightarrow H^{k+l}_{DR}(M),$$

$$([\omega], [\eta]) \mapsto [\omega \wedge \eta].$$

Equipped with this product structure, $H^{*}_{DR}(M)$ is called the de Rham cohomology algebra (page 113 of [3]). It is an algebra over $\mathbb{R}$. 

2
Let $f : M_1 \longrightarrow M_2$ be a smooth map between manifolds $M_1$ and $M_2$. Then since

$$
\begin{array}{ccc}
\Omega^*(M_2) & \xrightarrow{f^*} & \Omega^*(M_1) \\
\downarrow d & & \downarrow d \\
\Omega^*(M_2) & \xrightarrow{f^*} & \Omega^*(M_1),
\end{array}
$$

we have that $f^*\omega$ is closed if $\omega \in \Omega^*(M_2)$ is closed; and $f^*\omega$ is exact if $\omega$ is exact. Generally, the converse does not hold. Moreover, since

$$
\Omega^*(M_2) \times \Omega^*(M_2) \xrightarrow{f^*} \Omega^*(M_1) \times \Omega^*(M_1)
$$

$f^*$ induces an $\mathbb{R}$-algebra homomorphism

$$
f^* : H^*(M_2) \longrightarrow H^*(M_1),
\quad [\omega] \longmapsto [f^*\omega].
$$

Let $S^\infty_k(M) = \{(S^\infty_k(M), \partial) \mid k = 0, 1, 2, \cdots\}$ be the $C^\infty$ singular chain complex with coefficients in $\mathbb{R}$ of a manifold $M$. We denote its dual complex $\text{Hom}(S^\infty_k(M), \mathbb{R})$ by $S^\ast_k(M) = \{(S^\ast_k(M), \delta) \mid k = 0, 1, 2, \cdots\}$ and call this the $C^\infty$ singular cochain complex of $M$ with coefficients in $\mathbb{R}$. Let $\omega \in \Omega^k(M)$. Then we can define an $\mathbb{R}$-linear map

$$
I(\omega) : \quad S^\infty_k \longrightarrow \mathbb{R},
\quad \sigma \longmapsto \sum_i r_i \int_{\Delta^k} \sigma_i^* \omega,
$$

where

$$
\sigma = \sum_i r_i \sigma_i \in S^\infty_k(M), \quad r_i \in \mathbb{R},
\quad \sigma_i : \Delta^k \longrightarrow M \quad \text{is of class $C^\infty$.}
$$

3
Hence we obtain a map

\[ I : \Omega^k(M) \longrightarrow S^k_\infty(M), \]
\[ \omega \longmapsto I(\omega) \]

for \( k = 0, 1, 2, \cdots \). By properties of integrals, the map \( I \) is \( \mathbb{R} \)-linear. Furthermore, let \( c \in S^\infty_{k+1}(M) \). Then by the Stokes Theorem on chains,

\[ I(d\omega)(c) = \int_c d\omega = \int_{\partial c} \omega = I(\omega)\partial c = (\delta I(\omega))c. \]

Hence the following diagram commutes

\[
\begin{array}{ccc}
\Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M) \\
\downarrow I & & \downarrow I \\
S^k_\infty(M) & \xrightarrow{\delta} & S^{k+1}_\infty(M).
\end{array}
\]

Therefore, the map \( I \) induces a group homomorphism

\[ I : H^*_{DR}(M) \longrightarrow H^*(S_\infty^*(M)). \]

**Lemma 1** (page 104 of [3]). Let \( M \) be an \( n \)-dimensional manifold. Let \( S_\ast(M) \) be the singular chain complex of \( M \) with coefficients in \( \mathbb{R} \). Then the homology group of this chain complex is \( H_\ast(M; \mathbb{R}) \). The inclusion map \( i : S^\infty_* (M) \hookrightarrow S_\ast(M) \) induces a natural isomorphism

\[ H_\ast(S^\infty_* (M)) \cong H_\ast(S_\ast(M)) = H_\ast(M; \mathbb{R}). \]

Therefore, \( H^\ast(S^\infty_* (M)) \) is naturally isomorphic to the singular cohomology group \( H^\ast(M; \mathbb{R}) \) of \( M \).
Theorem 2 (de Rham Theorem, 114, 115, 131 of [3]). Let $M$ be an $n$-dimensional manifold. Then the homomorphism $I$ is a group isomorphism

$$I : H^*_\text{DR}(M) \cong H^*(S^*_\infty(M)).$$

Hence $H^*_\text{DR}(M) \cong H^*(M; \mathbb{R})$ as groups. Moreover, if we denote the group isomorphism from $H^*(S^*_\infty(M))$ to $H^*(M; \mathbb{R})$ as $I'$, then $I'$ preserves the product structure

$$H^*_\text{DR}(M) \xrightarrow{I'} H^*(M; \mathbb{R})$$

$$\wedge \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
Such an assignment is a $k$-cochain of $X$ with coefficients in $\mathbb{R}$ with respect to $\mathcal{U}$. Let $\tilde{C}^k(X;\mathbb{R};\mathcal{U})$ be the collection of all $k$-cochains. Then $\tilde{C}^k(X;\mathbb{R};\mathcal{U})$ is a vector space over $\mathbb{R}$. Let $\delta : \tilde{C}^k(X;\mathbb{R};\mathcal{U}) \to \tilde{C}^{k+1}(X;\mathbb{R};\mathcal{U})$
\[
(\delta c)(\alpha_0, \alpha_1, \cdots, \alpha_{k+1}) = \sum_{i=0}^{k+1} (-1)^i c(\alpha_0, \cdots, \hat{\alpha}_i, \cdots, \alpha_{k+1}),
\]
for any $c \in \tilde{C}^k(X;\mathbb{R};\mathcal{U})$.

Then $\delta \circ \delta = 0$ and we obtain the cochain complex $\{(\tilde{C}^k(X;\mathbb{R};\mathcal{U}), \delta) \mid k = 0, 1, \cdots \}$ with
\[
\check{H}^k(X;\mathbb{R};\mathcal{U}) = \frac{\text{Ker}(\delta : \tilde{C}^k(X;\mathbb{R};\mathcal{U}) \to \tilde{C}^{k+1}(X;\mathbb{R};\mathcal{U}))}{\text{Im}(\delta : \tilde{C}^{k-1}(X;\mathbb{R};\mathcal{U}) \to \tilde{C}^k(X;\mathbb{R};\mathcal{U}))}.
\]

**Definition 1.** Let $X$ be a topological space and $\mathcal{U} = \{U_\alpha\}$ an open covering of $X$. Then $\mathcal{U}$ is called a contractible open covering if the intersections of a finite number of open sets $U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}$ are all contractible.

**Theorem 3** (page 121 of [3]). Let $M$ be an $n$-dimensional manifold. Then
(1). there exists a contractible open covering $\mathcal{U}$ of $M$;
(2). for an arbitrary contractible open covering $\mathcal{U}$ of $M$, we have an $\mathbb{R}$-algebra natural isomorphism
\[
H^*_\text{DR}(M) \cong \check{H}^*(M;\mathbb{R};\mathcal{U}).
\]

**Corollary 4.** Let $M$ be an $n$-dimensional compact manifold. Then $H^*(M;\mathbb{R})$ is finite-dimensional.

**Proof of Corollary.** Let $\mathcal{U}$ be a contractible open covering of $M$. Since $M$ is compact, there exists a finite subset $\mathcal{V} = \{U_1, \cdots, U_k\} \subseteq \mathcal{U}$ such that $\mathcal{V}$ is an open covering of $M$. Since $\mathcal{U}$ is contractible, $\mathcal{V}$ is still contractible. Hence
\[
\check{H}^*(M;\mathbb{R};\mathcal{V}) \cong H^*_\text{DR}(M) \cong \check{H}^*(M;\mathbb{R};\mathcal{U}).
\]

Since $\mathcal{V}$ is a finite set, we obtain that for each $i = 0, 1, \cdots, k$, $\check{H}^i(M;\mathbb{R};\mathcal{V})$ is finite dimensional. Thus their direct sum $\check{H}^*(M;\mathbb{R};\mathcal{V})$ is finite dimensional and hence is $H^*(M;\mathbb{R})$. \qed
Remark 5. The converse is not true in general ($\mathbb{R}^n$ is a counterexample). Can we strengthen the corollary to the statements:

(i). Any compact manifold $M$ is homeomorphic to a finite CW-complex;

(ii). Any compact manifold $M$ is homotopy equivalent to a finite CW-complex? I will study later.

3 Hodge theorem

In this section we assume that $M$ is an oriented compact $n$-dimensional Riemannian manifold without boundary.

For $k = 0, 1, \cdots, n$, let the collection of all harmonic $k$-forms on $M$ be

$$H^k(M) = \{ \omega \in \Omega^k(M) \mid \Delta \omega = 0 \}.$$ 

Let $\omega \in H^k(M)$. Then $\delta \omega = d\omega = 0$. Hence we can define a map

$$j : H^k(M) \longrightarrow H^k_{DR}(M),$$

$$\omega \mapsto \omega + B^k(M).$$

Furthermore, if $\omega + B^k(M) = 0$, then $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(M)$. Thus

$$(\omega, \omega) = (d\eta, \omega) = (\eta, \delta \omega) = (\eta, 0) = 0$$

which means $\omega = 0$. Hence $i$ is injective. It is direct to see that $i$ is $\mathbb{R}$-linear.

By Corollary 4, $H^k_{DR}(M)$ is finite dimensional. Hence $H^k(M)$ is also finite dimensional.

We consider $H^k(M)$ together with another two spaces $d\Omega^{k-1}(M), \delta \Omega^{k+1}(M)$.

Let $\omega \in H^k(M), \eta \in \Omega^{k-1}(M), \theta \in \Omega^{k+1}(M)$. Then

$$(\omega, d\eta) = (\delta \omega, \eta) = 0,$$

$$(\omega, \delta \theta) = (d\omega, \theta) = 0,$$

$$(d\eta, \delta \theta) = (d^2 \eta, \theta) = 0.$$ 

Hence $H^k(M), d\Omega^{k-1}(M)$ and $\delta \Omega^{k+1}(M)$ are pairwise orthogonal. Thus their sum is a direct sum.
Let \( \omega \in \Omega^k(M) \). Suppose \( \omega \) is orthogonal to \( \mathbb{H}^k(M) \oplus d\Omega^{k-1}(M) \oplus \delta\Omega^{k+1}(M) \). Then for any \( \eta \in \Omega^{k-1}(M) \), \( (\delta \omega, \eta) = 0 \). And for any \( \theta \in \Omega^{k+1}(M) \), \( (d \omega, \theta) = 0 \). Hence \( \omega \in \mathbb{H}^k(M) \). Hence by our assumption, \( \omega = 0 \). Therefore,

\[
\{ \mathbb{H}^k(M) \oplus d\Omega^{k-1}(M) \oplus \delta\Omega^{k+1}(M) \}^\perp = \{ 0 \}.
\]

**Remark 6.** Since \( \Omega^k(M) \) is infinite dimensional over \( \mathbb{R} \), we cannot conclude from the above that the direct sum is \( \Omega^k(M) \). An example that a proper subspace of an inner product space whose orthogonal complement is zero is given as follows.

**Example 2.** Let \((\Omega, \mu)\) be a measure space. Let \( L^2(\Omega, \mu) \) be the vector space of all square-integrable functions from \( \Omega \) to \( \mathbb{R} \). Define the inner product on \( L^2(\Omega, \mu) \) by \( (f, g) = \int fg d\mu \) for \( f, g \in L^2(\Omega, \mu) \). Then

\[
|(f, g)| \leq \left( \int f^2 d\mu \right)^{\frac{1}{2}} \left( \int g^2 d\mu \right)^{\frac{1}{2}} < \infty.
\]

Let \( V \) be the subspace of all simple functions in \( L^2(\Omega, \mu) \). If \( f \) is not zero on a subset of positive measure, then there exists integer \( n \geq 1 \) and measurable subset \( E \subseteq \Omega \) such that \( \mu(E) > 0 \) and \( f(x) \geq \frac{1}{n} \) (or \( f(x) \leq -\frac{1}{n} \), in which case we replace \( f \) with \(-f\)) for any \( x \in E \). Let \( g = n\chi_E \in V \). Then \( (f, g) = \int f g d\mu = \int_E nf d\mu \geq \mu(E) > 0 \). Therefore, for any \( h \in V^\perp \), \( h \) is zero almost everywhere. This means \( V^\perp = 0 \). On the other hand, we can choose \((\Omega, \mu)\) suitable (for example, we can choose \( \mathbb{R}^k \) with Lebesgue measure) such that there exists a sequence of pairwise disjoint measurable subsets \( \{ E_n \}_{n=1}^\infty \) with \( 0 < \mu(E_n) < \infty \). Let \( g = \sum_{n=1}^\infty \frac{1}{n^2 \mu(E_n)} \chi_{E_n} \). Then \( g \in L^2(\Omega, \mu) \) but \( g \not\in V \). Hence \( V \) is a proper subspace with \( V^\perp = 0 \).

Nevertheless, the following theorems hold.

**Theorem 7** (Hodge decomposition, page 160 of [3]). *An arbitrary k-form can be uniquely written as the sum of a harmonic form, an exact form and a dual exact form. That is, we have the orthogonal direct sum*

\[
\Omega^k(M) = \mathbb{H}^k(M) \oplus d\Omega^{k-1}(M) \oplus \delta\Omega^{k+1}(M).
\]
Let $\omega \in Z^k(M)$. Then $\omega$ has the decomposition

$$\omega = \omega_H + d\eta + \delta \theta, \quad \omega_H \in \mathbb{H}^k(M), \eta \in \Omega^{k-1}(M), \theta \in \Omega^{k+1}(M).$$

Since $\omega \in Z^k(M)$,

$$0 = (d\omega = d\omega_H + d^2\eta + d\delta \theta, \theta) = (d\delta \theta, \theta) = (\delta \theta, \delta \theta).$$

Hence $\delta \theta = 0$. Hence $\omega = \omega_H + d\eta$. Hence $j(\omega_H) = \omega_H + B^k(M) = \omega + B^k(M)$. Hence $j$ is surjective. On the other hand, by the direct sum decomposition, $\mathbb{H}^k(M) \cap B^k(M) = \{0\}$. Hence $j$ is injective.

**Theorem 8** (Hodge theorem, page 159 of [3]). The map $j : \mathbb{H}^k(M) \rightarrow H^k_{DR}(M)$ is an isomorphism.

**Corollary 9.** Let $M$ be an oriented compact $n$-dimensional Riemannian manifold without boundary. Then $H^n_{DR}(M) = \mathbb{R}$.

**Proof.** By the last proposition of reading report I, we see that $\mathbb{H}^n(M) = \text{Span}_\mathbb{R}\{v_M\} \cong \mathbb{R}$. Hence $H^n_{DR}(M) \cong \mathbb{H}^n(M) \cong \mathbb{R}$. $\square$

In fact, we can generalize this corollary to get the Poincaré duality of cohomology groups of $M$. For $0 \leq k \leq n$, we define a $\mathbb{R}$-bilinear map

$$\Phi : H^k_{DR}(M) \times H^{n-k}_{DR}(M) \rightarrow \mathbb{R},$$

$$([\omega], [\eta]) \mapsto \int_M \omega \wedge \eta.$$

Here $\omega$ and $\eta$ are closed $k$-form and $(n-k)$-form representing elements in $H^k_{DR}(M)$ and $H^{n-k}_{DR}(M)$ respectively. This map is well-defined since for $\omega' = \omega + d\alpha$ and $\eta' = \eta + d\beta$, it can be verified by Stokes’ Theorem that $\int_M \omega' \wedge \eta' = \int_M \omega \wedge \eta$.

Now we equip a Riemannian metric $g$ on $M$. Let an arbitrary nonzero $[\omega] \in H^k_{DR}(M)$. Then by Hodge Theorem, we may assume $\omega$ harmonic. Let $\eta = *\omega$. Then since $\Delta$ and $*$ commutes, $\eta \in \mathbb{H}^{n-k}(M)$. Hence

$$\Phi([\omega], [\eta]) = \int_M \omega \wedge \eta = \int_M \omega \wedge *\omega = (\omega, \omega) > 0.$$
Therefore, $\Phi$ is a non-degenerate bilinear form. It follows that $\Phi$ induces a linear isomorphism (called the Poincaré duality)

$$\Phi' : H^{n-k}_{DR}(M) \rightarrow H^k_{DR}(M)^* = H_k(M; \mathbb{R}),$$

$$[\eta] \mapsto \Phi(\cdot, [\eta]).$$

Since $H^k_{DR}(M)$ is finite-dimensional, $H^k_{DR}(M)^* \cong H^k_{DR}(M)$. Hence as vector spaces over $\mathbb{R}$ (depending on choices of bases),

$$H^{n-k}_{DR}(M) \cong H^k_{DR}(M).$$

Now we turn to define the Euler characteristic.

Let $G$ be an abelian group. The rank $\text{rank}(G)$ of $G$ is defined to be the size of the largest set of $\mathbb{Z}$-linearly independent elements in $G$. In particular, when $G$ is finitely generated, by the fundamental theorem of finitely generated abelian groups, $G$ can be uniquely expressed as

$$G = \mathbb{Z}^r \times \mathbb{Z}/p_1^{a_1} \times \cdots \times \mathbb{Z}/p_n^{a_n},$$

where $p_1 \leq \cdots \leq p_n$ are primes and $r \geq 0, a_1, \cdots, a_n \geq 1$, are integers such that $a_i \leq a_{i+1}$ whenever $p_i = p_{i+1}$. In this case, $\text{rank}(G) = r$.

**Definition 3** (page 146 of [1]). For a finite CW-complex $X$, the Euler characteristic $\chi(X)$ is defined to be the alternating sum $\sum_n (-1)^n c_n$ where $c_n$ is the number of $n$-cells of $X$. Moreover, it can be proved

$$\chi(X) = \sum_n (-1)^n \text{rank} H_n(X; \mathbb{Z}) = \sum_n (-1)^n \dim H_n(X; \mathbb{R})$$

is independent of the choice of CW-structures of $X$ hence is well-defined.

**Corollary 10.** Let $M$ be an oriented compact $n$-dimensional Riemannian manifold without boundary. If $n$ is odd, then $\chi(M) = 0$.

**Proof.** From the above discussion, $\chi(X) = \sum_{k=0}^n (-1)^k \dim H_k(M; \mathbb{R}) = \sum_{k=0}^n (-1)^k \dim H^k(M; \mathbb{R}) = \sum_{k=0}^n (-1)^k \dim H^k_{DR}(M)$. Since $H^{n-k}_{DR}(M) \cong H^k_{DR}(M)$ for all $k = 0, 1, \cdots, n$ and $n$ is odd, the sum must be zero. $\square$
4 Applications

In this section we will discuss some applications of de Rham Theorem and Hodge Theorem.

4.1 Mapping degree

In this subsection we talk about the mapping degree of maps between manifolds.

Let $M, N$ be compact oriented $n$-dimensional manifolds, where $M$ is assumed to be path-connected and $N$ is not assumed to be path-connected. Let $N = N_1 \sqcup \cdots \sqcup N_k$, where $N_1, \cdots, N_k$ are the path-connected components of $N$. Let a map $f : N \to M$. We denote $f_i = f|_{N_i}$ for $i = 1, \cdots, k$.

For each $i$, $f_i^* : \Omega^*(M) \to \Omega^*(N_i)$ induces an algebra homomorphism $f_i^* : H^*(M; \mathbb{R}) \to H^*(N_i; \mathbb{R})$. In particular, we define the number $\deg f_i$, called the degree of $f_i$, by the following commutative diagram

$$
\begin{array}{ccc}
H^n(M; \mathbb{R}) & \xrightarrow{f_i^*} & H^n(N_i; \mathbb{R}) \\
\cong & & \cong \\
\mathbb{R} & \xrightarrow{\text{scalar multiplication by } \deg f_i} & \mathbb{R}
\end{array}
$$

We define the degree of $f$ to be the number

$$
\deg f = \sum_{i=1}^k \deg f_i.
$$

Let $\omega \in \Omega^n(M)$. Then from the commutative diagram and with the help of Hodge decomposition, we have

$$
\int_N f^* \omega = \sum_{i=1}^k \int_{N_i} f_i^* \omega = \sum_{i=1}^k \deg f_i \int_M \omega = \deg f \int_M \omega.
$$
Hence if we choose \( \omega \) to be non-vanishing on \( M \), then

\[
\deg f = \frac{\int_N f^* \omega}{\int_M \omega}.
\]

**Remark 11.** If we choose the coefficients to be \( \mathbb{Z} \), we still have the commutative diagram

\[
\begin{array}{ccc}
H^n(M; \mathbb{Z}) & \xrightarrow{f_i^*} & H^n(N_i; \mathbb{Z}) \\
\cong & & \cong \\
\mathbb{Z} & \text{scalar multiplication by} & \mathbb{Z} \\
\downarrow & deg f_i & \downarrow \\
\end{array}
\]

Hence \( \deg f_i \) as well as \( \deg f \) are integers in fact. In contrast with Hopf invariant in the next subsection, we consider the special case \( f : S^n \rightarrow S^n \):

\[
\begin{array}{ccc}
H^n(S^n; \mathbb{Z}) & \xrightarrow{f^*} & H^n(S^n; \mathbb{Z}) \\
\cong & & \cong \\
\mathbb{Z} & \text{deg f} & \mathbb{Z} \\
\downarrow & \downarrow & \downarrow \\
\end{array}
\]

\[
\begin{array}{ccc}
H_n(S^n; \mathbb{Z}) & \xrightarrow{f_*} & H_n(S^n; \mathbb{Z}) \\
\cong & & \cong \\
\mathbb{Z} & \text{deg f} & \mathbb{Z} \\
\downarrow & \downarrow & \downarrow \\
\end{array}
\]

### 4.2 Hopf invariant

In this subsection we talk about Hopf invariant. We state two methods to construct Hopf invariant.

**Construction 1 (page 427-428 of [1])**. Let a continuous map \( f : S^m \rightarrow S^n \) with \( m \geq n \). We can form a CW-complex \( C_f \) by attaching a cell \( e^{m+1} \) to \( S^n \) via \( f \). The homotopy type of \( C_f \) depends only on the homotopy class of \( f \). Thus for maps \( f, g : S^n \rightarrow S^n \), any invariant of homotopy type that distinguishes \( C_f \) from \( C_g \) will show that \( f \) is not homotopic to \( g \).

In particular, when \( m = n \), from the cellular chain complex of \( C_f \), we have \( H_n(C_f; \mathbb{Z}) = \mathbb{Z}/|\deg f| \). Hence we can use \( \deg f \), up to sign, as such an invariant distinguishing \( C_f \) from \( C_g \).
Now let $m = 2n - 1$. Then
\[
H^n(C_f; \mathbb{Z}) = \text{Span}_{\mathbb{Z}}\{\alpha\} \cong \mathbb{Z},
\]
\[
H^{2n}(C_f; \mathbb{Z}) = \text{Span}_{\mathbb{Z}}\{\beta\} \cong \mathbb{Z},
\]
\[
H^i(C_f; \mathbb{Z}) = 0, \text{ for } i \neq 0, n, 2n,
\]
\[
H^*(C_f; \mathbb{Z}) = \text{Algebra}_{\mathbb{Z}}\{1, \alpha, \beta \mid \alpha \sim \alpha = H(f)\beta \text{ for some } H(f) \in \mathbb{Z}\}.
\]

Here we use the generalized definition of algebra over a commutative ring rather than merely over a field.

The map $f$ induces a map $\tilde{f} : (D^{2n}, \partial D^{2n} = S^{2n-1}) \rightarrow (C_f, S^n)$. This induces a group homomorphism $\tilde{f}^* : H^{2n}(C_f, S^n; \mathbb{Z}) \rightarrow H^{2n}(D^{2n}, S^{2n-1}; \mathbb{Z})$.

We consider the characteristic map of the cell $e^{2n}$
\[
\chi : (D^{2n}, S^{2n-1}) \rightarrow (C_f, S^n),
\]
\[
\chi|_{e^{2n}} = \text{Id} : e^{2n} \rightarrow e^{2n},
\]
\[
\chi|_{S^{2n-1}} = f.
\]

Then $\chi$ induces an isomorphism
\[
H^{2n}(C_f, S^n; \mathbb{Z}) = \tilde{H}^{2n}(C_f/S^n; \mathbb{Z}) = \tilde{H}^{2n}(S^n; \mathbb{Z}) = \mathbb{Z}
\]
\[
\chi^* : H^{2n}(D^{2n}, S^{2n-1}; \mathbb{Z}) = \tilde{H}^{2n}(D^{2n}/S^{2n-1}; \mathbb{Z}) = \tilde{H}^{2n}(S^n; \mathbb{Z}) = \mathbb{Z}.
\]

Moreover, since $n \geq 2$, we have the isomorphism
\[
\phi : H^{2n}(C_f; \mathbb{Z}) \xrightarrow{\cong} H^{2n}(C_f, S^n; \mathbb{Z}).
\]

Fix a generator $\sigma$ of $H^{2n}(D^{2n}, S^{2n-1}; \mathbb{Z})$. Then $\phi^{-1}\chi^*\sigma$ is a generator of $H^{2n}(C_f; \mathbb{Z})$. We choose $\beta = \phi^{-1}\chi^*\sigma$. Hence once the generator of $H^{2n}(D^{2n}, S^{2n-1}; \mathbb{Z})$ is fixed, $\beta$ is uniquely determined.

**Definition 4.** The integer $H(f)$ is called the Hopf invariant of $f$. It only depends on the homotopy class of $f$ since it is determined by the cohomology algebra. Hence the Hopf invariant $H$ is an assignment of integers $H([f])$ to each element $[f] \in \pi_{2n-1}(S^n)$.

**Proposition 12** (page 428 of [1]). The Hopf invariant $H : \pi_{2n-1}(S^n) \rightarrow \mathbb{Z}$ is a group homomorphism.
Construction 2 (page 133-135 of [3]). Let a $C^\infty$ map $f : S^m \to S^n$ with $m \geq n$. Let $\theta \in \Omega^n(S^n)$ such that $\int_{S^n} \theta = 1$. Then since $d(f^*\theta) = f^*(d\theta) = 0$, $f^*\theta$ is a closed $n$-form on $S^m$.

Let $m = 2n - 1$ and $n \geq 2$. Then since $H^n(S^{2n-1}; \mathbb{R}) = 0$, by de Rham Theorem, there exists a $(n - 1)$-form $\eta \in \Omega^{n-1}(S^{2n-1})$ such that $f^*\theta = d\eta$. Let

$$H(f) = \int_{S^{2n-1}} \eta \wedge d\eta.$$ 

Then the real number $H(f)$ is called the Hopf invariant of $f$. We need to verify that $H(f)$ is well-defined, i.e., it does not depend on the choice of $\theta$ and $\eta$.

Let $\theta' \in \Omega^n(S^n)$ such that $\int_{S^n} \theta' = 1$. Let $f^*\theta' = d\eta'$. Then since $H^n(S^n; \mathbb{R}) = \mathbb{R}$, by de Rham Theorem, there exists $\tau \in \Omega^{n-1}(S^n)$ such that $\theta' = \theta + d\tau$. Then since $d(\eta' - \eta - f^*\tau) = f^*(\theta' - \theta - d\tau) = 0$, and $H^{n-1}(S^{2n-1}; \mathbb{R}) = 0$, by de Rham Theorem, there exists $\zeta \in \Omega^{n-2}(S^{2n-1})$ such that $\eta' - \eta - f^*\tau = d\zeta$. Therefore

$$\eta' \wedge d\eta' = (\eta + f^*\tau + d\zeta) \wedge (\eta + f^*\tau + d\zeta)$$
$$= (\eta + f^*\tau + d\zeta) \wedge (d\eta + f^*d\tau)$$
$$= \eta \wedge d\eta + \eta \wedge (f^*\tau + f^*\tau) \wedge (f^*\theta + f^*d\tau) + d\zeta \wedge (d\eta + f^*d\tau)$$
$$= \eta \wedge d\eta + \eta \wedge (f^*(\tau \wedge (\theta + d\tau))) + d(\zeta \wedge (d\eta + f^*d\tau))$$
$$= \eta \wedge d\eta + \eta \wedge (f^*(\tau + d\zeta)) + d(\zeta \wedge (d\eta + f^*d\tau)).$$

The last equality follows because $\tau \wedge (\theta + d\tau) \in \Omega^{2n-1}(S^n)$ hence is zero. By Stokes’ Theorem,

$$\int_{S^{2n-1}} \eta' \wedge d\eta' = \int_{S^{2n-1}} \eta \wedge d\eta + \int_{S^{2n-1}} \eta \wedge (f^*\tau)$$
$$= \int_{S^{2n-1}} \eta \wedge d\eta + \int_{S^{2n-1}} (-1)^{n-1} (d\eta) \wedge f^*\tau$$
$$= \int_{S^{2n-1}} \eta \wedge d\eta + (-1)^{n-1} \int_{S^{2n-1}} f^*(\theta \wedge \tau)$$
$$= \int_{S^{2n-1}} \eta \wedge d\eta.$$
Hence $H(f)$ does not depend on the choice of $\theta$ and $\eta$ hence is well-defined.

Next we need to verify that $H(f)$ is a homotopy invariant. That is, if two $C^\infty$ maps $f_0, f_1 : S^{2n-1} \to S^n$ are homotopic, then $H(f_0) = H(f_1)$.

By our assumption, there exists a continuous map $F : S^{2n-1} \times [0, 1] \to S^n$ such that $F(p, 0) = f_0(p), F(p, 1) = f_1(p)$ for any $p \in S^{2n-1}$. A continuous map can be approximated by a $C^\infty$ map. Hence without loss of generality, we may assume $F$ is of class $C^\infty$. By Kunneth formula, $H^n(S^{2n-1} \times [0, 1]; \mathbb{R}) = 0$. Hence by de Rham Theorem, there exists $\tilde{\eta} \in \Omega^{n-1}(S^{2n-1} \times \mathbb{R})$ such that $d\tilde{\eta} = F^*\theta$. Let $i_0, i_1$ be defined by the commutative diagram

\[
\begin{array}{ccc}
S^{2n-1} \times \{1\} & \xrightarrow{i_1} & S^{2n-1} \times [0, 1] \\
\downarrow F & & \downarrow F \\
S^{2n-1} \times \{0\} & \xrightarrow{i_0} & S^n
\end{array}
\]

Then $d(i_0^*\tilde{\eta}) = i_0^*(d\tilde{\eta}) = i_0^*(F^*\theta) = f_0^*\theta$, $d(i_1^*\tilde{\eta}) = i_1^*(d\tilde{\eta}) = i_1^*(F^*\theta) = f_1^*\theta$.

By applying Stokes’ Theorem, we obtain

\[
0 = \int_{S^{2n-1} \times [0, 1]} F^*(\theta \wedge \theta) = \int_{S^{2n-1} \times [0, 1]} F^*\theta \wedge F^*\theta = \int_{S^{2n-1} \times [0, 1]} d(i_0^*\tilde{\eta}) = \int_{S^{2n-1} \times [0, 1]} \tilde{\eta} \wedge d\tilde{\eta} = \int_{\partial(S^{2n-1} \times [0, 1])} i_0^*\tilde{\eta} \wedge d_i^*\tilde{\eta} = \int_{S^{2n-1} \times \{0\}} i_0^*\tilde{\eta} \wedge \partial i_0^*\tilde{\eta} = H(f_1) - H(f_0).
\]

Therefore we have proved $H(f)$ is homotopy invariant. $\square$

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Question. Why the Hopf invariants constructed by these two methods are equivalent? I may study in the future.

4.3 Linking number

In this subsection we will discuss linking number of submanifolds in Euclidean spaces.

Definition 5 (page 102 of [2]). Let \( J^d \) and \( K^l \) be two disjoint compact oriented path-connected submanifolds of \( \mathbb{R}^{n+1} \), where \( \dim J^d = d \), \( \dim K^l = l \), \( d + l = n \). The linking number is the integer

\[
\text{lk}(J, K) = \deg(\Psi_{J,K})
\]

where

\[
\Psi_{J,K} : J \times K \to S^n \subset \mathbb{R}^{n+1}, \quad (x, y) \mapsto \frac{y - x}{\|y - x\|}.
\]

Here \( J \times K \) is equipped with the product orientation and \( S^n \) is oriented as the boundary of \( D^{n+1} \subset \mathbb{R}^{n+1} \) with the standard orientation of \( \mathbb{R}^{n+1} \).

Proposition 13 (page 103 of [2]). The linking number has the following properties:

(i). \( \text{lk}(J, K) \) changes sign when the orientation of either \( J \) or \( K \) is reversed.

(ii). \( \text{lk}(J, K) = (-1)^{(d+1)(l+1)} \text{lk}(K, J) \).

(iii). If \( J \) and \( K \) are separated by a hyperplane in \( \mathbb{R}^{n+1} \) then \( \text{lk}(J, K) = 0 \).

(iv). Let \( g_t \) and \( h_t \) be homotopies of the inclusions \( g_0 : J \to \mathbb{R}^{n+1} \) and \( h_0 : K \to \mathbb{R}^{n+1} \) to smooth embeddings \( g_1 \) and \( h_1 \) respectively. If \( g_t(J) \cap h_t(K) = \emptyset \) for all \( t \in [0,1] \), then \( \text{lk}(J, K) = \text{lk}(g_1(J), h_1(K)) \).

Now we assume \( n = 2 \) and \( J, K \) are disjoint submanifolds of \( \mathbb{R}^3 \) diffeomorphic to \( S^1 \). Let \( \alpha : [0,1] \to J, \beta : [0,1] \to K \) be parametrizations such that the smallest \( t \in [0,1] \) (resp. \( s \in [0,1] \)) satisfying \( \alpha(t) = \alpha(0) \) (resp. \( \beta(s) = \beta(0) \)) is \( t = 1 \) (resp. \( s = 1 \)) and \( |\alpha'(t)| \) (resp. \( |\beta'(s)| \)) is constant. Let
\[ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \text{ and } \alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t)) \text{ for all } t \in [0, 1] \text{ (resp. } \beta(s) = (\beta_1(s), \beta_2(s), \beta_3(s)) \text{ for all } s \in [0, 1]). \text{ Let}
\]
\[ \omega = \frac{1}{||x||^3} (x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2) \]
and denote the restriction of \( \omega \) to \( S^2 \) by \( \omega_0 \). Then
\[
\begin{align*}
d\omega &= - \frac{3d||x||}{||x||^4} (x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2) \\
&\quad + \frac{1}{||x||^3} (dx_1 \wedge dx_2 \wedge dx_3 - dx_2 \wedge dx_1 \wedge dx_3 + dx_3 \wedge dx_1 \wedge dx_2) \\
&= - \frac{3}{||x||^3} (x_1 dx_1 + x_2 dx_2 + x_3 dx_3) \cdot \\
&(x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2) \\
&\quad + \frac{3}{||x||^3} dx_1 \wedge dx_2 \wedge dx_3 \\
&= 0,
\end{align*}
\]
\[
\int_{S^2} \omega = \int_{S^2} \omega_0 \\
= \int_{\partial D^3} (x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2) \\
= \int_{D^3} d(x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2) \\
= 3 \int_{D^3} dx_1 \wedge dx_2 \wedge dx_3 \\
= 4\pi.
\]
Since $\Psi : J \times K \cong S^1 \times S^1 = T^2 \rightarrow S^2$, we have
\[
\deg \Psi = \frac{\int_{T^2} \Psi^* \omega}{\int_{S^2} \omega} = \frac{1}{4\pi \int_{T^2} \Psi^* \omega} = \frac{1}{4\pi \int_{T^2} \Psi^* \omega}.
\]
\[
= \frac{1}{4\pi} \int_{T^2} \left\{ \Psi^*(\frac{x_1}{||x||})(\alpha'_2 dt + \beta'_2 ds) \wedge (\alpha'_3 dt + \beta'_3 ds)
- \Psi^*(\frac{x_2}{||x||})(\alpha'_1 dt + \beta'_1 ds) \wedge (\alpha'_3 dt + \beta'_3 ds)
+ \Psi^*(\frac{x_3}{||x||})(\alpha'_1 dt + \beta'_1 ds) \wedge (\alpha'_2 dt + \beta'_2 ds) \right\}
= \frac{1}{4\pi} \int_{T^2} \left\{ \frac{x_1(t,s)}{||x(t,s)||^2} (\alpha'_2 \beta'_3 - \alpha'_3 \beta'_2) dt \wedge ds
- \frac{x_2(t,s)}{||x(t,s)||^2} (\alpha'_1 \beta'_3 - \alpha'_3 \beta'_1) dt \wedge ds
+ \frac{x_3(t,s)}{||x(t,s)||^2} (\alpha'_1 \beta'_2 - \alpha'_2 \beta'_1) dt \wedge ds \right\}
= \frac{1}{4\pi} \int_{T^2} \left( \frac{1}{||x(t,s)||^3} \begin{vmatrix} \alpha_1(t) - \beta_1(s) & \alpha'_1(t) & \beta'_1(s) \\ \alpha_2(t) - \beta_2(s) & \alpha'_2(t) & \beta'_2(s) \\ \alpha_3(t) - \beta_3(s) & \alpha'_3(t) & \beta'_3(s) \end{vmatrix} dt \wedge ds \right)
= \frac{1}{4\pi} \int_0^1 \left( \int_0^1 \frac{\det(\alpha(t) - \beta(s), \alpha'(t), \beta'(s))}{||\alpha(t) - \beta(s)||^3} dt \right) ds.
\]
Hence we obtain the explicit expression of $\text{lk}(J, K)$ for the case $J \cong S^1 \cong K$ in $\mathbb{R}^3$.

**Question.** For general $J^d, K^l$ in $\mathbb{R}^{n+1}$, $n = d + l$, is there any analogous formula? In the general case we are still able to let
\[
\omega = \frac{1}{||x||^{n+1}} \sum_{i=1}^{n+1} (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1}
\]
such that $d\omega = 0$ and compute the integral $\int_{S^n} \omega$. And $\Psi^* dx_i$ is still able to be computed, although it concerns partial derivatives of $d + l$ variables hence complicated. However, is it possible to get $\int_{K \times J} \Psi^* \omega$, since we do not know
the shape of $K$ and $L$ thus the integral domain is unknown? In particular, if $J = S^d$ and $K = S^l$, then is it possible to get the explicit expression of $\int_{S^d \times S^l} \Psi^* \omega$ by using polar coordinates on spheres, thus compute out the linking number? □

References

