The proof of the Bill conjecture

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The math problem is Bill conjecture:

**Guess one:** if \( A^x + B^y = C^z \), and A, B, C, x, y, z are all positive integers, and x, y, z are greater than 2, then the A, B, C must have a common prime factor.

**Guess two:** if A, B, C are positive integers and overall coprime. Then the equation \( A^x + B^y = C^z \), no x, y, z all positive integer solutions of greater than 2.

The essence of these two kinds of conjecture is the same.

**Prove:**
Let x, y, z is a positive integer, and greater than 2.
Let \( x = y, z = x + 1 \), have to

\[
2^x + 2^y = 2^z \quad (1)
\]

Obviously 2 and 2 are not coprime.

Let \( n = y = z, D, E, F \) is a positive integer, and \( D > E \). Then \( F = D^n - E^n \), according to the Catalan (Eugne Charles Catalan) theorem, F will not equal to 1.

Have to

\[
F + E^n = D^n \quad (2)
\]

Both sides also multiplied by \( F^n \) term, have to

\[
F^{(n+1)} + (EF)^n = (DF)^n
\]

Let \( F = GH, G \) is a positive integer, H is a prime.

Have to

\[
(GH)^{(n+1)} + (EGH)^n = (DGH)^n \quad (3)
\]
Let $A = GH$, $B = EGH$, $C = DGH$, and $x = n + 1$, $y = z = n$. Because $E$, $D$ can be any value, $n$ also can be any value, for the type (2) both sides only multiplied by $F^n$ term can turn into the indefinite equation form $A^x + B^y = C^z$.

So in addition to the type (1), the type (3) is the indefinite equation $A^x + B^y = C^z$ solution form only. In this kind of indefinite equation $A$, $B$, $C$ must have a common a prime factor $H$.

So for $y \neq z$ case? In addition to $y|z$ or $z|y$, for $F + E^y = D^z$ equation on both sides, no matter what multiplied by about $y$, $z$, $D$, $E$, $F$ term can’t turn into the indefinite equation form $A^x + B^y = C^z$. When $y|z$ or $z|y$, $F = D^z - E^y$ can be written as $F = (D^p)^y - E^y$ or $F = D^z - (E^q)^z$ ($p$, $q$ is an integer greater than 1, and $yp = z$ or $zq = y$), then the type (3) is the same indefinite equation $A^x + B^y = C^z$ solution form, just the $D$ turn into $D^p$ or the $E$ turn into $E^q$.

By the discussion above all, the Bill conjecture is proved.