Associative space - time sedenions

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We present an alternative type of sixteen-component hypercomplex scalar-vector values named "space-time sedenions", generating associative noncommutative space-time Clifford algebra.

1. Introduction

The eight-component hypercomplex numbers such as biquaternions and octonions [1-6] enclosing scalar, pseudoscalar, vector and pseudovector components are widely used for the generalization of equations of quantum mechanics and field theory. These structures take into account the space symmetry with respect to the spatial inversion. However, a consistent relativistic approach requires taking into consideration full time and space symmetries that leads to the sixteen-component space-time algebras.

The well known sixteen-component hypercomplex numbers, sedenions, are obtained from octonions by the Cayley-Dickson extension procedure [7,8]. In this case the sedenion is defined as

$$S = O_1 + O_2 \boldsymbol{e}, \tag{1}$$

where O_i is an octonion and the parameter of duplication $\boldsymbol{e}$ is similar to imaginary unit $\boldsymbol{e}^2 = -1$. The algebra of sedenions has the specific rules of multiplication. The product of two sedenions

$$S_1 = O_{11} + O_{12}\boldsymbol{e},$$

$$S_2 = O_{21} + O_{22}\boldsymbol{e},$$

is defined as

$$S_1 S_2 = \left(O_{11} + O_{12}\boldsymbol{e}\right)\left(O_{21} + O_{22}\boldsymbol{e}\right) = \left(O_{11}O_{21} - \bar{O}_{22}O_{12}\right) + \left(O_{22}O_{11} + O_{12}\bar{O}_{21}\right)\boldsymbol{e}, \tag{2}$$

where $\bar{O}_i$ is conjugated octonion. The sedenionic multiplication (2) allows one to introduce a well defined norm of sedenion. However such procedure of constructing the higher hypercomplex numbers leads to the fact that the sedenions as well as octonions generate normed but nonassociative algebra [9-11]. This greatly complicates the use of the Cayley-Dickson sedenions in the physical applications.

Recently we have developed an alternative approach to constructing the multicomponent values based on our scalar-vector conception realized in associative eight-component octons [12-14] and sixteen-component sedeons [15-19]. In this paper we present an alternative version of the sixteen-component associative space-time algebra.

2. Sedenionic space-time algebra

It is known, the quaternion is a four-component object

$$\hat{q} = q_0\mathbf{a_0} + q_1\mathbf{a_1} + q_2\mathbf{a_2} + q_3\mathbf{a_3}, \tag{3}$$

where components q_ν (Greek indexes $\nu = 0, 1, 2, 3$) are numbers (complex in general), $\mathbf{a_0} \equiv 1$ is scalar units and values $\mathbf{a_m}$ (Latin indexes $\mathrm{m} = 1, 2, 3$) are quaternionic units, which are interpreted as unit vectors. The rules of multiplication and commutation for $\mathbf{a_m}$ are presented in Table 1. We introduce also the space-time basis $\mathbf{e_t}, \mathbf{e_r}, \mathbf{e_{tr}}$, which is responsible for the space-time inversions. The indexes **t** and **r** indicate the transformations (**t** for time inversion and **r** for spatial inversion), which change the corresponding values. The value $\mathbf{e_0} \equiv 1$ is a scalar unit. For convenience we introduce numerical designations $\mathbf{e_1} \equiv \mathbf{e_t}$ (time scalar unit); $\mathbf{e_2} \equiv \mathbf{e_r}$ (space scalar unit) and $\mathbf{e_3} \equiv \mathbf{e_{tr}}$ (space-time scalar unit). The rules of multiplication and commutation for this basis we choose similar to the rules for quaternionic units (see Table 2).

Table 1.

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$
$\mathbf{a}_1$	-1	$\mathbf{a}_3$	$-\mathbf{a}_2$
$\mathbf{a}_2$	$-\mathbf{a}_3$	-1	$\mathbf{a}_1$
$\mathbf{a}_3$	$\mathbf{a}_2$	$-\mathbf{a}_1$	-1

Table 2.

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{e}_1$	-1	$\mathbf{e}_3$	$-\mathbf{e}_2$
$\mathbf{e}_2$	$-\mathbf{e}_3$	-1	$\mathbf{e}_1$
$\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{e}_1$	-1

Note that the unit vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and the space-time units $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ generate the anticommutative algebras:

$$\begin{aligned} \mathbf{a_n a_m} &= -\mathbf{a_m a_n}, \\ \mathbf{e_n e_m} &= -\mathbf{e_m e_n}, \end{aligned} \tag{4}$$

for $\mathbf{n} \neq \mathbf{m}$, but $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ commute with $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$:

$$\mathbf{e_n a_m} = \mathbf{a_m e_n}, \tag{5}$$

for any $\mathbf{n}$ and $\mathbf{m}$. Then we can introduce the sixteen-component space-time sedenion $\tilde{V}$ in the following form:

$$\begin{aligned} \tilde{V} = {} & \mathbf{e_0}\left(V_{00}\mathbf{a_0} + V_{01}\mathbf{a_1} + V_{02}\mathbf{a_2} + V_{03}\mathbf{a_3}\right) + \mathbf{e_1}\left(V_{10}\mathbf{a_0} + V_{11}\mathbf{a_1} + V_{12}\mathbf{a_2} + V_{13}\mathbf{a_3}\right) \\ & + \mathbf{e_2}\left(V_{20}\mathbf{a_0} + V_{21}\mathbf{a_1} + V_{22}\mathbf{a_2} + V_{23}\mathbf{a_3}\right) + \mathbf{e_3}\left(V_{30}\mathbf{a_0} + V_{31}\mathbf{a_1} + V_{32}\mathbf{a_2} + V_{33}\mathbf{a_3}\right). \end{aligned} \tag{6}$$

The sedenionic components $V_{\nu\mu}$ are numbers (complex in general). Introducing designation of scalar and vector values in accordance with the following relations

$$\begin{aligned} & V = \mathbf{e_0} V_{00} \mathbf{a_0}, \\ & \vec{V} = \mathbf{e_0}\left(V_{01}\mathbf{a_1} + V_{02}\mathbf{a_2} + V_{03}\mathbf{a_3}\right), \\ & V_{\mathbf{t}} \equiv V_{\mathbf{1}} = \mathbf{e_1} V_{10} \mathbf{a_0}, \\ & \vec{V}_{\mathbf{t}} \equiv \vec{V}_{\mathbf{1}} = \mathbf{e_1}\left(V_{11}\mathbf{a_1} + V_{12}\mathbf{a_2} + V_{13}\mathbf{a_3}\right), \\ & V_{\mathbf{r}} \equiv V_{\mathbf{2}} = \mathbf{e_2} V_{20} \mathbf{a_0}, \\ & \vec{V}_{\mathbf{r}} \equiv \vec{V}_{\mathbf{2}} = \mathbf{e_2}\left(V_{21}\mathbf{a_1} + V_{22}\mathbf{a_2} + V_{23}\mathbf{a_3}\right), \\ & V_{\mathbf{tr}} \equiv V_{\mathbf{3}} = \mathbf{e_3} V_{30} \mathbf{a_0}, \\ & \vec{V}_{\mathbf{tr}} \equiv \vec{V}_{\mathbf{3}} = \mathbf{e_3}\left(V_{31}\mathbf{a_1} + V_{32}\mathbf{a_2} + V_{33}\mathbf{a_3}\right), \end{aligned} \tag{7}$$

We can represent the sedenion in the following scalar-vector form:

$$\tilde{V} = V + \vec{V} + V_{\mathbf{t}} + \vec{V}_{\mathbf{t}} + V_{\mathbf{r}} + \vec{V}_{\mathbf{r}} + V_{\mathbf{tr}} + \vec{V}_{\mathbf{tr}} . \tag{8}$$

Thus, the sedenionic algebra encloses four groups of values, which are differed with respect to spatial and time inversion.

(1) Absolute scalars (V) and absolute vectors ($\vec{V}$) are not transformed under spatial and time inversion.

(2) Time scalars ($V_{\mathbf{t}}$) and time vectors ($\vec{V}_{\mathbf{t}}$) are changed (in sign) under time inversion and are not transformed under spatial inversion.

(3) Space scalars ($V_{\mathbf{r}}$) and space vectors ($\vec{V}_{\mathbf{r}}$) are changed under spatial inversion and are not transformed under time inversion.

(4) Space-time scalars ($V_{\mathbf{tr}}$) and space-time vectors ($\vec{V}_{\mathbf{tr}}$) are changed under spatial and time inversion.

Further we will use the symbol 1 instead units $\mathbf{a_0}$ and $\mathbf{e_0}$ for simplicity. Introducing the designations of scalar-vector values

$$\begin{aligned}
\bar{V}_0 &= V_{00} + V_{01}\mathbf{a_1} + V_{02}\mathbf{a_2} + V_{03}\mathbf{a_3}, \\
\bar{V}_1 &= V_{10} + V_{11}\mathbf{a_1} + V_{12}\mathbf{a_2} + V_{13}\mathbf{a_3}, \\
\bar{V}_2 &= V_{20} + V_{21}\mathbf{a_1} + V_{22}\mathbf{a_2} + V_{23}\mathbf{a_3}, \\
\bar{V}_3 &= V_{30} + V_{31}\mathbf{a_1} + V_{32}\mathbf{a_2} + V_{33}\mathbf{a_3},
\end{aligned} \tag{9}$$

we can write the sedenion (6) in the following compact form

$$\tilde{\boldsymbol{V}} = \bar{\boldsymbol{V}}_0 + \mathbf{e_1}\bar{\boldsymbol{V}}_1 + \mathbf{e_2}\bar{\boldsymbol{V}}_2 + \mathbf{e_3}\bar{\boldsymbol{V}}_3 . \tag{10}$$

On the other hand, introducing designations of space-time sedenion-scalars

$$\begin{aligned}
\boldsymbol{V}_0 &= (V_{00} + \mathbf{e_1}V_{10} + \mathbf{e_2}V_{20} + \mathbf{e_3}V_{30}), \\
\boldsymbol{V}_1 &= (V_{01} + \mathbf{e_1}V_{11} + \mathbf{e_2}V_{21} + \mathbf{e_3}V_{31}), \\
\boldsymbol{V}_2 &= (V_{02} + \mathbf{e_1}V_{12} + \mathbf{e_2}V_{22} + \mathbf{e_3}V_{32}), \\
\boldsymbol{V}_3 &= (V_{03} + \mathbf{e_1}V_{13} + \mathbf{e_2}V_{23} + \mathbf{e_3}V_{33}),
\end{aligned} \tag{11}$$

we can write the sedenion (6) as

$$\tilde{\boldsymbol{V}} = \boldsymbol{V}_0 + \boldsymbol{V}_1\mathbf{a_1} + \boldsymbol{V}_2\mathbf{a_2} + \boldsymbol{V}_3\mathbf{a_3}, \tag{12}$$

or introducing the sedenion-vector

$$\vec{\boldsymbol{V}} = \vec{V} + \vec{V}_{\mathbf{t}} + \vec{V}_{\mathbf{r}} + \vec{V}_{\mathbf{tr}} = \boldsymbol{V}_1\mathbf{a_1} + \boldsymbol{V}_2\mathbf{a_2} + \boldsymbol{V}_3\mathbf{a_3}, \tag{13}$$

we can rewrite the sedenion in following compact form

$$\tilde{\boldsymbol{V}} = \boldsymbol{V}_0 + \vec{\boldsymbol{V}} . \tag{14}$$

Further we will indicate sedenion-scalars and sedenion-vectors with the bold capital letters.

Let us consider the sedenionic multiplication in detail. The sedenionic product of two sedenions $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{B}}$ can be represented in the following form

$$\tilde{\boldsymbol{A}}\tilde{\boldsymbol{B}} = \left(\boldsymbol{A}_0 + \vec{\boldsymbol{A}}\right)\left(\boldsymbol{B}_0 + \vec{\boldsymbol{B}}\right) = \boldsymbol{A}_0\boldsymbol{B}_0 + \boldsymbol{A}_0\vec{\boldsymbol{B}} + \vec{\boldsymbol{A}}\boldsymbol{B}_0 + \left(\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}}\right) + \left[\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}\right] \tag{15}$$

Here we denoted the sedenionic scalar multiplication of two sedenion-vectors (internal product) by symbol "$\cdot$" and round brackets

$$\left(\vec{A}\cdot\vec{B}\right)=-A_1B_1-A_2B_2-A_3B_3, \tag{16}$$

and sedenionic vector multiplication (external product) by symbol "×" and square brackets,

$$\left[\vec{A}\times\vec{B}\right]=\left(A_2B_3-A_3B_2\right)\mathbf{a_1}+\left(A_3B_1-A_1B_3\right)\mathbf{a_2}+\left(A_1B_2-A_2B_1\right)\mathbf{a_3}. \tag{17}$$

In (16) and (17) the multiplication of sedenionic components is performed in accordance with (11) and Table 2. Thus the sedenionic product

$$\tilde{F}=\tilde{A}\tilde{B}=F_0+\vec{F} \tag{18}$$

has the following components:

$$\begin{aligned} F_0 &= A_0B_0-A_1B_1-A_2B_2-A_3B_3, \\ F_1 &= A_1B_0+A_0B_1+\left(A_2B_3-A_3B_2\right), \\ F_2 &= A_2B_0+A_0B_2+\left(A_3B_1-A_1B_3\right), \\ F_3 &= A_3B_0+A_0B_3+\left(A_1B_2-A_2B_1\right). \end{aligned} \tag{19}$$

Note that in the sedenionic algebra the square of vector is defined as

$$\vec{A}^2=\left(\vec{A}\cdot\vec{A}\right)=-A_1^2-A_2^2-A_3^2, \tag{20}$$

and the square of modulus of vector is

$$\left|\vec{A}\right|^2=-\left(\vec{A}\cdot\vec{A}\right)=A_1^2+A_2^2+A_3^2. \tag{21}$$

3. Spatial rotation and space-time inversion

The rotation of sedenion $\tilde{V}$ on the angle θ around the absolute unit vector $\vec{n}$ is realized by uncompleted sedenion

$$\tilde{U}=\cos(\theta/2)+\vec{n}\sin(\theta/2) \tag{22}$$

and by conjugated sedenion $\tilde{U}^*$:

$$\tilde{U}^*=\cos(\theta/2)-\vec{n}\sin(\theta/2) \tag{23}$$

with

$$\tilde{U}\tilde{U}^*=\tilde{U}^*\tilde{U}=1. \tag{24}$$

The transformed sedenion $\tilde{V}'$ is defined as sedenionic product

$$\tilde{V}'=\tilde{U}^*\,\tilde{V}\,\tilde{U}\quad, \tag{25}$$

Thus the transformed sedenion $\tilde{V}'$ can be written as

$$\begin{aligned} \tilde{V}' &= \left[\cos(\theta/2)-\vec{n}\sin(\theta/2)\right]\left(V_0+\vec{V}\right)\left[\cos(\theta/2)+\vec{n}\sin(\theta/2)\right] \\ &= V_0+\vec{V}\cos\theta-\vec{n}\left(\vec{n}\cdot\vec{V}\right)(1-\cos\theta)-\left[\vec{n}\times\vec{V}\right]\sin\theta. \end{aligned} \tag{26}$$

It is clearly seen that rotation does not transform the sedenion-scalar part, but the sedenionic vector $\vec{V}$ is rotated on the angle θ around $\vec{n}$.

The operations of time inversion ($\hat{R}_{\mathbf{t}}$), space inversion ($\hat{R}_{\mathbf{r}}$) and space-time inversion ($\hat{R}_{\mathbf{tr}}$) are connected with transformations in $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ basis and can be presented as

$$\begin{aligned}
\hat{R}_{\mathrm{t}}\tilde{\boldsymbol{V}} &= -\mathbf{e}_2\tilde{\boldsymbol{V}}\mathbf{e}_2 = \bar{\boldsymbol{V}}_{\boldsymbol{0}} - \mathbf{e}_1\bar{\boldsymbol{V}}_{\boldsymbol{1}} + \mathbf{e}_2\bar{\boldsymbol{V}}_{\boldsymbol{2}} - \mathbf{e}_3\bar{\boldsymbol{V}}_{\boldsymbol{3}}, \\
\hat{R}_{\mathrm{r}}\tilde{\boldsymbol{V}} &= -\mathbf{e}_1\tilde{\boldsymbol{V}}\mathbf{e}_1 = \bar{\boldsymbol{V}}_{\boldsymbol{0}} + \mathbf{e}_1\bar{\boldsymbol{V}}_{\boldsymbol{1}} - \mathbf{e}_2\bar{\boldsymbol{V}}_{\boldsymbol{2}} - \mathbf{e}_3\bar{\boldsymbol{V}}_{\boldsymbol{3}}, \\
\hat{R}_{\mathrm{tr}}\tilde{\boldsymbol{V}} &= -\mathbf{e}_3\tilde{\boldsymbol{V}}\mathbf{e}_3 = \bar{\boldsymbol{V}}_{\boldsymbol{0}} - \mathbf{e}_1\bar{\boldsymbol{V}}_{\boldsymbol{1}} - \mathbf{e}_2\bar{\boldsymbol{V}}_{\boldsymbol{2}} + \mathbf{e}_3\bar{\boldsymbol{V}}_{\boldsymbol{3}}.
\end{aligned} \tag{27}$$

4. Sedenionic Lorentz transformations

The relativistic event four-vector can be represented in the follow sedenionic form:

$$\tilde{\boldsymbol{S}} = \mathbf{e}_1 ct + \mathbf{e}_2\vec{r}\,. \tag{28}$$

The square of this value is the Lorentz invariant

$$\tilde{\boldsymbol{S}}\,\tilde{\boldsymbol{S}} = -\,c^2t^2 + x^2 + y^2 + z^2. \tag{29}$$

The Lorentz transformation of event four-vector is realized by uncompleted sedenions

$$\begin{aligned}
\tilde{\boldsymbol{L}} &= \mathrm{ch}\,\vartheta + \mathbf{e}_3\vec{m}\,\mathrm{sh}\,\vartheta, \\
\tilde{\boldsymbol{L}}^* &= \mathrm{ch}\,\vartheta - \mathbf{e}_3\vec{m}\,\mathrm{sh}\,\vartheta,
\end{aligned} \tag{30}$$

where $\mathrm{th}\,2\vartheta = \mathrm{v}/c$, v is velocity of motion along the absolute unit vector $\vec{m}$. Note that

$$\tilde{\boldsymbol{L}}^*\tilde{\boldsymbol{L}} = \tilde{\boldsymbol{L}}\,\tilde{\boldsymbol{L}}^* = 1. \tag{31}$$

The transformed event four-vector $\tilde{\boldsymbol{S}}'$ is written as

$$\begin{aligned}
\tilde{\boldsymbol{S}}' = \tilde{\boldsymbol{L}}^*\,\tilde{\boldsymbol{S}}\,\tilde{\boldsymbol{L}} &= \left(\mathrm{ch}\,\vartheta - \mathbf{e}_3\mathrm{sh}\,\vartheta\,\vec{m}\right)\left(\mathbf{e}_1 ct + \mathbf{e}_2\vec{r}\right)\left(\mathrm{ch}\,\vartheta + \mathbf{e}_3\mathrm{sh}\,\vartheta\,\vec{m}\right) = \\
&\mathbf{e}_1 ct\,\mathrm{ch}2\vartheta + \mathbf{e}_1\left(\vec{m}\cdot\vec{r}\right)\mathrm{sh}2\vartheta \\
&+\mathbf{e}_2\vec{r} - \mathbf{e}_2 ct\vec{m}\,\mathrm{sh}2\vartheta - 2\mathbf{e}_2\left(\vec{m}\cdot\vec{r}\right)\vec{m}\,\mathrm{sh}^2\vartheta + \mathbf{e}_2\left(\vec{m}\cdot\vec{r}\right)\vec{m}\left(1-\mathrm{ch}2\vartheta\right).
\end{aligned} \tag{32}$$

Separating the values with $\mathbf{e}_1$ and $\mathbf{e}_2$ we get the well known formulas for time and coordinates transformation [20]:

$$t' = \frac{t - x\,\mathrm{v}/c^2}{\sqrt{1-\mathrm{v}^2/c^2}},\quad x' = \frac{x - t\,\mathrm{v}}{\sqrt{1-\mathrm{v}^2/c^2}},\quad y' = y,\quad z' = z, \tag{33}$$

where x is the coordinate along the $\vec{m}$ vector.

Let us also consider the Lorentz transformation of the full sedenion $\tilde{\boldsymbol{V}}$.

The transformed sedenion $\tilde{\boldsymbol{V}}'$ can be written as sedenionic product

$$\tilde{\boldsymbol{V}}' = \tilde{\boldsymbol{L}}^*\,\tilde{\boldsymbol{V}}\,\tilde{\boldsymbol{L}}\,. \tag{34}$$

$$\begin{aligned}
\tilde{\boldsymbol{V}}' &= \left(\mathrm{ch}\,\vartheta - \mathbf{e}_{\mathbf{tr}}\mathrm{sh}\,\vartheta\,\vec{m}\right)\left(\boldsymbol{V}_0 + \vec{\boldsymbol{V}}\right)\left(\mathrm{ch}\,\vartheta + \mathbf{e}_{\mathbf{tr}}\mathrm{sh}\,\vartheta\,\vec{m}\right) \\
&= \boldsymbol{V}_0\mathrm{ch}^2\vartheta + \mathbf{e}_{\mathbf{tr}}\boldsymbol{V}_0\mathbf{e}_{\mathbf{rt}}\mathrm{sh}^2\vartheta - \left(\mathbf{e}_{\mathbf{tr}}\boldsymbol{V}_0 - \boldsymbol{V}_0\mathbf{e}_{\mathbf{tr}}\right)\vec{m}\,\mathrm{ch}\,\vartheta\,\mathrm{sh}\,\vartheta \\
&+ \vec{\boldsymbol{V}}\mathrm{ch}^2\vartheta - \mathbf{e}_{\mathbf{tr}}\vec{m}\,\vec{\boldsymbol{V}}\vec{m}\,\mathbf{e}_{\mathbf{tr}}\mathrm{sh}^2\vartheta - \left(\mathbf{e}_{\mathbf{tr}}\vec{m}\,\vec{\boldsymbol{V}} - \vec{\boldsymbol{V}}\vec{m}\mathbf{e}_{\mathbf{tr}}\right)\mathrm{ch}\,\vartheta\,\mathrm{sh}\,\vartheta\,.
\end{aligned} \tag{35}$$

Rewriting the expression (35) with scalar (16) and vector (17) products we get

$$\tilde{V}' = V_0 \mathrm{ch}^2\vartheta + \mathbf{e}_{\mathbf{tr}} V_0 \mathbf{e}_{\mathbf{tr}} \mathrm{sh}^2\vartheta - \left(\mathbf{e}_{\mathbf{tr}} V_0 - V_0 \mathbf{e}_{\mathbf{tr}}\right)\vec{m}\,\mathrm{ch}\vartheta\,\mathrm{sh}\vartheta$$
$$+ \vec{\boldsymbol{V}}\mathrm{ch}^2\vartheta - \mathbf{e}_{\mathbf{tr}}\vec{\boldsymbol{V}}\mathbf{e}_{\mathbf{tr}}\mathrm{sh}^2\vartheta - 2\mathbf{e}_{\mathbf{tr}}\left(\vec{m}\cdot\vec{\boldsymbol{V}}\right)\mathbf{e}_{\mathbf{tr}}\vec{m}\,\mathrm{sh}^2\vartheta \tag{36}$$
$$-\left(\mathbf{e}_{\mathbf{tr}}\left(\vec{m}\cdot\vec{\boldsymbol{V}}\right) - \left(\vec{\boldsymbol{V}}\cdot\vec{m}\right)\mathbf{e}_{\mathbf{tr}}\right)\mathrm{ch}\vartheta\,\mathrm{sh}\vartheta - \left(\mathbf{e}_{\mathbf{tr}}\left[\vec{m}\times\vec{\boldsymbol{V}}\right] - \left[\vec{\boldsymbol{V}}\times\vec{m}\right]\mathbf{e}_{\mathbf{tr}}\right)\mathrm{ch}\vartheta\,\mathrm{sh}\vartheta.$$

Thus, the transformed sedenion has the following components:

$$\begin{aligned}
&V' = V\,,\\
&V'_{\mathbf{tr}} = V_{\mathbf{tr}}\,,\\
&V'_{\mathbf{r}} = V_{\mathbf{r}}\,\mathrm{ch}\,2\vartheta - \mathbf{e}_{\mathbf{tr}}\left(\vec{m}\cdot\vec{V}_{\mathbf{t}}\right)\mathrm{sh}\,2\vartheta\,,\\
&V'_{\mathbf{t}} = V_{\mathbf{t}}\,\mathrm{ch}\,2\vartheta - \mathbf{e}_{\mathbf{tr}}\left(\vec{m}\cdot\vec{V}_{\mathbf{r}}\right)\mathrm{sh}\,2\vartheta\,,\\
&\vec{V}' = \vec{V}\,\mathrm{ch}\,2\vartheta + 2\left(\vec{m}\cdot\vec{V}\right)\vec{m}\,\mathrm{sh}^2\vartheta - \mathbf{e}_{\mathbf{tr}}\left[\vec{m}\times\vec{V}_{\mathbf{rt}}\right]\mathrm{sh}2\vartheta\,,\\
&\vec{V}'_{\mathbf{tr}} = \vec{V}_{\mathbf{tr}}\,\mathrm{ch}\,2\vartheta + 2\left(\vec{m}\cdot\vec{V}_{\mathbf{tr}}\right)\vec{m}\,\mathrm{sh}^2\vartheta - \mathbf{e}_{\mathbf{tr}}\left[\vec{m}\times\vec{V}\right]\mathrm{sh}2\vartheta\,,\\
&\vec{V}'_{\mathbf{r}} = \vec{V}_{\mathbf{r}} - 2\left(\vec{m}\cdot\vec{V}_{\mathbf{r}}\right)\vec{m}\,\mathrm{sh}^2\vartheta - \mathbf{e}_{\mathbf{tr}}V_{\mathbf{t}}\,\vec{m}\,\mathrm{sh}\,2\vartheta\,,\\
&\vec{V}'_{\mathbf{t}} = \vec{V}_{\mathbf{t}} - 2\left(\vec{m}\cdot\vec{V}_{\mathbf{t}}\right)\vec{m}\,\mathrm{sh}^2\vartheta - \mathbf{e}_{\mathbf{tr}}V_{\mathbf{r}}\,\vec{m}\,\mathrm{sh}\,2\vartheta\,.
\end{aligned} \tag{37}$$

4. Subalgebras of space-time quaternions and octonions

The sedenionic basis introduced above enables constructing different types of low-dimensional hypercomplex numbers. For example one can introduce space-time complex numbers

$$\begin{aligned}
&Z_{\mathbf{t}} = z_1 + \mathbf{e}_{\mathbf{t}} z_2\,,\\
&Z_{\mathbf{r}} = z_1 + \mathbf{e}_{\mathbf{r}} z_2\,,\\
&Z_{\mathbf{tr}} = z_1 + \mathbf{e}_{\mathbf{tr}} z_2\,,
\end{aligned} \tag{38}$$

which are transformed under space and time conjugation. Moreover we can consider the space-time quaternions, which differ in their properties with respect to the operations of the spatial and time inversion.

$$\hat{q} = q_0\mathbf{a_0} + \mathbf{e_0}\left(q_1\mathbf{a_1} + q_2\mathbf{a_2} + q_3\mathbf{a_3}\right), \tag{39}$$

$$\hat{q}_{\mathbf{t}} = q_0\mathbf{a_0} + \mathbf{e_t}\left(q_1\mathbf{a_1} + q_2\mathbf{a_2} + q_3\mathbf{a_3}\right), \tag{40}$$

$$\hat{q}_{\mathbf{r}} = q_0\mathbf{a_0} + \mathbf{e_r}\left(q_1\mathbf{a_1} + q_2\mathbf{a_2} + q_3\mathbf{a_3}\right), \tag{41}$$

$$\hat{q}_{\mathbf{tr}} = q_0\mathbf{a_0} + \mathbf{e_{tr}}\left(q_1\mathbf{a_1} + q_2\mathbf{a_2} + q_3\mathbf{a_3}\right), \tag{42}$$

The absolute quaternion (39) is the sum of the absolute scalar and absolute vector. It remains constant under the transformations of space and time inversion (27). Time quaternion $\hat{q}_{\mathbf{t}}$, space quaternion $\hat{q}_{\mathbf{r}}$ and space-time quaternion $\hat{q}_{\mathbf{tr}}$ are transformed under inversions in accordance with the commutation rules for the basis elements $\mathbf{e_t}, \mathbf{e_r}, \mathbf{e_{tr}}$. For example, performing the operation of time inversion (see (27)) with the quaternion $\hat{q}_{\mathbf{t}}$ we obtain the conjugated quaternion

$$\bar{\hat{q}}_{\mathbf{t}} = -\mathbf{e_r}\hat{q}_{\mathbf{t}}\mathbf{e_r} = q_0\mathbf{a_0} - \mathbf{e_t}\left(q_1\mathbf{a_1} + q_2\mathbf{a_2} + q_3\mathbf{a_3}\right). \tag{43}$$

On the other hand, the sedenionic basis allows one to construct various types of space-time eight-component octonions:

$$\breve{G}_{\mathbf{t}} = G_{00} + G_{01}\mathbf{a_1} + G_{02}\mathbf{a_2} + G_{03}\mathbf{a_3} + \mathbf{e_t}G_{10} + \mathbf{e_t}\left(G_{11}\mathbf{a_1} + G_{12}\mathbf{a_2} + G_{13}\mathbf{a_3}\right), \tag{44}$$

$$\breve{G}_{\mathbf{r}} = G_{00} + G_{01}\mathbf{a_1} + G_{02}\mathbf{a_2} + G_{03}\mathbf{a_3} + \mathbf{e_r}G_{20} + \mathbf{e_r}\left(G_{21}\mathbf{a_1} + G_{22}\mathbf{a_2} + G_{23}\mathbf{a_3}\right), \tag{45}$$

$$\breve{G}_{\mathbf{tr}} = G_{00} + G_{01}\mathbf{a_1} + G_{02}\mathbf{a_2} + G_{03}\mathbf{a_3} + \mathbf{e_{tr}}G_{30} + \mathbf{e_{tr}}\left(G_{31}\mathbf{a_1} + G_{32}\mathbf{a_2} + G_{33}\mathbf{a_3}\right). \tag{46}$$

6. Concluding remarks

The algebra of sedenions proposed in this article is the anticommutative associative space-time Clifford algebra. The sedenionic basis elements $\mathbf{a_n}$ are responsible for the spatial rotation, while the elements $\mathbf{e_n}$ are responsible for the space-time inversions. Mathematically, these two bases are equivalent, and the different physical properties attributed to them are an important physical essence of our sedenionic hypothesis.

In contrast to the previously discussed sedeonic algebra [15-19], which uses the multiplication rules of basic elements $\mathbf{a}'_\mathbf{n}$ and $\mathbf{e}'_\mathbf{n}$ proposed by A.Macfarlane [21], the multiplication rules for sedenionic basis elements $\mathbf{a_n}$ and $\mathbf{e_n}$ coincide with the rules for quaternion units introduced by W.R.Hamilton [22]. There is a close connection between these two basses. The transition from the sedeonic basis to sedenionic basis is performed by following replacement:

$$\mathbf{a}'_\mathbf{n} = i\mathbf{a_n},$$
$$\mathbf{e}'_\mathbf{n} = i\mathbf{e_n}.$$

There is one disadvantage of sedenions connected with the fact that the square of the vector is a negative value. However, on the other side the sedenionic rules of cross-multiplying do not contain the imaginary unit and this leads to the considerable simplifications in the calculations. But of course, the physical results do not depend on the choice of algebra, so these two algebras are equivalent.

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