ALGEBRAIC STRUCTURES USING [0, n)

W.B.VASANTHA KANDASAMY
FLORENTIN SMARANDACHE

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W. B. Vasantha Kandasamy
Florentin Smarandache

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PREFACE

In this book authors for the first time introduce a new method of building algebraic structures on the interval [0, n). This study is interesting and innovative. However, [0, n) is a semigroup under product, × modulo n and a semigroup under min or max operation. Further [0, n) is a group under addition modulo n.

We see [0, n) under both max and min operation is a semiring. [0, n) under + and × is not in general a ring. We define S = {[0, n), +, ×} to be a pseudo special ring as the distributive law is not true in general for all a, b ∈ S. When n is a prime, S is defined as the pseudo special interval domain which is of infinite order for all values of n, n a natural integer.

Several special properties about these structures are studied and analyzed in this book. Certainly these new algebraic structures will find several application in due course of time. All these algebraic structures built using the interval [0, n) is of infinite order. Using [0, n) matrices are built and operations such as + and × are performed on them. It is important to note in all places where semigroups and semirings and groups find their applications these new algebraic structures can be replaced and applied appropriately.

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W.B.VASANTHA KANDASAMY
FLORENTIN SMARANDACHE

Chapter One

INTRODUCTION

In this book we for the first time study algebraic structures built using the interval [0, n).

We see $Z_n = \{0, 1, 2, \ldots, n - 1\}$ is always a proper subset of [0, n). This study gives many new concepts for we get pseudo interval rings of infinite order. The semigroups can be built using [0, n) under $\times$ or max or min operations.

Each enjoys a special property. Matrices are built using [0, n) and the operations $\times$ or max or min are defined. Only in case $\times$ and min we have zero divisors. This study gives several nice properties. If $Z_n \subseteq [0, n)$ is a Smarandache semigroup then so is [0, n) under $\times$. However under max or min such concept cannot sustain.

We see $R = \{[0, n), +, \times\}$ is a pseudo ring. Study on these pseudo rings is carried out in a systematic way. We have studied the finite ring Z_n; $Z_n \subseteq [0, n)$ but when we include or transform the whole interval into a pseudo ring, the notion of this concept is interesting and innovative.

Why was study of this form was not done and what is the real problem faced in studying this [0, n) structure?

We see when p is a prime we do not get an interval integral domain. For decimals cannot have inverses in [0, p) under product $\times$.

Using S_{max} = {[0, n), max} we get a semigroup which is idempotent and this semigroup has no greatest element and the least element is 0 as max {0, t} = t for all t $\in$ [0, n) \ {0}.

Likewise S_{min} = {[0, n), min} has no greatest element and 0 is the least element so that min {x, 0} = 0 for all x $\in$ [0, n).

This gives an idempotent semigroup of infinite order and it has several interesting features. We study $S_{\times}$ = {(0, n), $\times$}; this gives a number of zero divisors and units.

If n is a prime we do not have even a single zero divisor or idempotent only (n–2) units. These semigroups are of infinite order and this study is an interesting one.

Now R = {[0, n), +, $\times$} be the pseudo ring as the distributive laws are not true in general in R. R is of infinite order if n = p, p a prime then R is not a pseudo integral domain of infinite order. R has units, zero divisors and idempotents. If n is not a prime R has zero divisors and R is not an integral domain, R is only a commutative pseudo ring with unit.

If $Z_n \subseteq$ [0, n) is a Smarandache pseudo ring so is the pseudo ring R = {[0, n), +, $\times$} (n, prime or otherwise); infact if n is a prime R is always a pseudo S-ring.

Study of pseudo ideals in case of R = {[0, n), +, $\times$} is an interesting problem.

If a matrix is built using this R, we see R has zero divisors, units and idempotents. We see R has finite subrings also; but those finite subrings are not ideals. Here these pseudo rings contains subrings which are not pseudo subrings.

Chapter Two

ALGEBRAIC STRUCTURES USING THE INTERVAL [0, n) UNDER SINGLE BINARY OPERATION

Here we use the half closed open interval [0, n), $n < \infty$; n an integer. On [0, n) four operations can be given so that under + mod n, [0, n) is the special interval group. [0, n) under $\times$ mod n is only a special interval semigroup and under max (or min) [0, n) is a special interval semigroup.

Study of this is innovative and interesting. This study throws light on how the interval [0, n) behaves under product and sum +; several special features about them are analysed.

Let S = {[0, 9), +} be the group under addition modulo 9. 0 is the additive inverse.

For every $x \in [0, 9)$ there is a unique $y \in [0, 9)$ such that $x + y \equiv 9 \equiv 0 \pmod 9$; so x is the inverse of y with respect to '+' and vice versa.

If $x = 3.029 \in S$; then $y = 5.971 \in [0, 9)$ and $x + y = 3.029 + 5.971 \in [0, 9)$ is such that $x + y = 3.029 + 5.971 = 9 \equiv 0 \pmod 9$ so x is the additive inverse of y and vice versa.

We will illustrate this situation by some examples.

Example 2.1: Let $S = \{[0, 4), +\}$ be the special interval group. This group has also finite subgroups. For $P = \{0, 1, 2, 3\} \subseteq S$ is a subgroup of S under +.

We call S as the special interval group.

$T = \{0, 2\} \subseteq S$ is a special interval subgroup of S.

Example 2.2: Let $S = \{[0, 12), +\}$ be the special interval group. $T = \{0, 6\} \subseteq S$ is a special interval subgroup of S.

$P = \{0, 2, 4, 6, 8, 10\} \subseteq S$ is also a special interval subgroup of S.

$M = \{0, 4, 8\} \subseteq S$ is also a special interval subgroup of S.

DEFINITION 2.1: *Let $S = \{[0, n), n \geq 2$, n an integer; $+\}$ be the special interval group under addition modulo n. S is a group; for if $a, b \in S$.*

(1) $a + b \pmod n \in S$.
(2) $0 \in S = [0, n)$ is such that for all $a \in S$, $a + 0 = 0 + a = a$.
(3) For every $a \in S$ there exist a unique b in S such that $a + b \equiv n = 0 \pmod n$, b is called the additive inverse of a and vice versa.
(4) $a + b = b + a$ for all $a, b \in S$.

Thus (S, +) is an abelian group under '+', defined as the special natural group on interval [0, n) under '+' or special interval group.

Clearly o(S) = ∞ for any n $\in$ N. This interval [0, n) give a group of infinite order under '+' modulo n.

We will give examples of them.

Example 2.3: Let S = {[0, 11), +} be the special natural group on interval [0, 11). o(S) = ∞ and S is a abelian. S has many finite order subgroups.

The subgroup generated by $\langle 0.1 \rangle$ = {0, 0.1, 0.9, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 1, 1.1, 1.2, …, 1.18, 1.9, 2, 2.1, …, 10.9} $\subseteq$ S is a finite subgroup of S under + modulo 11.

The subgroup generated by T = $\langle 1 \rangle$ is such that o(T) = 11 and so on. However [0, t); t < 11 is not a subgroup under +.

Example 2.4: Let S = {[0, 7), +} be the special natural interval group.

T_1 = {0, 1, 2, 3, 4, 5, 6} $\subseteq$ S is a subgroup of finite order.

S has only one group of finite order.

Can S have other subgroups?

T_2 = {0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, …, 6, 6.5} $\subseteq$ S is again a subgroup of finite order.

T_3 = {0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, …, 6.2, 6.4, 6.8, 6.6} $\subseteq$ S is again a subgroup of finite order.

Thus this is a special natural interval group which has many finite special natural interval subgroups.

Now [0, 7) $\underset{\neq}{\subset}$ [0, 7], 7 is prime yet we have subgroups for S = {[0, 7), +}.

Example 2.5: Let S = {[0, 16), +} be a special interval group under +.

T_1 = {0, 8} is a subgroup of S. T_2 = {0, 4, 8, 12} is again a subgroup of S.

Consider T_3 = {0, 2, 4, 6, 8, 10, 12, 14} $\subseteq$ S is again a subgroup of S.

T_4 = {0, 1, 2, …, 15} $\subseteq$ S is also a subgroup of S. Further $T_4 \cong Z_{16}$.

Now we consider
T_5 = {0, 0.0001, 0.0002, …, 15, 15.0001, …, 15.9999} $\subseteq$ S. T_5 is a subgroup of S of finite order.

Now having seen subgroups of finite order we proceed on to build algebraic groups using [0, n) under the operation +.

Example 2.6: Let S = {(a_1, a_2, a_3) | $a_i \in$ [0, 30), +} be the special interval group of infinite order.

This is of infinite order and is commutative. This has both subgroups; of finite and infinite order.

We will just illustrate this by the following.

T_1 = {(a_1, 0, 0) | $a_i \in$ [0, 30), +} $\subseteq$ S is a subgroup of infinite order.

T_2 = {(0, a_1, 0) | $a_1 \in$ [0, 30), +} $\subseteq$ S and

T_3 = {(0, 0, a_1) | $a_1 \in$ [0, 30), +} $\subseteq$ S are also subgroups of infinite order.

We see $T_i \cap T_j$ = {(0, 0, 0)} if $i \neq j$, $1 \leq i, j \leq 3$.

Consider P_1 = {(a_1, 0, 0) | $a_1 \in$ {0, 1, 2, …, 29}, +} $\subseteq$ S is a subgroup of S. We see P_1 is a finite subgroup and of order 30.

$P_2 = \{(0, a_1, 0) \mid a_1 \in \{0, 2, 4, 6, 8, 10, \ldots, 28\}, +\} \subseteq S$ is a finite subgroup of order 15.

$P_3 = \{(0, 0, a_1) \mid a_1 \in \{0, 10, 20\}, +\} \subseteq S$ is a finite subgroup of order 3.

$P_4 = \{(a_1, a_2, a_3) \mid a_1, a_2, a_3 \in \{0, 5, 10, 15, 20, 25\}, +\} \subseteq S$ is a finite subgroup of order 216.

Thus S has finite number of finite subgroups.

$B = \{(a_1, a_2, a_3) \mid a_i \in \{0, 10, 20\}, 1 \leq i \leq 3, +\} \subseteq S$ is a special interval subgroup of S of finite order.

$B' = \{(a_1, a_2, 0) \mid a_1, a_2 \in [0, 30), +\} \subseteq S$ is a subgroup of S of infinite order.

We can have subgroups of both finite and infinite order.

$B \cap B' = \{(a_1, a_2, 0) \mid a_1, a_2 = \{0, 10, 20\}\}$ and

$B \cup B' = \{(a, b, c) \mid a, b \in [0, 30)$ and $c \in \{0, 10, 20\}\}$ are again subgroups of S.

Example 2.7: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_8 \end{bmatrix} \middle|\; a_i \in [0, 19); 1 \leq i \leq 8 \right\}$$

be the special interval group of infinite order.

S has finite number of subgroups.

$$T_1 = \left\{\left.\begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right| a_1 \in [0, 19)\right\}$$

is the special interval subgroup.

$$T_2 = \left\{\left.\begin{bmatrix} 0 \\ a_1 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right| a_1, a_2 \in [0, 19)\right\}$$

be the special interval subgroup of infinite order.

$$T_3 = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ 0 \\ a_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right| a_1 \in [0, 19)\right\} \subseteq S$$

be the special interval subgroup of infinite order.

$$P_1 = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix}\right| a_i \in \{0, 1, 2, \ldots, 18\}; 1 \leq i \leq 9\right\} \subseteq S$$

be the finite special interval subgroup of S.

Let

$$B = \left\{ \left[\begin{array}{c} a_1 \\ 0 \\ a_2 \\ 0 \\ a_3 \\ 0 \\ a_4 \\ 0 \\ a_5 \end{array}\right] \middle| \; a_i \in [0, 19); 1 \leq i \leq 5 \right\} \subseteq S$$

be the special interval subgroup of S of infinite order.

$$B_1 = \left\{ \left[\begin{array}{c} a_1 \\ 0 \\ a_2 \\ 0 \\ a_3 \\ 0 \\ a_4 \\ 0 \\ a_5 \end{array}\right] \middle| \; a_i \in [0, 19); 1 \leq i \leq 5 \right\} \subseteq S$$

be the special interval subgroup of S of infinite order.

S has several finite subgroups as well as infinite subgroups.

Example 2.8: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle| \; a_i \in [0, 8); 1 \leq i \leq 15 \right\}$$

be the special interval group of infinite order.

Take

$$P_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1 \in \{0, 2, 4, 6\} \right\} \subseteq S;$$

P_1 is a special interval subgroup of order 4.

$$P_2 = \left\{ \begin{bmatrix} 0 & a_2 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_2 \in \{0, 2, 4, 6\} \right\} \subseteq S$$

is a special interval subgroup of order 4.

We have atleast 15 subgroups of order 4.

Let

$$T_1 = \left\{ \begin{bmatrix} a_1 & a_2 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1, a_2 \in \{0, 4\} \right\} \subseteq S$$

be another special interval subgroup of order 4.

$$T_1 = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 4 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 4 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}\right\} \subseteq S$$

is of order 4.

$$T_2 = \left\{\begin{bmatrix} 0 & a_1 & a_2 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1, a_2 \in \{0, 4\}\right\} \subseteq S$$

is the special interval subgroup.

$$T_3 = \left\{\begin{bmatrix} a_1 & 0 & a_2 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1, a_2 \in \{0, 4\}\right\} \subseteq S$$

is the special interval subgroup. $o(T_3) = 4$.

$$T_4 = \left\{\begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1, a_2 \in \{0, 4\}\right\} \subseteq S$$

be the special interval subgroup. $o(T_3) = 4$ and so on.

$$T_{15} = \left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & a_1 & a_2 \end{bmatrix} \middle|\; a_1, a_2 \in \{0, 4\}\right\} \subseteq S$$

be the special interval subgroup. $o(T_{15}) = 4$.

$$W_1 = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \middle|\; a_i \in \{0, 4\}, 1 \leq i \leq 3\right\}.$$

$$o(W_1) = \left\{\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 4 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix},\right.$$

$$\left.\begin{bmatrix} 4 & 4 & 4 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 4 & 4 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 0 & 4 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 4 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}\right\} \subseteq S$$

is a special interval subgroup of order 8.

We can find several subgroups of finite order. We see S has infinite subgroups also.

Example 2.9: Let

$$S = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix} \middle|\; a_i \in [0, 12), 1 \leq i \leq 14\right\}$$

be the special interval group.

S has several subgroups of finite order.

For $P_1 = \left\{\begin{bmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \middle| \; a_1 \in [0, 6)\right\} \subseteq S$,

$$P_2 = \left\{\begin{bmatrix} 0 & a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \middle| \; a_1 \in [0, 6)\right\} \subseteq S, \ldots,$$

$$P_{14} = \left\{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_1 \end{bmatrix} \middle| \; a_1 \in [0, 6)\right\} \subseteq S$$

are 14 subgroups of order two.

Take

$$B_1 = \left\{\begin{bmatrix} a_1 & a_2 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix} \middle| \; a_1, a_2 \in \{0, 6\}\right\} \subseteq S,$$

$$B_2 = \left\{\begin{bmatrix} a_1 & 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \middle| \; a_1, a_2 \in \{0, 6\}\right\} \subseteq S,$$

$$B_3 = \left\{\begin{bmatrix} a_1 & 0 & 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \middle| \; a_1, a_2 \in \{0, 6\}\right\}$$

and so on are all subgroups of S.

$$B_1 = \left\{\begin{bmatrix} 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}, \begin{bmatrix} 6 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix},\right.$$

$$\begin{bmatrix} 6 & 6 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 6 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}\Bigg\} \subseteq S$$

is a subgroup of order 4. There are atleast 66 subgroups of order 4.

We can get

$$D_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \middle| \; a_1, a_2, a_3 \in \{0, 6\} \right\} \subseteq S$$

be the subgroup of order eight.

$$D_1 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$

$$\begin{bmatrix} 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 6 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 6 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\left. \begin{bmatrix} 0 & 6 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 6 & 6 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\} \subseteq S$$

be the subgroup of order 8.

$$D_2 = \left\{ \begin{bmatrix} a_1 & 0 & a_2 & a_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \middle| \; a_1, a_2, a_3 \in \{0, 6\} \right\} \subseteq S,$$

$$\ldots, \; D_t = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1 & a_2 & a_3 \end{bmatrix} \middle| \; a_1, a_2, a_3 \in \{0, 6\} \right\} \subseteq S$$

are t (< ∞) special interval subgroups of order 8 (t = 220).

Likewise we can find subgroups of finite order.

S has also subgroups of infinite order for take

$$M_1 = \left\{\begin{bmatrix} a_1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{bmatrix} \middle| a_1 \in [0, 12)\right\} \subseteq S, \ldots,$$

$$M_{12} = \left\{\begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & a_1 \end{bmatrix} \middle| a_1 \in [0, 12)\right\} \subseteq S$$

are all subgroups of infinite order.

$$N_1 = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \middle| a_1, a_2, a_3, a_4 \in [0, 12)\right\} \subseteq S,$$

$$N_2 = \left\{\begin{bmatrix} a_1 & 0 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \middle| a_1, a_2, a_3, a_4 \in [0, 12)\right\} \subseteq S,$$

$$\ldots,\ N_r = \left\{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_1 & a_2 & a_3 & a_4 \end{bmatrix} \middle| a_i \in [0, 12), 1 \leq i \leq 4\right\} \subseteq S$$

(r < ∞) are all subgroups of infinite order.

We have atleast 495 such subgroups and so on.

Thus we have more number of finite subgroups than that of infinite subgroups (prove or disprove)!

Take

$$L_1 = \left\{\begin{bmatrix} a_1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \middle| a_1 \in \{0, 2, 4, 6, 8, 10\}\right\} \subseteq S$$

be the subgroup of S.

We see $o(L_1) = 6$. We have 12 subgroups of order 6.

$$L_2 = \left\{\begin{bmatrix} a_1 & a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \middle| a_1, a_2 \in \{0, 2, 4, 6, 8, 10\}\right\} \subseteq S$$

be the subgroup of S of finite order. $o(L_2) = 36$.

$$L_2 = \left\{\begin{bmatrix} 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \end{bmatrix},\right.$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \end{bmatrix}, \begin{bmatrix} 6 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \end{bmatrix},$$

$$\begin{bmatrix} 8 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \end{bmatrix}, \begin{bmatrix} 10 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 4 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}, \begin{bmatrix} 0 & 6 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}, \begin{bmatrix} 0 & 8 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 10 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}, \begin{bmatrix} 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 4 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix},$$

$$\begin{bmatrix} 6 & 6 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}, \begin{bmatrix} 8 & 8 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}, \begin{bmatrix} 10 & 10 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 4 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}, \begin{bmatrix} 4 & 2 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}, \begin{bmatrix} 2 & 6 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix},$$

$$\begin{bmatrix} 6 & 2 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}, \begin{bmatrix} 2 & 8 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}, \begin{bmatrix} 8 & 2 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 10 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}, \begin{bmatrix} 10 & 2 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}, \begin{bmatrix} 4 & 6 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix},$$

$$\begin{bmatrix} 6 & 4 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}, \begin{bmatrix} 4 & 8 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix},$$

$$\begin{bmatrix} 8 & 4 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}, \begin{bmatrix} 4 & 10 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}, \begin{bmatrix} 10 & 4 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix},$$

$$\begin{bmatrix} 6 & 8 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}, \begin{bmatrix} 8 & 6 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}, \begin{bmatrix} 6 & 10 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix},$$

$$\left.\begin{bmatrix} 10 & 6 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}, \begin{bmatrix} 8 & 10 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}, \begin{bmatrix} 8 & 10 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}\right\}.$$

Clearly $o(L_2) = 36$. We have atleast 66 such subgroups of order 36.

Likewise we can find

$$W_1 = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \middle| a_i \in \{0, 2, 4, 6, 8, 10\}, 1 \le i \le 4 \right\} \subseteq S$$

to be subgroup of finite order.

We have atleast 495 subgroups of this type.

Further we using the subgroup {0, 3, 6, 9}; get finite order special interval subgroups of S.

Example 2.10: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 \\ \vdots & \vdots \\ a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in [0, 13); 1 \leq i \leq 12, + \right\}$$

be the special interval group of infinite order.

Clearly [0, 13) has finite subgroups under addition say F = {0, 1, 2, 3, …, 12}.

We can get atleast $S_{12} = {}_{12}C_1 + {}_{12}C_2 + \ldots + {}_{12}C_{12}$ number of special interval subgroups finite order using F.

We have at least $S_{12} = {}_{12}C_1 + {}_{12}C_2 + \ldots + {}_{12}C_{11}$ number of subgroups of infinite order.

THEOREM 2.1: *Let S = {n × m matrices with entries from [0, t)} (t a prime) be the special interval group of infinite order.*

(i) S has atleast $S_t = {}_{n\times m}C_1 + {}_{n\times m}C_2 + \ldots + {}_{n\times m}C_{n\times m}$ number of finite subgroups where the matrix takes its entries from F = {0, 1, 2, …, t–1} (m × n = mn).
(ii) S has atleast $S_t - 1$ number of subgroups of infinite order.

Proof is direct and hence left as an exercise to the reader.

THEOREM 2.2: *Let S = {Collection of n × m matrices with entries from [0, t); t not a prime} be the special interval group under addition.*

[0, t) has subgroups of finite order and these contribute to special interval subgroups of S of finite order apart from the finite groups mentioned in theorem 2.1.

Proof is left as an exercise to the reader.

Now, can we have any other group under + using intervals of the form [0, n)?

This is answered by examples.

Example 2.11: Let S = {[0, 3) × [0, 7), +} be a special interval group of infinite order.

Take P = {{0, 1, 2} × {0, 1, 2, 3, 4, 5, 6}} $\subseteq$ S, P is a special interval subgroup of S of finite order.

T = {{0, 1, 2} × {0}} $\subseteq$ S is a subgroup of S of finite order.

W = {{0} × {0, 1, 2, 3, 4, 5, 6}} $\subseteq$ S is again a subgroup of S of finite order.

We have many finite groups.

L = {[0, 3) × {0}} is a subgroup of infinite order and M = {{0} × [0, 7)} $\subseteq$ S is again a subgroup of infinite order.

Thus S has both subgroups of finite and infinite order.

Example 2.12: Let S = {[0, 6) × [0, 10) × [0, 12) × [0, 20) = (a_1, a_2, a_3, a_4) where $a_1 \in [0, 6)$, $a_2 \in [0, 10)$, $a_3 \in [0, 12)$ and $a_4 \in [0, 20)$} be the special interval group of infinite order. S has subgroups of finite order as well as of infinite order.

(0, 0, 0, 0) acts as the additive identity. Let x = (3.5, 5.9, 10.2, 5) $\in$ S the additive inverse of x is y = (2.5, 4.1, 1.8, 15) $\in$ S for x + y = (0, 0, 0, 0).

Now let x = (5.2, 7.39, 10.4, 15.9) and y = (3.5, 4.8, 5.1, 8.2) $\in$ S.

We find x + y = (5.2, 7.39, 10.4, 15.9) + (3.5, 4.8, 5.1, 8.2)
= (2.7, 2.89, 5.5, 4.1) $\in$ S.

This is the way ‘+’ operation is performed on S.

Thus by using the direct product of groups notion, we are in a position to get more and more special interval groups. As these groups are of infinite order and under the operation ‘+’ and as they are commutative we are not in a position to study several other properties.

Example 2.13: Let S = {[0, 4) × [0,9) × [0, 21) × [0, 7)} be the special interval group under ‘+’. S is commutative.

Take P_1 = {([0, 4) × {0} × {0} × {0}) = {(a, 0, 0, 0)} where a ∈ [0, 4)} ⊆ S is a subgroup of infinite order in S.

Now P_2 = {(0, a, 0, 0) | a ∈[0, 9)} ⊆ S is again a subgroup of infinite order in S.

P_3 = {(0, 0, a, 0) | a ∈ [0, 21)} ⊆ S is a subgroup of infinite order in S.

P_4 = {(0, 0, 0, a) | a ∈ [0, 7)} ⊆ S is a subgroup of infinite order in S.

Thus S has several subgroups of infinite order.

Consider M_4 = {(a, 0, 0, 0) | a ∈ {0, 1, 2, 3}} ⊆ S: M_4 is a subgroup of S of finite order.

We see S has several subgroups of finite order. Also S has several subgroups of infinite order. Infact S = $P_1 + P_2 + P_3 + P_4$ is a direct sum of subgroups.

We see $P_i \cap P_j$ = {(0, 0, 0, 0)} if i ≠ j; 1 ≤ i, j ≤ 4 for every element a ∈ S has a unique representation from P_1, P_2, P_3 and P_4.

Let M_3 = {(0, a, 0, 0) | a ∈ {0, 1, 2, 3, 4, …, 8}} ⊆ S is also a subgroup of S and o(M_3) = 9.

Likewise $M_2 = \{(0, 0, a, 0) \mid a \in \{0, 1, 2, 3, \ldots, 20\}\} \subseteq S$ is a subgroup of S and $o(M_2) = 21$ and $M_1 = \{(0, 0, 0, a) \mid a \in \{0, 1, 2, \ldots, 6\}\} \subseteq S$ is a subgroup of order 7.

Clearly $M_i \cap M_j = \{(0, 0, 0, 0)\}$ if $i \neq j$; $1 \leq i, j \leq 4$ but $M_1 + M_2 + M_3 + M_4 \neq S$ and $M_1 + M_2 + M_3 + M_4 = \{(a, b, c, d) \mid a \in \{0, 1, 2, 3\}, b \in \{0, 1, 2, 3, 4, \ldots, 8\}, c \in \{0, 1, 2, 3, 4, \ldots, 20\}$ and $d \in \{0, 1, 2, \ldots, 6\}\} \subseteq S$ is a subgroup of finite order in S.

Now $N_1 = \{(a, b, 0, c) \mid a \in \{0, 1, 2, 3\}, b \in \{0, 1, 2, 3, \ldots, 8\}$ and $c \in \{0, 1, 2, 3, \ldots, 6\}\} \subseteq S$ is a subgroup of finite order in S.

$N_2 = \{(a, b, 0, 0) \mid a \in \{0, 2\}, b \in \{0, 3, 6\}\} \subseteq S$ is again a subgroup of finite order in S. $P = \{(a, 0, b, 0) \mid a \in [0, 4)$ and $b \in [0, 21)\} \subseteq S$ is again a subgroup of infinite order.

Thus we can have groups constructed using different intervals $[0, a_i)$ where a_i are integers and a_i's different.

We will proceed onto give some more examples.

Example 2.14: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \middle| \; a_1 \in [0, 8), a_2 \in [0, 3), a_3 \in [0, 12) \text{ and } a_4, a_5, a_6, a_7 \in [0, 48) \right\}$$

be the special interval group under addition.

$$\text{Let } x = \begin{bmatrix} 3.3 \\ 1 \\ 10 \\ 5.7 \\ 7.8 \\ 12.1 \\ 40.4 \end{bmatrix} \text{ and } y = \begin{bmatrix} 7 \\ 2 \\ 5.1 \\ 44.5 \\ 38.6 \\ 40.2 \\ 30 \end{bmatrix} \in S,$$

$$x + y = \begin{bmatrix} 3.3 \\ 1 \\ 10 \\ 5.7 \\ 7.8 \\ 12.1 \\ 40.4 \end{bmatrix} + \begin{bmatrix} 7 \\ 2 \\ 5.1 \\ 44.5 \\ 38.6 \\ 40.2 \\ 30 \end{bmatrix}$$

$$= \begin{bmatrix} 3.3 + 7 \,(\text{mod}\, 8) \\ (1 + 2)\,(\text{mod}\, 3) \\ (10 + 5.1)\,(\text{mod}\, 12) \\ (5.7 + 44.5)\,(\text{mod}\, 48) \\ (7.8 + 38.6)\,(\text{mod}\, 48) \\ (12.1 + 40.2)\,(\text{mod}\, 48) \\ (40.4 + 30)\,(\text{mod}\, 48) \end{bmatrix} = \begin{bmatrix} 2.3 \\ 0 \\ 3.1 \\ 4.2 \\ 46.4 \\ 4.3 \\ 22.4 \end{bmatrix} \in S.$$

This is the way addition is performed on S.

Let

$$x = \begin{bmatrix} 6.3 \\ 2.1 \\ 10.7 \\ 46.3 \\ 3.5 \\ 7.8 \\ 9.62 \end{bmatrix} \in S$$

the additive inverse of x is y ∈ S where

$$y = \begin{bmatrix} 1.7 \\ 0.9 \\ 1.3 \\ 1.7 \\ 44.5 \\ 40.2 \\ 38.38 \end{bmatrix} \in S \text{ is such that } x + y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix};$$

the additive identity of x in S.

S has both infinite and finite order special interval subgroups.

$$\text{Let } T_1 = \left\{ \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle|\, a \in [0, 8) \right\} \subseteq S;$$

T_1 is a subgroup of S of infinite order.

Let

$$M_1 = \left\{ \left.\begin{bmatrix} a \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right| a \in \{0, 1, 2, 3, \ldots, 7\} \subseteq [0, 8)\right\} \subseteq S$$

be a subgroup of S of order 8.

Consider

$$T_2 = \left\{ \left.\begin{bmatrix} 0 \\ a \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right| a \in [0, 3)\right\} \subseteq S,$$

T_2 is a subgroup of infinite order.

$$M_2 = \left\{ \left.\begin{bmatrix} 0 \\ a \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right| a \in \{0, 1, 2\}\right\} \subseteq S$$

is a subgroup of finite order and $o(M_2) = 3$.

$$x = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in S \text{ are such that}$$

$$x + y = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$M_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| a \in [0, 12) \right\} \subseteq S$$

is an infinite special interval subgroup of S.

$$T_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| a \in \{0, 1, 2, 3, \ldots, 11\} \right\} \subseteq S$$

is a subgroup of S order 12.

M_2 has no subgroups but T_3 has subgroups given by

$$T_3^1 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| a \in \{0, 3, 6, 9\} \right\} \subseteq T_3,$$

$$T_3^2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| a \in \{0, 6\} \right\} \subseteq T_3 ,$$

$$T_3^3 = \left\{\begin{bmatrix} 0 \\ 0 \\ a \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| a \in \{0, 2, 4, 6, 8, 10\}\right\} \subseteq T_3 \text{ and}$$

$$T_3^4 = \left\{\begin{bmatrix} 0 \\ 0 \\ a \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| a \in \{0, 4, 8\}\right\} \subseteq T_3$$

are the four subgroups of the subgroup T_3 of S.

Let

$$B_1 = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| a_1 \in [0, 48)\right\} \subseteq S$$

be the special interval subgroup of S under '+' of infinite order.

$$B_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ 0 \\ 0 \end{bmatrix} \middle| a_1 \in [0, 48) \right\} \subseteq S$$

is a subgroup of S different from B_1.

$$B_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ 0 \end{bmatrix} \middle| a_1 \in [0, 48) \right\} \subseteq S$$

be the subgroup of S different from B_1 and B_2 of infinite order.

$$B_4 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_1 \end{bmatrix} \middle| a_1 \in [0, 48) \right\} \subseteq S$$

is a subgroup of S of infinite order.

Clearly

$$B_i \cap B_j = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ for } i \neq j, 1 \leq i, j \leq 4.$$

Let

$$D_1 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| a_1 \in \{0, 24\} \right\} \subseteq S$$

be a subgroup of order two in S.

$$D_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ 0 \\ 0 \end{bmatrix} \middle| a_1 \in \{0, 12, 24, 36\} \right\} \subseteq S$$

is again a subgroup of order four in S.

We have subgroups of order 2, 3, 4, 6, 8, 12 and so on in S.

Example 2.15: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| a_i \in [0, 8), 1 \leq i \leq 16, + \right\}$$

be a special interval group.

S has several subgroups of infinite order and also several subgroups of finite order.

$$P_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| a_1 \in [0, 8), + \right\} \subseteq S$$

is an infinite special interval subgroup of S.

$$M_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| a_1 \in \{0, 2, 4, 6\}, + \right\} \subseteq S$$

is a finite special interval subgroup of S.

$$P_2 = \left\{ \begin{bmatrix} 0 & a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| a_1 \in [0, 8), + \right\} \subseteq S$$

is an infinite subgroup of S.

$$M_2 = \left\{\begin{bmatrix} 0 & a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| a_1 \in \{0, 2, 4, 6\}, +\right\} \subseteq S$$

is a finite subgroup of S.

$$P_3 = \left\{\begin{bmatrix} 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| a_1 \in [0, 8), +\right\} \subseteq S$$

is a subgroup of S of infinite order.

$$M_3 = \left\{\begin{bmatrix} 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| a_1 \in \{0, 2, 4, 6\}, +\right\} \subseteq S$$

is a subgroup of finite order.

$$P_4 = \left\{\begin{bmatrix} 0 & 0 & 0 & a_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| a_1 \in [0, 8), +\right\} \subseteq S$$

be the subgroup of S of infinite order.

$$M_4 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 & a_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right| a_1 \in \{0, 2, 4, 6\}, +\right\} \subseteq S$$

is a subgroup of finite order.

$$P_5 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right| a_1 \in [0, 8), +\right\} \subseteq S$$

be the subgroup of S is of infinite order.

$$M_5 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right| a_1 \in \{0, 2, 4, 6\}, +\right\} \subseteq S$$

is a subgroup of order four and so on with

$$P_{16} = \left\{\left.\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{16} \end{bmatrix}\right| a_{16} \in [0, 8), +\right\} \subseteq S$$

is a subgroup of infinite order and

$$M_{16} = \left\{\left.\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{16} \end{bmatrix}\right| a_1 \in \{0, 2, 4, 6\}, +\right\} \subseteq S$$

is a subgroup of order four.

Now

$$N_{1,2} = \left\{\left.\begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right| a_1, a_2 \in [0, 8), \ +\right\} \subseteq S$$

is a subgroup of infinite order.

$$R_{1,2} = \left\{\left.\begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right| a_1, a_2 \in [0, 2, 4, 6\}\right\} \subseteq S$$

is a subgroup of finite order and so on.

$$R_{1,16} = \left\{\left.\begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{16} \end{bmatrix}\right| a_1, a_{16} \in \{0, 2, 4, 6\}\right\} \subseteq S$$

is a subgroup of finite order.

$$N_{1,16} = \left\{\left.\begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{16} \end{bmatrix}\right| a_1, a_{16} \in [0, 8)\right\} \subseteq S$$

is a subgroup of infinite order.

$$R_{5,12} = \left\{\begin{bmatrix} 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{12} \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| a_1, a_{12} \in \{0, 2, 4, 6\}\right\} \subseteq S$$

is a subgroup of finite order.

$$N_{5,12} = \left\{\begin{bmatrix} 0 & 0 & 0 & 0 \\ a_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{12} \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| a_5, a_{12} \in [0, 8)\right\} \subseteq S$$

is a subgroup of infinite order.

$$N_{3,15} = \left\{\begin{bmatrix} 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{15} & 0 \end{bmatrix} \middle| a_1, a_{15} \in [0, 8)\right\} \subseteq S$$

is a subgroup of S of infinite order.

Let

$$R_{3,15} = \left\{\begin{bmatrix} 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{15} & 0 \end{bmatrix} \middle| a_1, a_{15} \in \{0, 2, 4, 6\}\right\} \subseteq S$$

be a subgroup of S of finite order.

Likewise

$$T_{1,5,8} = \left\{\begin{bmatrix} a_1 & 0 & 0 & 0 \\ a_5 & 0 & 0 & a_8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| a_1, a_5, a_8 \in \{0, 2, 4, 8\}\right\} \subseteq S$$

is a subgroup of finite order.

$$J_{1,5,8} = \left\{\begin{bmatrix} a_1 & 0 & 0 & 0 \\ a_5 & 0 & 0 & a_8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| a_1, a_5, a_8 \in \{0, 2, 4, 8\}\right\} \subseteq S$$

is a subgroup of finite order.

$$W_{7,12,14} = \left\{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a_7 & 0 \\ 0 & 0 & 0 & a_{12} \\ 0 & a_{14} & 0 & 0 \end{bmatrix} \middle| a_7, a_{12}, a_{14} \in \{0, 2, 4, 8\}\right\} \subseteq S$$

is a subgroup of finite order and so on.

Let

$$E_{1, 2, 5, 7, 11, 16} = \left\{\begin{bmatrix} a_1 & a_2 & 0 & 0 \\ a_5 & 0 & a_7 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{16} \end{bmatrix} \middle| a_1, a_2, a_5, a_7, a_{11}, a_{16} \in [0, 8)\right\} \subseteq S$$

be a special interval subgroup of infinite order.

$$F_{1,2,5,7,11,16} = \left\{\begin{bmatrix} a_1 & a_2 & 0 & 0 \\ a_5 & 0 & a_7 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{16} \end{bmatrix} \middle| \; a_1, a_2, a_5, \right.$$

$$\left. a_7, a_{11}, a_{16} \in \{0, 2, 4, 6\}\right\} \subseteq S$$

is a subgroup of finite order.

This S has several but finite number of finite subgroups and infinite subgroups.

Example 2.16: Let

$$S = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{25} & a_{26} & a_{27} & a_{28} \\ a_{29} & a_{30} & a_{31} & a_{32} \end{bmatrix} \middle| \; a_i \in [0, 19); 1 \leq i \leq 32, +\right\}$$

be a special interval group of infinite order.

Let

$$T_1 = \left\{\begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| \; a_1 \in [0, 19)\right\} \subseteq S$$

be a subgroup of infinite order.

$$T_7 = \left\{\left.\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a_1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}\right| a_1 \in [0, 19)\right\} \subseteq S$$

is a subgroup of infinite order.

$$T_{10} = \left\{\left.\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}\right| a_1 \in [0, 19)\right\} \subseteq S$$

be a subgroup of infinite order.

$$T_{15} = \left\{\left.\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}\right| a_1 \in [0, 19)\right\} \subseteq S$$

be a subgroup of infinite order.

$$T_{26} = \left\{ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ 0 & a_{26} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \middle| \; a_{26} \in [0, 19) \right\} \subseteq S$$

be a subgroup of infinite order.

$$T_{31} = \left\{ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_1 & 0 \end{array} \right] \middle| \; a_1 \in [0, 19) \right\} \subseteq S$$

be a subgroup of infinite order.

$$Q_{3,11} = \left\{ \left[\begin{array}{cccc} 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{array} \right] \middle| \; a_1, a_3 \in [0, 19) \right\} \subseteq S$$

be a subgroup of infinite order.

$$Q_{7,20} = \left\{\left.\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a_7 & 0 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & a_{20} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right| a_7, a_{20} \in [0, 19)\right\} \subseteq S$$

be a subgroup of infinite order.

$$Y_{3,10,17,31} = \left\{\left.\begin{bmatrix} 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a_{10} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{17} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & a_{31} & 0 \end{bmatrix} \right| a_3, a_{10}, a_{17}, a_{31} \in [0, 19)\right\} \subseteq S$$

is a subgroup of infinite order.

S has subgroups of infinite order. S can have subgroups of finite order also.

Example 2.17: Let

$$S = \left\{\left.\begin{bmatrix} a_1 & a_2 & ... & a_9 \\ a_{10} & a_{11} & ... & a_{18} \\ a_{19} & a_{20} & ... & a_{27} \end{bmatrix} \right| a_i \in [0, 15), 1 \le i \le 27\right\}$$

be the special interval group.

This has subgroups of both finite and infinite order.

$$A_1 = \left\{ \begin{bmatrix} a_1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{bmatrix} \middle| a_1 \in \{0, 5, 10\} \right\} \subseteq S$$

is a special interval subgroup of order three.

$$A_7 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & a_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \middle| a_7 \in \{0, 5, 10\} \right\} \subseteq S$$

is a special interval subgroup of order three and so on.

$$A_{19} = \left\{ \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ a_{19} & 0 & \ldots & 0 \end{bmatrix} \middle| a_{19} \in \{0, 5, 10\} \right\} \subseteq S$$

is a subgroup of order three and

$$A_{27} = \left\{ \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & a_{27} \end{bmatrix} \middle| a_{27} \in \{0, 5, 10\} \right\} \subseteq S$$

is a subgroup of order three.

$$\text{Let } P_1 = \left\{ \begin{bmatrix} a_1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{bmatrix} \middle| a_1 \in [0, 15) \right\} \subseteq S$$

be a subgroup of infinite order.

$$P_2 = \left\{\begin{bmatrix} 0 & a_2 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix} \middle| a_2 \in [0, 15)\right\} \subseteq S$$

is again a subgroup of S of infinite order and so on.

$$P_{26} = \left\{\begin{bmatrix} 0 & 0 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 \\ 0 & 0 & ... & a_{26} & 0 \end{bmatrix} \middle| a_{26} \in [0, 15)\right\} \subseteq S$$

is a subgroup of infinite order in S.

Suppose

$$A_{3,5,9} = \left\{\begin{bmatrix} 0 & 0 & a_3 & 0 & a_5 & ... & a_9 \\ 0 & 0 & 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & 0 & 0 & ... & 0 \end{bmatrix} \middle| a_3, a_5, a_9 \in [0, 15)\right\} \subseteq S$$

is a special interval a subgroup of infinite order in S.

$$F_{3,5,9} = \left\{\begin{bmatrix} 0 & 0 & a_3 & 0 & a_5 & ... & a_9 \\ 0 & 0 & 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & 0 & 0 & ... & 0 \end{bmatrix} \middle| a_3, a_5, a_9 \in \{0, 5, 10\}\right\} \subseteq S$$

is a special interval subgroup of infinite order in S.

Thus S has both finite and infinite order subgroups in S.

Let

$$V_{r_2} = \left\{ \begin{bmatrix} 0 & 0 & \ldots & 0 \\ a_1 & a_2 & \ldots & a_9 \\ 0 & 0 & \ldots & 0 \end{bmatrix} \middle| a_i \in \{0, 5, 10\}, 1 \leq i \leq 9 \right\} \subseteq S$$

be the subgroup of S. V_{r_2} is of finite order.

$$W_{r_2} = \left\{ \begin{bmatrix} 0 & 0 & \ldots & 0 \\ a_1 & a_2 & \ldots & a_9 \\ 0 & 0 & \ldots & 0 \end{bmatrix} \middle| a_i \in [0, 15), 1 \leq i \leq 9 \right\} \subseteq S$$

be the subgroup of S of infinite order.

$$W_{r_3} = \left\{ \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ a_1 & a_2 & \ldots & a_9 \end{bmatrix} \middle| a_i \in [0, 15), 1 \leq i \leq 9 \right\} \subseteq S$$

be the subgroup of S of infinite order.

$$V_{r_3} = \left\{ \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ a_1 & a_2 & \ldots & a_9 \end{bmatrix} \middle| a_i \in \{0, 5, 10\}, 1 \leq i \leq 9 \right\} \subseteq S$$

is a subgroup of S of finite order.

$$E_{C_5} = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 & a_1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & a_2 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & a_3 & 0 & \ldots & 0 \end{bmatrix} \middle| a_i \in [0, 15), 1 \leq i \leq 3 \right\} \subseteq S$$

is a subgroup of S of infinite order.

$$D_{C_5} = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 & a_1 & 0 & ... & 0 \\ 0 & 0 & 0 & 0 & a_2 & 0 & ... & 0 \\ 0 & 0 & 0 & 0 & a_3 & 0 & ... & 0 \end{bmatrix} \middle| a_i \in \{0, 3, 6, 9, 12\}, 1 \leq i \leq 3 \right\} \subseteq S$$

is a finite subgroup of S.

We can have using the 9 columns; 9 subgroups of finite order and 9 subgroups of infinite order.

Thus we have several subgroups of finite order and infinite order in S.

Example 2.18: Let

$$S = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \hline a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ \hline a_{19} & a_{20} & a_{21} \\ a_{22} & a_{23} & a_{24} \\ \hline a_{25} & a_{26} & a_{27} \end{array} \right] \middle| a_i \in [0, 23); 1 \leq i \leq 27 \right\}$$

be the special interval super matrix group under +. S is of infinite order and is commutative.

To the best of authors knowledge S has subgroups of finite order. However S has several subgroups of infinite order.

Consider

$$P_{r_1} = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{bmatrix} \middle| \, a_i \in [0, 23); 1 \leq i \leq 3 \right\} \subseteq S$$

P_{r_1} is a subgroup of S of infinite order.

$$T_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{bmatrix} \middle| \, a_i \in [0, 23) \right\} \subseteq S$$

is a special interval subgroup of infinite order.

$$T_7 = \left\{ \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right] \middle| \; a_1 \in [0, 23) \right\} \subseteq S$$

is a subgroup of infinite order.

$$T_{18} = \left\{ \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right] \middle| \; a \in [0, 23) \right\} \subseteq S$$

is a subgroup of infinite order of S.

Now

$$T_{26} = \left\{ \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & a & 0 \end{array} \right] \middle| \; a \in [0, 23) \right\} \subseteq S$$

is a subgroup of S of infinite order.

$$W_{c_1} = \left\{ \left[\begin{array}{ccc} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ \hline a_3 & 0 & 0 \\ a_4 & 0 & 0 \\ a_5 & 0 & 0 \\ a_6 & 0 & 0 \\ \hline a_7 & 0 & 0 \\ a_8 & 0 & 0 \\ \hline a_9 & 0 & 0 \end{array} \right] \middle| \; a_i \in [0, 23), 1 \leq i \leq 9 \right\} \subseteq S$$

is a subgroup of S of infinite order.

$$L_{a_1, a_7, a_{11}, a_{18}, a_{27}} = \left\{ \left[\begin{array}{ccc} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline a_7 & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{18} \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & a_{27} \end{array} \right] \middle| \; a_1, a_7, a_{11}, \right.$$

$$\left. a_{18}, a_{27} \in [0, 23) \right\} \subseteq S$$

is a subgroup of S of infinite order.

S has finitely many subgroups infinite order and finite order.

Example 2.19: Let

$$S = \left\{ \left[\begin{array}{c|cc|c} a_1 & a_2 & a_3 & a_4 \\ a_5 & \ldots & \ldots & a_8 \\ \hline a_9 & \ldots & \ldots & a_{12} \\ a_{13} & \ldots & \ldots & a_{16} \\ a_{17} & \ldots & \ldots & a_{20} \\ \hline a_{21} & \ldots & \ldots & a_{24} \\ a_{25} & \ldots & \ldots & a_{28} \end{array} \right] \middle| \; a_i \in [0, 6); 1 \leq i \leq 28, + \right\}$$

be the special interval group of infinite order.

This group has several subgroups of finite order and several subgroups of infinite order. Z_6, {0, 2, 4} and {0, 3} are subgroups of [0, 6) which help in getting finite subgroups.

Let

$$T = \left\{ \left[\begin{array}{c|cc|c} a_1 & a_2 & a_3 & a_4 \\ a_5 & \dots & \dots & a_8 \\ \hline a_9 & \dots & \dots & a_{12} \\ a_{13} & \dots & \dots & a_{16} \\ a_{17} & \dots & \dots & a_{20} \\ \hline a_{21} & \dots & \dots & a_{24} \\ a_{25} & \dots & \dots & a_{28} \end{array} \right] \middle| \; a_i \in \{0, 3\}, 1 \leq i \leq 28 \right\} \subseteq S$$

is a finite subgroup of S.

Likewise

$$P = \left\{ \left[\begin{array}{c|cc|c} a_1 & a_2 & a_3 & a_4 \\ a_5 & \dots & \dots & a_8 \\ \hline a_9 & \dots & \dots & a_{12} \\ a_{13} & \dots & \dots & a_{16} \\ a_{17} & \dots & \dots & a_{20} \\ \hline a_{21} & \dots & \dots & a_{24} \\ a_{25} & \dots & \dots & a_{28} \end{array} \right] \middle| \; a_i \in \{0, 2, 4\}, 1 \leq i \leq 28 \right\} \subseteq S$$

is a finite subgroup of S.

Let

$$M_{1,2,3} = \left\{ \left[\begin{array}{c|cc|c} a_1 & a_2 & a_3 & 0 \\ 0 & \dots & \dots & 0 \\ \hline 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 \\ \hline 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 \end{array} \right] \middle| \; a_1, a_2, a_3 \in [0, 6) \right\} \subseteq S$$

is a subgroup of S of infinite order.

$$W_{1,2,3} = \left\{ \left[\begin{array}{c|cc|c} a_1 & a_2 & a_3 & 0 \\ 0 & \cdots & \cdots & 0 \\ \hline 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ \hline 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \end{array}\right] \middle| \; a_1, a_2, a_3 \in \{0, 2, 4\} \right\} \subseteq S$$

is a subgroup of finite order in S.

Thus S has only finite number of subgroups of finite order.

Let us now give one or two examples of special interval super row matrix groups (super column matrix) groups.

Example 2.20: Let

$$S = \left\{ \left[\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ \hline a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ \hline a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{25} & a_{26} & a_{27} & a_{28} \\ a_{29} & a_{30} & a_{31} & a_{32} \\ a_{33} & a_{34} & a_{35} & a_{36} \\ a_{37} & a_{38} & a_{39} & a_{40} \end{array}\right] \middle| \; a_i \in [0, 13);\ 1 \leq i \leq 40, + \right\}$$

be the special interval group of infinite order.

S has several or equivalently $n = {}_{13}C_1 + {}_{13}C_2 + \ldots + {}_{13}C_{13}$ number of subgroups all of them are of infinite order.

$$A_1 = \left\{ \left[\begin{array}{cccc} a_1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \right| a_1 \in [0, 13)\Big\} \subseteq S$$

be the subgroup of S of infinite order so on.

$$A_{27} = \left\{ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \right| a \in [0, 13)\Big\} \subseteq S$$

is a subgroup of S of infinite order.

$$A_{39} = \left\{ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \end{array}\right] \middle|\ a \in [0, 13) \right\} \subseteq S$$

is a special interval subgroup of the special interval super column matrix subgroup of S of infinite order and so on.

$$B_{r_5} = \left\{ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a_2 & a_3 & a_4 \end{array}\right] \middle|\ a_i \in [0, 13), 1 \leq i \leq 4 \right\} \subseteq S$$

is a special interval super column matrix subgroup of S of infinite order.

$$B_{10} = \left\{\left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \end{array}\right] \right| a_i \in [0, 13), 1 \leq i \leq 4\} \subseteq S$$

is again a special interval super column matrix subgroup of infinite order.

Now consider

$$D_{c_1} = \left\{\left[\begin{array}{cccc} a_1 & 0 & 0 & 0 \\ \hline a_2 & 0 & 0 & 0 \\ a_3 & 0 & 0 & 0 \\ \hline a_4 & 0 & 0 & 0 \\ a_5 & 0 & 0 & 0 \\ a_6 & 0 & 0 & 0 \\ \hline a_7 & 0 & 0 & 0 \\ a_8 & 0 & 0 & 0 \\ a_9 & 0 & 0 & 0 \\ a_{10} & 0 & 0 & 0 \end{array}\right] \right| a_i \in [0, 11), 1 \leq i \leq 10\} \subseteq S$$

is a subgroup of S of infinite order.

$$D_{C_3} = \left\{\left[\begin{array}{cccc} 0 & 0 & a_1 & 0 \\ \hline 0 & 0 & a_2 & 0 \\ 0 & 0 & a_3 & 0 \\ \hline 0 & 0 & a_4 & 0 \\ 0 & 0 & a_5 & 0 \\ 0 & 0 & a_6 & 0 \\ \hline 0 & 0 & a_7 & 0 \\ 0 & 0 & a_8 & 0 \\ 0 & 0 & a_9 & 0 \\ 0 & 0 & a_{10} & 0 \end{array}\right] \middle|\; a_i \in [0, 13), 1 \leq i \leq 10\right\} \subseteq S$$

is a subgroup of S of infinite order.

So we can have 14 such subgroups given by D_{C_i} and B_{r_j}; $1 \leq i \leq 4$ and $1 \leq j \leq 10$, however these subgroups will find their place in the n subgroups mentioned.

Example 2.21: Let

$$S = \left\{\left[\begin{array}{cc|ccc|c|ccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \end{array}\right] \middle|\; a_i \in [0, 11), 1 \leq i \leq 27, +\right\}$$

be the special interval row matrix group.

S is of infinite order S has only subgroups of infinite order barring

$$Q = \left\{\left[\begin{array}{cc|ccc|c|ccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \end{array}\right] \right|$$

$$a_i \in \{0, 1, 2, \ldots, 10\}\}, a_i \in [0, 11) \} \subseteq S$$

is a subgroup of infinite order. We have 27 such subgroups. Each $T_i \cong \{[0, 11), +\}$ that is T_i is isomorphic with the special interval group, for $1 \leq i \leq 27$.

Let

$$P_{C_2} = \left\{\left[\begin{array}{cc|ccc|c|ccc} 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \right| a_1, a_2,$$

$$a_3 \in [0, 11), + \} \subseteq S$$

be a subgroup of infinite order we have 9 such subgroups.

$$B_{r_2} = \left\{\left[\begin{array}{cc|ccc|c|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \right| a_i \in [0, 11),$$

$$1 \leq i \leq 9\} \subseteq S$$

is a subgroup of infinite order. We have 3 such subgroups.

Now we give polynomial groups using intervals.

Example 2.22: Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \right| a_i \in [0, 17)\}$$

under + be the special interval group of polynomials of infinite order.

Example 2.23: Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 22), + \right\}$$

be a group. S is of infinite order. S has finite subgroups.

For take

$$M = \left\{ \sum_{i=0}^{10} a_i x^i \,\middle|\, a_i \in \{0, 11\}, 0 \leq i \leq 10, + \right\} \subseteq S$$

is a finite subgroup of S.

$p(x) \in M$ has coefficients either 0 or 11 only and each $p(x) \in M$ is such that $p(x) + p(x) = (0)$; zero polynomial as $11 + 11 \equiv 0 \pmod{22}$.

So S has subgroups of order two, three and so on. S has also subgroups of infinite order.

$$N = \left\{ \sum_{i=0}^{8} a_i x^i \,\middle|\, a_i \in [0, 22), 0 \leq i \leq 8 \right\} \subseteq S$$

is a subgroup of infinite order S has also infinitely many subgroups of finite order.

S has also infinitely many subgroups of finite order.

Example 2.24: Let

$$S = \left\{ \sum_{i=0}^{27} a_i x^i \,\middle|\, a_i \in [0, 19), 0 \leq i \leq 27 \right\}$$

be a special interval polynomial group. S is of infinite order.

The subgroup of finite order being;

$$P = \left\{ \sum_{i=0}^{27} a_i x^i \,\middle|\, a_i \in \{0, 1, 2, 3, 4, 5, \ldots, 18\}, 0 \le i \le 27 \right\} \subseteq S.$$

$$T = \left\{ \sum_{i=0}^{10} a_i x^i \,\middle|\, a_i \in [0, 19), 0 \le i \le 10, + \right\} \subseteq S$$

is a subgroup of infinite order.

Let M = {a + bx | a, b ∈[0, 19), +} ⊆ S is also a subgroup of infinite order.

N = {a + bx + cx^2 + dx^3 | a, b, c, d ∈ [0, 19), +} ⊆ S is a subgroup of infinite order.

Example 2.25: Let

$$S = \left\{ \sum_{i=0}^{15} a_i x^i \,\middle|\, a_i \in [0, 3), 0 \le i \le 15, + \right\}$$

be a special interval group of polynomials of infinite order.

Let

$$X_1 = \left\{ \sum_{i=0}^{5} a_i x^i \,\middle|\, a_i \in \{0, 1, 2\}, 0 \le i \le 5, + \right\} \subseteq S$$

be a subgroup of finite order.

$$X_2 = \left\{ \sum_{i=0}^{8} a_i x^i \,\middle|\, a_i \in \{0, 0.5, 1, 1.5, 2, 2.5\}, 0 \le i \le 8, + \right\} \subseteq S$$

is also a subgroup of finite order.

$$X_3 = \left\{\sum_{i=0}^{10} a_i x^i \,\middle|\, a_i \in \{0, 0.25, 0.50, 0.75, 1, 1.25, 1.50,\right.$$

$$\left. 1.75, 2, 2.25, 2.50, 2.75\}, 0 \leq i \leq 10, +\right\} \subseteq S$$

is a subgroup of finite order.

$Y_1 = \{a + bx \mid a, b \in [0, 3)\} \subseteq S$ is a subgroup of infinite order.

$Y_2 = \{a + bx^2 + cx^4 \mid a, b, c \in [0, 3), +\} \subseteq S$ is a subgroup of infinite order.

$Y_3 = \{a + bx^7 + cx^{10} \mid a, b, c \in [0, 3), +\} \subseteq S$ is a subgroup of infinite order.

Example 2.26: Let

$$S = \left\{\sum_{i=0}^{30} a_i x^i \,\middle|\, a_i \in [0, 2), 0 \leq i \leq 30\right\}$$

be the special interval polynomial group of infinite order. This has several finite subgroups.

Let $X_1 = \{a + bx \mid a, b \in \{0, 1\}, +\} \subseteq S$ be a subgroup of finite order $|X_1| = 4$.

$X_2 = \{a + bx \mid a, b \in \{0, 0.5, 1, 1.5\}, +\} \subseteq S$ is also a subgroup of finite order.

$X_1 \subseteq X_2$. $o(X_1) < o(X_2)$

$X_3 = \{a + bx \mid a, b \in \{0, 0.25, 0.50, 0.75, 1, 1.25, 1.50, 1.75\} +\} \subseteq S$ is a subgroup of finite order.

$X_1 \subseteq X_2 \subseteq X_3$ and $o(X_3) \geq o(X_2) \geq o(X_1)$.

$X_4 = \{a + bx \mid a, b \in \{0.125, 0.250, 0.375, 0.5, 0.625, 0.750, 1, 1.125, 1.250, 1.375, 1.5, 1.675, 1.750\}, +\} \subseteq S$ is a subgroup of finite order.

$o(X_4) > o(X_3) > o(X_2) > o(X_1)$ and

$X_1 \subseteq X_2 \subseteq X_3 \subseteq X_4$.

We can get a chain of subgroups.

We have several such chains.

Let $Y_1 = \{a + bx + cx^2 \mid a, b, c \in \{0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, \ldots, 1.8\} \subseteq [0, 2), +\}$ be a finite subgroup of S.

$Y_2 = \{a + bx + cx^2 \mid a, b, c \in \{0, 0.1, 0.2, \ldots, 1, 1.1, \ldots, 1.9\}, +\} \subseteq S$ is a finite subgroup of S.

Infact S has infinitely many finite subgroups. For let $Y_n = \{a + bx \mid a, b \in \{0, 0.001, 0.002, 0.003, \ldots, 1.001, \ldots, 1.999\} \subseteq [0, 2)$ be a subgroup of finite order.

Thus S has infinitely many finite subgroups.

It is the main advantage of using the interval [0, p) even if p is a prime [0, p) has infinitely many subgroups of finite order under '+'.

THEOREM 2.3: *Let S = {[0, p), +} be the special interval group (p a prime). S has infinitely many subgroups of finite order.*

Proof follows from the fact S_n = {0.0005 or 0.001 or 0.0002 or 0.00002} generates a finite subgroup under addition.

Corollary 2.1: Let p be any composite number in Theorem 2.3. Then also S has infinite number of finite subgroups.

Example 2.27: Let S = {[0, 7), +} be a group under '+'; S has infinitely many subgroups of finite order.

Example 2.28: Let S = {[0, 15), +} be a group. S has infinitely many subgroups of finite order.

Example 2.29: Let S = {[0, 3), × [0, 8), +} be a group. S has infinitely many subgroups of finite order.

Example 2.30: Let S = {[0, 7) × [0, 11) × [0, 29), +} be the special interval group of infinite order. S has infinitely many subgroups of finite order.

Example 2.31: Let S = $\{(a_1, a_2, a_3) \mid a_i \in [0, 3), 1 \leq i \leq 3\}$ be a special interval group. S has infinitely many subgroups of finite order.

We can have the usual notion of group homomorphism ϕ, kernel of the homomorphism ϕ and other properties.

As the group is under addition and the groups are of infinite order it is difficult to arrive more properties about them.

However we see if S = {[0, n), +} be the special interval group we get $Z_n \subseteq S$ as a subgroup of finite order.

Thus we have the following theorem.

THEOREM 2.4: *Let S = {[0, n), +} be the special interval group. $\{Z_n, +\} \subseteq S$ is always a finite subgroup of S.*

The proof is direct and hence left as an exercise to the reader.

Example 2.32: Let S = {[0, 7) × [0, 12) × [0, 17) × [0, 36), +} be a special interval group. Clearly $T = Z_7 \times Z_{12} \times Z_{17} \times Z_{36} \subseteq S$ is a subgroup of finite order.

Also $P_1 = Z_7 \times \{0, 3, 6, 9\} \times Z_{17} \times \{0, 12, 24\} \subseteq S$ is a subgroup of finite order.

$P_2 = Z_7 \times \{0\} \times \{0\} \times \{0, 6, 12, 18, 24, 30\} \subseteq S$ is a subgroup of finite order.

So in a way we call the special interval group under + as the extended modulo integer group under +.

Example 2.33: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \middle| \, a_i \in [0, 15), 1 \leq i \leq 9, + \right\}$$

be the special interval matrix group.

$$T = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \middle| \, a_i \in Z_{15} \subseteq \{0, 1, 2, \ldots, 14\}, 1 \leq i \leq 9, + \right\}$$

be the subgroup of S.

Infact S has infinite number of subgroups of finite order.

Example 2.34: Let

$$S = \left\{ \sum_{i=0}^{10} a_i x^i \,\middle|\, a_i \in [0, 4), 0 \leq i \leq 10, + \right\}$$

be a group of infinite order. S has infinite number of subgroups of finite order.

$$P_1 = \left\{ \sum_{i=0}^{10} a_i x^i \,\middle|\, a_i \in \{0, 2\}, 0 \leq i \leq 10, + \right\} \subseteq S$$

is a subgroup of finite order.

For all $p(x) \in P_1$ we have $p(x) + p(x) = 0$.

$$P_2 = \left\{ \sum_{i=0}^{10} a_i x^i \,\middle|\, a_i \in \{0, 1, 2, 3\}, 0 \leq i \leq 10, +\right\} \subseteq S$$

is a subgroup of finite order.

Let
$T_1 = \{a + bx \mid a, b \in \{0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5\}; +\} \subseteq S$ be the finite subgroup of S.

S has infinitely many subgroups of finite order.

$T_2 = \{a + bx + cx^2 \mid a, b, c \in \{0, 0.2, 0.4, \ldots, 3, 3.2, 3.4, 3.6, 3.8\} \subseteq [0, 4), +\} \subseteq S$.

$T_3 = \{a + bx^2 \mid a, b \in \{0, 0.1, 0.2, \ldots, 0.9, 1, 1.1, \ldots, 3.1, \ldots, 3.9\} \subseteq [0, 4), +\} \subseteq S$ is a subgroup of finite order.

Let

$$R = \left\{ \sum_{i=0}^{3} a_i x^i \,\middle|\, a_i \in \{0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5\} \subseteq [0, 4), 0 \leq i \leq 3\right\} \subseteq S$$

be a subgroup of finite order.

Infact S has infinitely many subgroups of finite order and this infinite groups has infinite number of finite subgroups.

It is an interesting observation for R or Q or Z under addition has no finite subgroups.

We suggest the following problems for the reader.

Problems

1. Find some special and interesting properties associated with special interval groups
   G = {[0, a), +, a a positive integer}.

2. If in a problem 1, a is a prime can G have infinite number of subgroups?

3. Can G in problem 1 have subgroups of infinite order?

4. Prove if a is a composite number in G given in problem 1 then G has many subgroups of finite order.

5. Let S = {[0, 11), +} be a special interval group.

   (i) Can S have subgroups of infinite order?
   (ii) Can S have infinite number of subgroups of finite order?
   (iii) Can S have infinite number of subgroups of infinite order?

6. Let S = {[0, 18), +} be a special interval group.

   Study questions (i) to (iii) of problem 5 for this S.

7. Let S = {[0, 24), +} be the special interval group.

   Study questions (i) to (iii) of problem 5 for this S.

8. Let S = {[0, p^2), p a prime +} be the special interval group.

   Study questions (i) to (iii) of problem 5 for this S.

9. Let S_1 = {[0, pq), p and q primes, +} be the special interval group.

   Study questions (i) to (iii) of problem 5 for this S.

10. Let $S_2 = \{[0,\ p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_n^{\alpha_n})\ \alpha_1 \geq 1, 1 \leq i \leq n$, p_j's prime and all of them are distinct $1 \leq j \leq n\}$ be the special interval group.

    Study questions (i) to (iii) of problem 5 for this S_2.

11. Let $S = \{[a_1, a_2, \ldots, a_9] \mid a_i \in [0, 19), 1 \leq i \leq 19\}$ be a special interval group.

    Study questions (i) to (iii) of problem 5 for this S.

12. Let T = {[0, 13), +} be the special interval group.

    (i) Can T have infinite subgroups other than T?
    (ii) Prove T has infinite number of finite subgroups.
    (iii) What is the smallest order of the finite subgroup?

13. Let S = {[0, 12), +} be the special interval group.

    (i) Find all infinite order subgroups of S.
    (ii) Prove S has infinitely many subgroups of finite order.
    (iii) Is two the order of the smallest subgroup of S?

14. Let S = {[0, p), +, p a prime} be the special interval group.

    (i) Find all infinite order subgroups of S.
    (ii) Prove the order of the smallest subgroup is p.

15. Let S = {[0, 24), +} be the special interval group.

    (i) Prove S has finite subgroups of order 2, 3, 4, 6, 8, 12 and so on.
    (ii) Can S have finite subgroups of order 5, 7, 9, 11, …, p, p a prime?
    (iii) Can S have infinite order subgroups?

16. Let $S = \{(a_1, a_2) \mid a_i \in [0, 11), 1 \leq i \leq 2\}$ be a special interval group under addition +.

    Study questions (i) to (iii) of problem 15 for this S.

17. Let $S_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \middle| \; a_i \in [0, 19); 1 \leq i \leq 9, + \right\}$

    be the special interval group.

    Study questions (i) to (iii) of problem 15 for this S_1.

18. Let $S_2 = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{24} \\ a_{25} & a_{26} & \ldots & a_{36} \\ a_{37} & a_{38} & \ldots & a_{48} \end{bmatrix} \middle| \; a_i \in [0, 29); 1 \leq i \leq 48, \right.$

    $+\}$ be the special interval group.

    Study questions (i) to (iii) of problem 15 for this S_2.

19. Let $S_3 = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{16} \\ a_{17} & a_{18} & \ldots & a_{32} \\ a_{33} & a_{34} & \ldots & a_{48} \\ a_{49} & a_{50} & \ldots & a_{64} \end{bmatrix} \middle| \; a_i \in [0, 43); 1 \leq i \leq 64, \right.$

    $+\}$ be the special interval group.

    Study questions (i) to (iii) of problem 15 for this S_3.

20. Let M = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{bmatrix} \right|$ $a_i \in [0, 30)$;

$1 \leq i \leq 25$, +} be the special interval group.

Study questions (i) to (iii) of problem 15 for this M.

21. Let V = {(a_1, a_2, a_3, a_4, a_5) | $a_1 \in [0, 5)$, $a_2 \in [0, 11)$, $a_3 \in [0, 15)$ $a_4 \in [0, 6)$ and $a_5 \in [0, 12)$, +} be the special interval group.

Study questions (i) to (iii) of problem 15 for this V.

22. Let V_1 = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \right|$ $a_1 \in [0, 30)$, $a_2 \in [0, 5)$, $a_3 \in$

[0, 14), $a_4 \in [0, 11)$, and $a_5 \in [0, 15)$, $a_6 \in [0, 19)$, a_7 , a_8, a_9, $a_{10} \in [0, 25)$, a_{11}, $a_{12} \in [0, 10)$ +} be the special interval group under +.

Study questions (i) to (iii) of problem 15 for this V_1.

23. Let S_1 = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \right|$ $a_1, a_2, a_3 \in [0, 24)$; $a_4, a_5, a_6 \in [0, 18)$ and

$a_7, a_8, a_9 \in [0, 31)$, +} be the special interval group.

Study questions (i) to (iii) of problem 15 for this S.

24. Let S = $\left\{\left[\begin{array}{c} a_1 \\ \hline a_2 \\ a_3 \\ \hline a_4 \\ a_5 \\ a_6 \\ \hline a_7 \\ a_8 \\ a_9 \\ a_{10} \\ \hline a_{11} \\ a_{12} \end{array}\right] \middle| \; a_i \in [0, 41); 1 \leq i \leq 12, +\right\}$ be the special interval group.

Study questions (i) to (iii) of problem 15 for this S.

25. Let S = $\{(a_1\ a_2 \mid a_3\ a_4\ a_5 \mid a_6) \mid a_i \in [0, 7), 1 \leq i \leq 6, +\}$ be the special interval group.

Study questions (i) to (iii) of problem 15 for this S.

26. Let S = $\left\{\left[\begin{array}{c|cc|ccc|c} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_8 & \dots & \dots & \dots & \dots & \dots & a_{14} \\ a_{15} & \dots & \dots & \dots & \dots & \dots & a_{21} \end{array}\right] \middle| \; a_i \in [0, 23); 1 \leq i \leq 21, +\right\}$ be the special interval group.

Study questions (i) to (iii) of problem 15 for this S.

27. Let S = $\left\{\left[\begin{array}{c|ccc|c|cc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ \hline a_8 & \ldots & \ldots & \ldots & \ldots & \ldots & a_{14} \\ a_{15} & \ldots & \ldots & \ldots & \ldots & \ldots & a_{21} \\ a_{22} & \ldots & \ldots & \ldots & \ldots & \ldots & a_{28} \\ \hline a_{29} & \ldots & \ldots & \ldots & \ldots & \ldots & a_{35} \\ a_{36} & \ldots & \ldots & \ldots & \ldots & \ldots & a_{42} \end{array}\right] \right|$ $a_i \in [0, 49)$;

$1 \leq i \leq 42, +\}$ be the special interval group.

Study questions (i) to (iii) of problem 15 for this S.

28. Let S = $\left\{\left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \hline a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ \hline a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \\ a_{22} & a_{23} & a_{24} \\ a_{25} & a_{26} & a_{27} \\ \hline a_{28} & a_{29} & a_{30} \end{array}\right] \right|$ $a_i \in [0, 192)$; $1 \leq i \leq 30, +\}$

be the special interval group.

Study questions (i) to (iii) of problem 15 for this S.

29. Let
S = $\{(a_1 \mid a_2 \; a_3 \; a_4 \; a_5 \mid a_6 \; a_7 \mid a_8) \mid a_i \in [0, 28), 1 \leq i \leq 8, +\}$
be the special interval group.

Study questions (i) to (iii) of problem 15 for this S.

30. Let S = $\left\{\left[\begin{array}{c|ccc|c|ccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ a_9 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & a_{16} \\ a_{17} & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & a_{24} \end{array}\right] \middle| \right.$ $a_i \in$ $[0, 28); 1 \leq i \leq 24\}$ be the special interval group.

    Study questions (i) to (iii) of problem 15 for this S.

31. Let S = $\left\{\sum_{i=0}^{\infty} a_i x^i \middle| \right.$ $a_i \in [0, 121)\}$ be the special interval polynomial of group of infinite order.

    Study questions (i) to (iii) of problem 15 for this S.

32. Let S = $\left\{\sum_{i=0}^{7} a_i x^i \middle| \right.$ $a_i \in [0, 18), 0 \leq i \leq 7\}$

    be the special interval polynomial of group of infinite order.

    (i) Prove S has several subgroups of finite order.
    (ii) Is the number of subgroups of S of finite order infinite or finite?
    (iii) Study questions (i) to (iii) of problem 15 for this S.

33. Let S = $\left\{\sum_{i=0}^{\infty} a_i x^i \middle| \right.$ $a_i \in [0, 36)\}$

    be the special interval group of polynomials.

    (i) Find all subgroups of finite order.
    (ii) Study questions (i) to (iii) of problem 15 for this S.

Chapter Three

SPECIAL INTERVAL SEMIGROUPS ON [0,n)

In this chapter for the first time authors introduce 3 different operations on the interval [0, n); $n < \infty$.

Thus $S_{min} = \{[0, n); \min\}$, $S_{max} = \{[0, n); \max\}$ and $S_{\times} = \{[0, n), \times\}$, $(n < \infty)$ are semigroups.

We study the algebraic substructures enjoyed by them and derive several interesting properties.

Let $S_{min} = \{[0, n), \min\}$ be a semigroup. Infact S_{min} is a semilattice and is of infinite order. S_{min} is commutative and S_{min} is an idempotent semigroup of infinite order. We call S_{min} as the special interval semigroup.

We will first give some examples of them.

Example 3.1: Let $S_{min} = \{[0, 24), \min\}$ be the semigroup of infinite order. Every singleton element is an idempotent.

For if $x = 9.23 \in S_{min}$ then $\min \{x, x\} = x$. Let $t_1 = 8.92$ and $t_2 = 12.03 \in S_{min}$, then $\min \{t_1, t_2\} = 8.92 = t_1$ and $P = \{t_1, t_2\} \subseteq$

S_{min} is a subsemigroup of order two. Infact we can get subsemigroups of order 1, 2, 3, …, any integer.

S_{min} has also subsemigroups of infinite order.

For $T_5 = \{[0, 5), \min\} \subseteq S_{min}$ is a subsemigroup of infinite order and T_5 is also an idempotent subsemigroup. S_{min} has no zero divisors.

Example 3.2: Let $S_{min} = \{[0, 17), \min\}$ be the special interval semigroup of infinite order. S_{min} has infinite number of subsemigroups of finite and infinite order. Every element in S_{min} is an idempotent.

Now using S_{min} we construct semigroups.

Example 3.3: Let $S_{min} = \{(a_1, a_2, a_3) \mid a_i \in [0, 4), 1 \leq i \leq 3\}$ be the special interval semigroup. S_{min} has several semigroups and infact zero divisors.

We call x in S_{min} to be a zero divisor if there exists a y in S_{min} with $\min \{x, y\} = (0, 0, 0)$. We see if $x = (0.32, 0, 0)$ and $y = (0, 0.9, 3.2) \in S_{min}$ then $\min \{x, y\} = (0, 0, 0)$.

Infact S_{min} has infinitely many zero divisors.

S has subsemigroups of infinite order.

Let $M_1 = \{(a_1, 0, 0) \mid a_1 \in [0, 4)\} \subseteq S_{min}$,
$M_2 = \{(0, a_1, 0) \mid a_1 \in [0, 4)\} \subseteq S_{min}$ and
$M_3 = \{(0, 0, a_1) \mid a_1 \in [0, 4)\} \subseteq S_{min}$ be three distinct subsemigroups of S_{min}.

We see $\min \{M_i, M_j\} = \{(0, 0, 0)\}$ if $i \neq j$, $1 \leq i, j \leq 3$. Every element in S_{min} is an idempotent and hence is a subsemigroup.

However we cannot say every pair of elements in S_{min} is a subsemigroup. For if $x = (0.3, 2, 3.4)$ and $y = (0.1, 3, 0.2) \in$

S_{min}. We see min {x, y} = {(0.3, 2, 3.4), (0.1, 3, 0.2)} = (min{0.3, 0.1}, min{2, 3}, min{3.4, 0.2} = (0.1, 2, 0.2) ≠ x or y.

Thus a pair of elements in S_{min} in general is not a subsemigroup under min operation.

Let X = {(0, 0, 0) (a, b, c) | a, b, c ∈[0, ,4) and a, b, c are fixed} ⊆ S_{min}. This pair of X is a subsemigroup.

Thus every pair {x, y} with x = (0, 0, 0) is always a subsemigroup of S_{min}. Let x = (a, b, c) and y = (d, e, f) ∈ S_{min} we say x $\leq_{min}$ y if min (a, d) = a, min (b, e) = b and min {c, f} = c.

Thus if T = $\{x_1, x_2, \ldots, x_n\}$ such that
$x_1 \leq_{min} x_2 \leq_{min} \ldots \leq_{min} x_n$, then T is a subsemigroup.

We call this order $\leq_{min}$ as “special min order”.

Infact S_{min} is not a special min orderable but T ⊆ S_{min} is special min orderable.

A natural question is can we have subsemigroups in S_{min} which are not special min orderable?

The answer is yes and S_{min} itself is not special min orderable.

For take x = (0.2, 1, 2.3) and y = (0.7, 0.9, 1.3) ∈ S_{min}.

We see min {x, y} = {(2, 0.9, 1.3)} ≠ x (or y).

Let min{x, y} = z we see x $\not\leq_{min}$ y but z $\leq_{min}$ x and z $\leq_{min}$ y and M = {x, y, z} is a special interval subsemigroup of S_{min}.

Thus a set which is not special min orderable is a subsemigroup. We can only say S_{min} is a partially special min ordered set.

This concept can help to get trees when the subsemigroups in S_{min} are of finite order.

Let P = {x = (0, 0, 0), x_1 = (0.3, 0.7, 1.1), x_2 = (0.4, 0.93, 0.84), x_3 = (3, 2, 0.2)} $\subseteq S_{min}$; min {x, x_i} = x for i = 1, 2, 3. min{x_1, x_2} = (0.3, 0.7, 0.84) = x_4; min {x_1, x_3} = (0.3, 0.7, 0.2) = x_5, min{x_2, x_3} = (0.4, 0.93, 0.2) = x_6 so P is not a subsemigroup.

We see P is partially min ordered set yet P is not a subsemigroup.

P_1 = {x, x_1, x_2, x_3, x_4, x_5, x_6} $\subseteq S_{min}$ is a subsemigroup of S_{min}.

Several interesting properties can be derived on subsets of S_{min}.

We see if P is only a subset of S_{min} and not a subsemigroup of S_{min} then we can complete it to get the subsemigroup in a finite number of steps if |P| < ∞ and only in infinite number of steps if |P| = ∞.

Example 3.4: Let

$$S_{min} = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \middle|\ a_i \in [0, 19), 1 \le i \le 7 \right\}$$

be the special interval semigroup.

Let us consider

$$M = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0.7 \\ 3 \\ 2.1 \\ 5.0 \\ 0.9 \\ 1.2 \end{bmatrix}, \begin{bmatrix} 0.1 \\ 0.3 \\ 4 \\ 0.2 \\ 3.2 \\ 0.6 \\ 1.2 \end{bmatrix} \right\} \subseteq S_{min};$$

clearly M is not a subsemigroup only a subset as

$$\min \left\{ \begin{bmatrix} 0 \\ 0.7 \\ 3 \\ 2.1 \\ 5.0 \\ 0.9 \\ 1.2 \end{bmatrix}, \begin{bmatrix} 0.1 \\ 0.3 \\ 4 \\ 0.2 \\ 3.2 \\ 0.6 \\ 1.2 \end{bmatrix} \right\} = \begin{bmatrix} 0 \\ 0.3 \\ 3 \\ 0.2 \\ 3.2 \\ 0.6 \\ 1.2 \end{bmatrix} \notin M$$

so M is only a subset as $\min \left\{ \begin{bmatrix} 0 \\ 0.7 \\ 3 \\ 2.1 \\ 5.0 \\ 0.9 \\ 1.2 \end{bmatrix}, \begin{bmatrix} 0.1 \\ 0.3 \\ 4 \\ 0.2 \\ 3.2 \\ 0.6 \\ 1.2 \end{bmatrix} \right\} \notin M.$

Now

$$W = \left\{\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0.1\\0.3\\4\\0.2\\3.2\\0.6\\1.2\end{bmatrix}, \begin{bmatrix}0\\0.7\\3\\2.1\\5.0\\0.9\\1.2\end{bmatrix}, \begin{bmatrix}0\\0.3\\3\\0.2\\3.2\\0.6\\1.2\end{bmatrix}\right\} \subseteq S_{min}$$

is a special interval subsemigroup of S_{min}.

Let

$$T = \left\{\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\end{bmatrix}, x_1 = \begin{bmatrix}0.2\\0.4\\5\\3.8\\7\\8\\9\end{bmatrix}, x_2 = \begin{bmatrix}6\\9\\2\\4\\4.3\\3.1\\2.5\end{bmatrix}, x_3 = \begin{bmatrix}0.5\\3\\4.3\\2.7\\2.5\\7\\5\end{bmatrix}\right\} \subseteq S_{min}.$$

We see $\min\{x_1, x_2\} = \left\{\begin{bmatrix}0.2\\0.4\\5\\3.8\\7\\8\\9\end{bmatrix}, \begin{bmatrix}6\\9\\2\\4\\4.3\\3.1\\2.5\end{bmatrix}\right\} = \begin{bmatrix}0.2\\0.4\\2\\3.8\\4.3\\3.1\\2.5\end{bmatrix} \notin W.$

$$\min\{x_1, x_3\} = \min\left\{\begin{bmatrix}0.2\\0.4\\5\\3.8\\7\\8\\9\end{bmatrix}, \begin{bmatrix}0.5\\3\\4.3\\2.7\\2.5\\7\\5\end{bmatrix}\right\} = \begin{bmatrix}0.2\\0.4\\4.3\\2.7\\2.5\\7\\5\end{bmatrix} \notin W.$$

$$\min\{x_2, x_3\} = \min\left\{\begin{bmatrix}6\\9\\2\\4\\4.3\\3.1\\2.5\end{bmatrix}, \begin{bmatrix}0.5\\3\\4.3\\2.7\\2.5\\7\\5\end{bmatrix}\right\} = \begin{bmatrix}0.5\\3\\2\\2.7\\2.5\\3.1\\2.5\end{bmatrix} \notin W.$$

Thus if we extend W by

$$W_1 = \left\{\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\end{bmatrix}, x_1, x_2, x_3, \begin{bmatrix}0.2\\0.4\\2\\3.8\\4.3\\3.1\\2.5\end{bmatrix}, \begin{bmatrix}0.2\\0.4\\4.3\\2.7\\2.5\\7\\5\end{bmatrix}, \begin{bmatrix}0.5\\3\\2\\2.7\\2.5\\3.1\\2.5\end{bmatrix}\right\} \subseteq S_{min}$$

is a special interval subsemigroup of $|W_1| = 7$.

If we have a set with 3 distinct elements we can extend W to W_1 and $|W_1| = 7$.

If the $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is added to W_1 we get order of W_1 is 7.

Likewise if $V = \{x_1, x_2, x_3, x_4, x_5\}$ such that $\min \{x_i, x_j\} \neq x_i$ or x_j if $i \neq j$ and $x_i \not\leq_{min} x_j$; if $i \neq j$ then V is not a subsemigroup we can complete V as follows:

$V \cup \{\min \{x_i, x_j\}; i \neq j, 1 \leq i, j \leq 5\} = V_1$; V_1 is a subsemigroup of S_{min}; $|V_1| = 5 + 5C_2 = 15$.

Thus if $A = \{x_1, x_2, \ldots, x_n\}$ with $\min \{x_i, x_j\} \neq x_i$ or x_j if $i \neq j$ then A is not a subsemigroup but we can complete A as A_1 and $|A_1| = n + nC_2$.

This is true for any finite n (This is true for infinite n also). Thus we can in a nice way complete a subset into a subsemigroup under min operation.

Example 3.5: Let

$$S_{min} = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \middle| \; a_i \in [0, 17), 1 \leq i \leq 30 \right\}$$

be a special interval matrix semigroup under min operation.

S_{min} has subsemigroups of order 1, 2, 3, 4, …, n; also n is infinite.

We can also for any given subset A $\subseteq S_{min}$ complete it to get a subsemigroup.

If A is a subset of S_{min} with n elements such that min {x, y} $\neq$ x or y and x $\neq$ y true for every x, y $\in$ A, then we can complete A to A_1 and A_1 will be subsemigroup of order n + nC_2.

Example 3.6: Let

$$S_{min} = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \right| a_i \in [0, 11), 1 \leq i \leq 15 \right\}$$

be the special interval semigroup.

Let

$$A = \left\{ \begin{bmatrix} 0.7 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 3.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 7.8 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2.98 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

$\subseteq S_{min}$.

A is only a subsemigroup.

A can be completed to A_1 only if x in A has a y in A with min {x, y} = x or y.

$$B = \left\{ \begin{bmatrix} 0.2 & 0.7 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0.9 & 0.2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0.3 & 9 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 7 & 6.8 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

$$\subseteq S_{min}$$

is not a subsemigroup. B can be completed to B_1 with $|B_1| = 4 + 4C_2 = 10$.

$$D = \left\{ \begin{bmatrix} 0.3 & 0 & 0 \\ 4 & 0 & 0 \\ 8 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 9 & 0 & 0 \\ 2 & 0 & 0 \\ 9.2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2.5 & 0 & 0 \\ 9.8 & 0 & 0 \\ 3.9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 10.5 & 0 & 0 \\ 2.9 & 0 & 0 \\ 7.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0.07 & 0 & 0 \\ 6.16 & 0 & 0 \\ 3.9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 4.3 & 0 & 0 \\ 7.1 & 0 & 0 \\ 6.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \subseteq S_{min};$$

be a subset of D and D can be made or completed into a subsemigroup D_1; $|D_1| = 6 + 6C_2 = 6 + 6.5 / 1.2 = 6 + 15 = 21$.

Let

$$E = \left\{ \begin{bmatrix} 9 & 0 & 2 \\ 0 & 4.3 & 0 \\ 7.1 & 0 & 9 \\ 0 & 3.5 & 0 \\ 0.1 & 0 & 0.6 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 4.5 \\ 0 & 3.7 & 6.3 \\ 9.6 & 0 & 9.9 \\ 0 & 0 & 7.2 \\ 6.5 & 0 & 0 \end{bmatrix} \right\} \subseteq S_{min},$$

E is only a subset of S_{min}. E can be completed to E_1 to be a subsemigroup of order three.

Example 3.7: Let

$$S_{min} = \left\{ \left[\begin{array}{c|ccc} a_1 & a_2 & a_3 & a_4 \\ \hline a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{array}\right] \middle| \; a_i \in [0, 9), 1 \leq i \leq 16 \right\}$$

be the special interval semigroup.

$$\text{Let } M = \left\{ \left[\begin{array}{c|ccc} 8 & 0.7 & 5.2 & 6.9 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{c|ccc} 6 & 6.3 & 0.2 & 0.7 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \right.$$

$$\left. \left[\begin{array}{c|ccc} 0.5 & 0.4 & 0.9 & 6.1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{c|ccc} 6.1 & 0.9 & 0.4 & 0.2 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \right\} \subseteq S_{min}.$$

M is only a subset and not a subsemigroup.

M can be completed to M_1 by adjoining all min$\{x, y\}$, $x \neq y$ where $x, y \in M$.

Thus $M \cup \{\min \{x, y\}\}; x \neq y, x, y \in M\} = M_1$ is a subsemigroup of S_{min}. $|M_1| = 4 + 4C_2$.

Example 3.8: Let

$$S_{\min} = \left\{ \left[\begin{array}{c|ccc|c} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & \ldots & \ldots & \ldots & a_{10} \\ a_{11} & \ldots & \ldots & \ldots & a_{15} \\ \hline a_{16} & \ldots & \ldots & \ldots & a_{20} \\ \hline a_{21} & \ldots & \ldots & \ldots & a_{25} \end{array} \right] \middle| \; a_i \in [0, 41), 1 \leq i \leq 25 \right\}$$

be a special interval semigroup of infinite order.

S_{min} has subsemigroups of all order and also subsets of S_{min} can be completed to get subsemigroups of both finite and infinite order.

We give the following theorem.

THEOREM 3.1: *Let S_{min} = {m $\times$ n matrix with entries from [0, s); s an integer; min} be the special interval semigroup of infinite order.*

If $P = \{x_1, x_2, \ldots, x_n\} \subseteq S_{min}$ (n finite or infinite) with min $\{x_i, x_j\} \neq x_i$ or x_j for $1 \leq i, j \leq n$ then the subset P can be completed to P_1 such that $P_1 = P \cup \{\min \{x_i, x_j\}; 1 \neq j, 1 \leq i, j \leq n\}$ and P_1 is a subsemigroup of S_{min}.

Proof follows from the fact that min operation in P_1 gives the desired subsemigroup.

Now we proceed onto study S_{max} = {[0, n), max}. S_{max} is also an infinite commutative semigroup which an idempotent semigroup.

We give examples of them and study their properties.

Example 3.9: Let S_{max} = {[0, 10), max} be the special interval semigroup of infinite order. S_{max} is an idempotent semigroup. We see if $T = \{x_1, x_2, x_3, x_4\} \subseteq S_{max}$, T is a subsemigroup.

S_{max} has subsemigroups of order 1, 2, 3, 4, ….

We see S_{max} is a chain for any two elements in S_{max} is max orderable that is if any x, y $\in S_{max}$ we have $x \leq_{max} y$ or $y \leq_{max} x$. Thus any subset of S_{max} is a subsemigroup. This is the special feature enjoyed by these special interval max semigroups.

Example 3.10 : Let S_{max} = {[0, 231), max} be a special interval max semigroup. Let x = 230.009 and y = 9.32 $\in S_{max}$.

T = {x, y} is a subsemigroup.

Hence these idempotent semigroups are max orderable semigroups with 0 as the least element. However this has no maximal or to be more precise the greatest element.

Now S_{max} cannot have zero divisors or the concept of units. These are semilattices of a perfect type.

Example 3.11: Let
$S_{max} = \{(a_1, a_2, a_3, a_4) \mid a_i \in [0, 15); 1 \leq i \leq 4\}$ be the special interval semigroup of infinite order.

Let x = (0.3, 6.9, 9.2, 0.7) and y = (12.1, 3, 4, 5.1) $\in S_{max}$. We see max {x, y} = max {(0.3, 6.9, 9.2, 0.7) (12.1, 3, 4, 5.1)}

= (max {0.3, 12.1}, max {6.9, 3} max {9.2, 4}, max {0.7, 5.1})

= (12.1, 6.9, 9.2, 5.1) ≠ x or y.

Thus P = {x, y} $\subseteq S_{max}$ is a subsemigroup.

However P_1 = {x, y, max{x, y}} ⊆ S is a subsemigroup. So in general a pair of elements in S_{max} is not a subsemigroup.

THEOREM 3.2: *Let*
S_{max} = {m × n matrix with entries from [0, s); max} be the special interval semigroup.

If x, y $\in S_{max}$ is such that $x \leq_{max} y$ ($y \leq_{max} x$) then S_{max} is a subsemigroup. Conversely if a pair of elements x, y $\in S_{max}$ is a subsemigroup, then $x \leq_{max} y$ ($y \leq_{max} x$) respectively.

Example 3.12: Let

$$S_{max} = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in [0, 18), 1 \leq i \leq 9 \right\}$$

be a special interval semigroup.

We see for

$$x = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \text{ and } y = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix} \in S_{max}$$

are ordered by $\leq_{max}$ if and only if $a_i \leq b_i$ for each i, i = 1, 2, …, 9.

Take

$$x = \begin{bmatrix} 0.3 & 7 & 2 \\ 4.2 & 3.1 & 11.8 \\ 12.3 & 5.001 & 7.09 \end{bmatrix} \text{ and}$$

$$y = \begin{bmatrix} 4 & 11. & 4 \\ 7.3 & 10.5 & 14.07 \\ 13.031 & 17.011 & 9.028 \end{bmatrix} \in S_{max}.$$

We see $x \leq_{max} y$ as we see each element x is strictly less than the corresponding element in y.

Now take

$$x = \begin{bmatrix} 9.2 & 2.3 & 0.3 \\ 11.2 & 1.5 & 3.92 \\ 7.3 & 17.5 & 16.5 \end{bmatrix} \text{ and } y = \begin{bmatrix} 3.7 & 9.2 & 10.31 \\ 9.73 & 3.4 & 1.82 \\ 4.7 & 10.5 & 17.891 \end{bmatrix} \in S_{max}.$$

We see $\max\{x, y\} = \begin{bmatrix} 9.2 & 9.2 & 10.31 \\ 11.2 & 3.4 & 3.92 \\ 7.3 & 17.5 & 17.891 \end{bmatrix} \neq x$

or y also $x \leq_{max} y$ and $y \leq_{max} x$.

So in general in this S_{max} we cannot order the matrix.

This is true in general for any $x, y, z \in S_{max}$.

If max $\{x, y\} \neq x$ or y or z we see $\{x, y, z\}$ does not form a subsemigroup.

However if $\{x, y, z\} \subseteq S_{max}$ is such that $\max\{x, y\} = z$ then $\{x, y, z\}$ forms a subsemigroup of S_{max}.

We have several subsemigroups in this S_{max} isomorphic with T = {[0, 18), max}.

Let

$$A_1 = \left\{\begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \, a_1 \in [0, 18), \max\right\} \subseteq T = \{[0, 18), \max\}$$

be a subsemigroup and is isomorphic with A_1.

We have at least 16 subsemigroup isomorphic to P = {[0, 18), max}.

Take

$$A_8 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a & 0 \end{bmatrix} \middle| \; a \in [0, 18) \right\} \subseteq S_{min} \, ;$$

A_8 is a subsemigroup and is isomorphic to P.

We see if the matrix in S_{min} has more than one entry and if we have more than one such matrices we see that subset in general will not be a subsemigroup so we have to make the completion of it.

Example 3.13: Let

$$S_{max} = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{19} & a_{20} \end{bmatrix} \middle| \; a_i \in [0, 17); 1 \leq i \leq 20 \right\}$$

be a special interval semigroup of infinite order and is commutative.

Let

$$P_1 = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \middle| \; a_1, a_2, a_3, a_4 \in [0, 17) \right\} \subseteq S_{max}$$

be a subsemigroup of infinite order.

However P_1 is not isomorphic to T = {[0, 17), max}.

$$P_2 = \left\{ \begin{bmatrix} a_1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \middle| \; a_1 \in [0, 17), \max \right\} \subseteq S_{max}$$

is a subsemigroup isomorphic with T = {[0, 17), max}. $P_2 \cong T$.

Let

$$P_4 = \left\{ \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \\ \vdots & \vdots \\ 0 & a_3 \end{bmatrix} \middle| \; a_1, a_2, a_3 \in [0, 17); \max \right\} \subseteq S_{max},$$

P_4 is a subsemigroup and is not isomorphic to T.

Likewise S_{max} has several subsemigroups which are not isomorphic to T.

Example 3.14: Let

$$S_{max} = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \hline a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ \hline a_{19} & a_{20} & a_{21} \\ a_{22} & a_{23} & a_{24} \\ a_{25} & a_{26} & a_{27} \\ \hline a_{28} & a_{29} & a_{30} \end{array} \right] \middle| \; a_i \in [0, 27); 1 \leq i \leq 30 \right\}$$

be the special interval super matrix semigroup.

S_{max} has infinite number of idempotents and infinite number of finite subsemigroups, infact infinite number of subsemigroups of order 1, order 2 and so on.

Recall semigroups of S is said to be a Smarandache semigroup if it has a subset P such that P under the operations of S is a group.

Clearly S_{max} in example 3.14 is not a Smarandache semigroup.

Inview of all those we have the following theorem.

THEOREM 3.3: *Let S_{max} or S_{min} be special interval semigroups. Both S_{max} and S_{min} are not Samrandache semigroups.*

The proof is direct and hence left as an exercise to the reader.

THEOREM 3.4: *Let S_{max} be a special interval matrix semigroup.*

(i) S_{max} has only a unique minimal element (least element)
(ii) S_{max} has no maximal element.

For proof (0), the zero matrix is the minimal element of S_{max}.

For (0) is the least element as max {(0), X} = X for every X $\in S_{max} \setminus \{0\}$.

THEOREM 3.5: *Let S_{min} be the special interval matrix semigroup.*

(i) (0) is the least element of S_{min}.
(ii) S_{min} has no greatest element.

Proof is direct and hence left as an exercise to the reader.

Example 3.15: Let

$$S_{max} = \left\{ \left[\begin{array}{c|ccc|c|c} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \end{array}\right] \middle| \; a_i \in [0, 27);\ 1 \leq i \leq 18 \right\}$$

be the special interval semigroup of super row matrix.

$$(0) = \left[\begin{array}{c|ccc|c|c} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$ is the least element of S_{max}.

However S_{max} has no greatest element.

Further S_{max} is not a Smarandache semigroup. $|S_{max}| = \infty$; S_{max} has infinite number of any finite order subsemigroup. S_{max} also has infinite number of infinite subsemigroup.

Example 3.16: Let

$$S_{min} = \left\{ \begin{bmatrix} a_1 & a_2 & ... & a_5 \\ a_6 & ... & ... & a_{10} \\ a_{11} & ... & ... & a_{15} \\ a_{16} & ... & ... & a_{20} \\ a_{21} & ... & ... & a_{25} \end{bmatrix} \middle| \; a_i \in [0, 49);\ 1 \leq i \leq 25 \right\}$$

be the special interval semigroup under min operation.

$$(0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & ... & ... & 0 \\ 0 & ... & ... & 0 \\ 0 & ... & ... & 0 \\ 0 & ... & ... & 0 \end{pmatrix}$$

is the least element of S_{min} and min{x, (0)} = {(0)} for all x $\in$ S_{min}.

Now we proceed onto describe semigroups using intervals under product.

Example 3.17: Let

$$S_{min} = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ \vdots & \vdots & \vdots & \vdots \\ a_{37} & a_{38} & a_{39} & a_{40} \end{bmatrix} \middle| \; a_i \in [0, 25); 1 \leq i \leq 40 \right\}$$

be the special interval semigroup.

We see max{(0), x} = x for all x $\in$ S_{max} \ {(0)}.

Example 3.18: Let $S_\times$ = {[0, 13), $\times$} be the special interval semigroup.

Let x = 0.001 and y = 2.01 $\in$ $S_\times$; x $\times$ y = 0.00201 $\in$ $S_\times$.

Let x = 5.002 and y = 0.005 $\in$ $S_\times$,
x $\times$ y = 5.002 $\times$ 0.0005 = 0025010 $\in$ $S_\times$.

We see $S_\times$ has zero divisors.

We see 1 $\in$ $S_\times$ is such that x $\times$ 1 = x for all x $\in$ $S_\times$.

Example 3.19: Let $S_\times$ = {[0, 15), $\times$} be the special interval semigroup under product $\times$.

Take x = 3 and y = 5 $\in$ $S_\times$; we see x $\times$ y = 3 $\times$ 5 = 0 (mod 15).

Let x = 4 $\in$ $S_\times$; x^2 = 4 $\times$ 4 = 1 (mod 15), so x is a unit.

Let $x = 2$ and $y = 8 \in S_\times$; $x \times y = 2 \times 8 = 16 = 1 \pmod{15}$. So $S_\times$ has units. $S_\times$ has zero divisors for $x = 3$ and $y = 10$ in $S_\times$ is such that

$x \times y = 3 \times 10 \equiv 0 \pmod{15}$ is a zero divisor.

Example 3.20: Let $S_\times = \{[0, 13), \times\}$ be the special interval semigroup. $S_\times$ has zero divisors. $S_\times$ has unit for $x = 7$ and $y = 2$ is such that $x \times y = 7 \times 2 = 14 \equiv 1 \pmod{13}$.

Example 3.21: Let $S_\times = \{[0, 24), \times\}$ be the special interval semigroup of infinite order. $S_\times$ has units for $5 \in S_\times$ is such that $5^2 = 1 \pmod{24}$. $7 \in S_\times$ is such that $7^2 \equiv 49 \equiv 1 \pmod{24}$ and $11 \in S_\times$ is such that $11^2 = 1 \pmod{24}$.

$S_\times$ has zero divisors for take $x = 6$, $y = 4 \in S_\times$, is such that $x \times y = 6 \times 4 \equiv 0 \pmod{24}$.

$x = 8$ and $y = 3 \in S_\times$ is such that $8 \times 3 \equiv 0 \pmod{24}$, $x = 2$ and $y = 12 \in S_\times$ is such that $x \times y = 2 \times 12 \equiv 0 \pmod{24}$.

$x = 4$ and $y = 12$ is such that $x \times y = 4 \times 12 \equiv 0 \pmod{24}$.

$x = 6$ and $y = 8 \in S_\times$ is such that $x \times y = 6 \times 8 = 0 \pmod{24}$. $x = 8$ and $y = 9 \in S_\times$ is such that $x \times y = 8 \times 9 = 0 \pmod{24}$. $x = 6$ and $y = 12 \in S_\times$ is such that $x \times y = 6 \times 12 = 72 = 0 \pmod{24}$ that $x \times y = 8 \times 12 = 0 \pmod{24}$ and so on.

$S_\times$ also has idempotents.

For $9 \in S_\times$ is such that $9 \times 9 = 81 = 9 \pmod{24}$ $16 \in S_\times$ is such that $16 \times 16 = 16 \pmod{24}$.

Further $S_\times$ also has nilpotent elements for $x = 12 \in S_\times$ is such that $x^2 \equiv 0 \pmod{12}$.

Thus $S_\times$ has units, idempotents, zero divisors and nilpotents $|S_\times| = \infty$.

$S_\times$ also has subgroups, for $P_1 = \{23, 1\} \subseteq S_\times$ is a subgroup of $S_\times$ so $S_\times$ is a Smarandache semigroup. $P_2 = \{7, 1\} \subseteq S_\times$ is also a group under $\times$. $P_3 = \{1, 5\} \subseteq S_\times$ is also a group under $\times$.

$P_3 = \{16, 8\} \subseteq S_\times$ is also a subgroup and so on.

Thus $S_\times$ is a S-semigroup.

Example 3.22: Let $S_\times = \{[0, 19), \times\}$ be a special interval semigroup under product. $S_\times$ has units and $S_\times$ has zero divisors. $S_\times$ is a S-subsemigroup as $P = \{1, 2, \ldots, 18\} \subseteq S_\times$ is a group under $\times$.

Every element in P is invertible and they are the only units of $S_\times$ and $S_\times$ has no idempotents.

Example 3.23: Let $S_\times = \{[0, 7), \times\}$ be a special interval semigroup under $\times$.

We see $2, 3, 4, 5, 6 \in S_\times$ are units but $S_\times$ has no idempotents. $S_\times$ is a Smarandache semigroup for $P_1 = \{1, 6\}$ and $P_2 = \{1, 2, 3, 4, 5, 6\} \subseteq S_\times$ are subgroups of $S_\times$.

$S_\times$ has infinitely many elements such that they are not units, for take $x = 0.31 \in S_\times$ we see $x^2 = 0.31 \times 0.31 = 0.0961$ and $x^3 = 0.0961 \times 0.31$ and so on. $x^n \to 0$.

Take $y = 6.1 \in S_\times$; $y^2 \cong 5.3154142$ that as $n \to \infty$ y^n may reach zero.

Thus $S_\times$ has infinite number of elements which are neither units nor idempotents, only finite number of units, has no idempotents but has zero divisors.

Example 3.24: Let $S_\times = \{[0, 6), \times\}$ be the special interval semigroup. $S_\times$ has finite number of idempotents for 3 and $4 \in S_\times$ are such that $3 \times 3 = 3 \pmod 6$ and $4 \times 4 = 4 \pmod 6$. Thus $S_\times$ has only two non trivial idempotents.

$S_\times$ has zero divisors for $2 \times 3 \equiv 0 \pmod 6$ and $4 \times 3 = 0 \pmod 6$. $S_\times$ has only two zero divisors. $S_\times$ has only one unit for $5 \in S_\times$ is such that $5^2 = 1 \pmod 6$.

$S_\times$ is a Smarandache semigroup as $P = \{1, 5\} \subseteq S_\times$ is a group. $S_\times$ has infinite number of elements which are not idempotents or units. Infact $S_\times$ contains the semigroup, $\{Z_6, \times\}$ as a proper subset which is a subsemigroup.

Example 3.25: Let $S_\times = \{[0, 16), \times\}$ be a special interval semigroup. $S_\times$ has only finite number of units, zero divisors and no idempotents.

For $x = 4 \in S_\times$ is a zero divisor as $4 \times 4 = x^2 = 0 \pmod{16}$, $y = 8 \in S_\times$ is a zero divisor for $8 \times 8 = 0 \pmod{16}$.

Also $x \times y = 0 \pmod{16}$.
Further $2 \times 8 = 0 \pmod{16}$.
We have $4 \times 8 = 0 \pmod{16}$.
$12 \times 4 = 0 \pmod{16}$.

$x = 11$ and $y = 3$ in $S_\times$ is such that $x \times y = 1 \pmod{16}$.
$7 \times 7 = 1 \pmod{16}$ in $S_\times$.
$13 \times 5 = 1 \pmod{16}$ in $S_\times$.
$9 \times 9 = 1 \pmod{16}$ in $S_\times$ are some of the units of $S_\times$.
$S_\times$ is a S-semigroup.

Example 3.26: Let $S_\times = \{[0, 30), \times\}$ be a special interval semigroup. $S_\times$ has units, idempotents and zero divisors. For $6 \in S_\times$ is such that $6 \times 6 = 6 \pmod{30}$, $10 \in S_\times$; $10 \times 10 = 10 \pmod{30}$,

$25 \times 25 \equiv 25 \pmod{30}$,
$15 \times 15 \equiv 15 \pmod{30}$,
$16 \times 16 \equiv 16 \pmod{30}$ and
$21 \times 21 \equiv 21 \pmod{30}$ are some idempotents of $S_\times$.

We see $10 \times 3 \equiv 0 \pmod{30}$,

$15 \times 2 \equiv 0 \pmod{30}$,
$10 \times 6 \equiv 0 \pmod{30}$,
$15 \times 4 \equiv 0 \pmod{30}$,
$10 \times 9 \equiv 0 \pmod{30}$,
$15 \times 6 \equiv 0 \pmod{30}$,

and $10 \times 12 \equiv 0 \pmod{30}$ and so on are all zero divisors of S. The units of $S_\times$ are $29 \in S_\times$ is such that $29 \times 29 \equiv 1 \pmod{30}$ and $11^2 \equiv 1 \pmod{3}$ units in $S_\times$. Thus $S_\times$ has only finite number of units, idempotents and zero divisors.

Example 3.27: Let $S_\times = \{[0, 25), \times\}$ be the semigroup of the special interval [0, 25).

$S_\times$ has units and zero divisors.
For $13 \times 2 \equiv 1 \pmod{25}$ is a unit
$17 \times 3 \equiv 1 \pmod{25}$ is a unit
$24 \times 24 \equiv 1 \pmod{25}$ is a unit and
$19 \times 4 \equiv 1 \pmod{25}$ is a unit.

Consider $5^2 \equiv 0 \pmod{25}$ is a zero divisor.

$10 \times 10 \equiv 0 \pmod{25}$. $15^2 \equiv 0 \pmod{25}$ and $20 \times 20 \equiv 0 \pmod{25}$ are some of the zero divisors of $S_\times$. However $S_\times$ has no nontrivial idempotents.

Example 3.28: Let $S_\times = \{[0, 14), \times\}$ be a special interval semigroup.

$7 \times 2 \equiv 0 \pmod{14}$; $4\times 7 \equiv 0 \pmod{14}$; $6\times 7 \equiv 0 \pmod{14}$; $8\times 7 \equiv 0 \pmod{14}$; $10 \times 7 \equiv 0 \pmod{14}$; and $12 \times 7 \equiv 0 \pmod{14}$ are zero divisors of $S_\times$.

$5 \times 3 \equiv 1 \pmod{14}$ and $13 \times 13 \equiv 1 \pmod{14}$ are units of $S\times$. $7^2 = 7 \pmod{14}$ and $8^2 = 8 \pmod{14}$ idempotents of $S_\times$.

Example 3.29: Let $S_\times = \{[0, 10), \times\}$ be a semigroup of the special interval [0, 10).

$S_\times$ has idempotents $5^2 \equiv 5 \pmod{10}$ and $6^2 = 6 \pmod{10}$ are idempotents of $S_\times$. $7 \times 3 \equiv 1 \pmod{10}$ and $9 \times 9 \equiv 1 \pmod{10}$ are units of $S_\times$.

$2 \times 5 \equiv 0 \pmod{10}$, $4 \times 5 \equiv 0 \pmod{10}$ $6 \times 5 \equiv 0 \pmod{10}$ and $8 \times 5 \equiv 0 \pmod{10}$ are zero divisors of $S_\times$.

So 5 and 6 can be used to construct dual like numbers of $S_\times$. $S_\times$ is a Smarandache semigroup as $\{1, 9\} = P \subseteq S_\times$ is a group.

Example 3.30: Let $S_\times = \{[0, 21), \times\}$ be the special interval semigroup. $11 \times 2 \equiv 1 \pmod{21}$, $13 \times 13 \equiv 1 \pmod{13}$, $20 \times 20 \equiv 1 \pmod{21}$ $8 \times 8 = 1 \pmod{21}$ $4 \times 16 \equiv 1 \pmod{21}$ and $17 \times 5 \equiv 1 \pmod{21}$ are units of $S_\times$.

$7^2 \equiv 7 \pmod{21}$ is an idempotent of $S_\times$. $15^2 \equiv 15 \pmod{21}$ is an idempotent and both 7, 15 can be used to build special dual like numbers of $S_\times$.

$3 \times 7 \equiv 0 \pmod{21}$ $6 \times 7 \equiv 0 \pmod{21}$, $9 \times 7 \equiv 0 \pmod{21}$, $12 \times 7 \equiv 0 \pmod{21}$ $15 \times 7 \equiv 0 \pmod{21}$ and $18 \times 7 \equiv 0 \pmod{21}$ are some of the zero divisors of $S_\times$.

Now in view of all this we have the following theorem.

THEOREM 3.6: *Let $S_\times = \{[0, p), \times\}$ be the special interval semigroup.*

(i) *If p is a prime, $S_\times$ has zero divisors but no idempotents and (p–2) number of units.*
(ii) *If p is a composite number $S_\times$ has zero divisors, units, idempotents and nilpotents.*
(iii) *$S_\times$ is always a S-semigroup.*
(iv) *$S_\times$ has finite subsemigroups.*

The proof is direct and hence left as an exercise to the reader.

Example 3.31: Let $S_\times = \{[0, 2p), \times, p \text{ a prime}\}$ be the special interval semigroup. $S_\times$ has non trivial idempotents.

Now we describe special interval matrix semigroup under product.

Example 3.32: Let $S_\times = \{(a_1, a_2, a_3, a_4) \mid a_i \in [0, 5), 1 \leq i \leq 4, \times\}$ be the special interval row matrix semigroup under product.

$S_\times$ has zero divisors and idempotents; for x = (0 1 0 1) in $S_\times$ is such that $x^2 = x$, (1 1 1 0) $\in S_\times$ is also an idempotent.

We see (1111) is the unit of $S_\times$.

$S_\times$ has units, x = (2 3 4 1) $\in S_\times$ and y = (3 2 4 1) $\in S_\times$ is such that x $\times$ y = (2 3 4 1) $\times$(3 2 4 1) = (1 1 1 1).

Let x = (0 2 0 0) and y = (0 0 4 3) $\in S_\times$ then x $\times$ y = (0 2 0 0) $\times$ (0 0 4 3) = (0 0 0 0) is a zero divisor.

$S_\times$ is of infinite order and $S_\times$ is a Smarandache semigroup. Infact $S_\times$ has finite number of finite subsemigroups.

The interesting feature is [0, 5) is an interval with prime 5 yet if we take row matrix under product we get $S_\times$ to have idempotents, zero divisors and units.

Example 3.33: Let
$S_\times = \{(a_1, a_2, \ldots, a_{10})$ where $a_i \in [0, 40), 1 \leq i \leq 40, \times\}$ be the special interval semigroup of infinite order. $S_\times$ is commutative has zero divisors, units and idempotents. $S_\times$ is a Smarandache semigroup.

Take M = {(1, 1, …, 1), (39, 39, …, 30)} $\subseteq S_\times$ is a group under $\times$, hence the claim.

T = {(1, 1, 1, …, 1), (11, 11, …, 11) $\subseteq S_\times$ is also a group.
W = {(1, 1, …, 1), (9, 9, …, 9)} $\subseteq S_\times$ is also a group.

Now $x = (0, 7, 10, 4, 8, 0, 5, 20, 10, 15)$ and $y = (9, 0, 4, 10, 5, 9, 8, 2, 8, 8) \in S_\times$ are such that $x \times y = (0, 0, \ldots, 0)$.

Example 3.34: Let

$$S_\times = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \middle| \; a_i \in [0, 24), 1 \le i \le 9, \times_n \right\}$$

be the special interval column matrix semigroup. $S_\times$ has idempotents, units, zero divisors and nilpotents.

$$x = \begin{bmatrix} 5 \\ 1 \\ 23 \\ 1 \\ 5 \\ 23 \\ 1 \\ 1 \\ 5 \end{bmatrix} \in S_\times \text{ is such that } x^2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\text{Clearly} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in S_\times \text{ is the unit of } S_\times.$$

$$\text{Let } y = \begin{bmatrix} 8 \\ 0 \\ 6 \\ 12 \\ 4 \\ 3 \\ 9 \\ 0 \\ 18 \end{bmatrix} \text{ and } z = \begin{bmatrix} 3 \\ 11 \\ 4 \\ 2 \\ 6 \\ 8 \\ 8 \\ 19 \\ 4 \end{bmatrix} \in S_\times$$

We see

$$y \times_n z = \begin{bmatrix} 8 \\ 0 \\ 6 \\ 12 \\ 4 \\ 3 \\ 9 \\ 0 \\ 18 \end{bmatrix} \times_n \begin{bmatrix} 3 \\ 11 \\ 4 \\ 2 \\ 6 \\ 8 \\ 8 \\ 19 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

is the zero divisor of $S_\times$.

Infact $S_\times$ has many zero divisors also.

We have infinite number of zero divisors.

Example 3.35: Let

$$S_\times = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{18} \end{bmatrix} \middle| \; a_i \in [0, 23), 1 \leq i \leq 18, \times_n \right\}$$

be the special interval column matrix.

$S_\times$ has idempotents which has only entries as 0 and 1 in the column matrix 18×1.

$S_\times$ has zero divisors, units and has no nilpotent element. Units are finite in number however zero divisors are infinite in number.

Further number of idempotents is also finite.

Example 3.36: Let

$$S_\times = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in [0, 12), 1 \leq i \leq 9, \times_n \right\}$$

be the special interval square matrix.

$S_\times$ has infinite number of zero divisors but only a finite number of idempotents and units. Infact $S_\times$ has idempotents.

Example 3.37: Let

$$S_\times = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{17} & a_{18} \end{bmatrix} \middle| \; a_i \in [0, 15), 1 \leq i \leq 18, \times_n \right\}$$

be the special interval semigroup. $S_\times$ has units, zero divisors and idempotents.

Only the number of zero divisors is infinite. Further $S_\times$ is a S-semigroup and $S_\times$ has several infinite subsemigroups also many finite subsemigroups.

Example 3.38: Let

$$S_\times = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ \vdots & \vdots & \vdots & \vdots \\ a_{45} & a_{46} & a_{47} & a_{48} \end{bmatrix} \right| a_i \in [0, 33), 1 \leq i \leq 48, \times_n \right\}$$

be the special interval matrix semigroup of infinite order.

$S_\times$ has infinite number of subsemigroups and finite number of finite subsemigroup. $S_\times$ is a S-semigroup. $S_\times$ has finite number of units and infinite number of zero divisors.

Next concept, one is interested in studying about these semigroups, is the ideals in them.

We will describe this by some examples.

Example 3.39: Let
$S_\times = \{(a_1, a_2, a_3, a_4) \mid a_i \in [0, 12), 1 \leq i \leq 4, \times_n\}$ be the special interval semigroup.

$P_1 = \{(a_1, 0, 0, 0) \mid a_1 \in [0, 12), \times\} \subseteq S_\times$ is a special interval subsemigroup of $S_\times$ which is also an ideal of $S_\times$.

$P_2 = \{(0, a_1, 0, 0) \mid a_1 \in [0, 12), \times\} \subseteq S_\times$ is a subsemigroup as well as an ideal of $S_\times$.

$B_1 = \{(a_1, 0, 0, 0) \mid a_1 \in \{0, 1, 2, \ldots, 11\} \subseteq S_\times$ is only a subsemigroup of $S_\times$ and is not an ideal of $S_\times$.

$B_2 = \{(0, a_1, a_2, 0, 0) \mid a_1\ a_2 \in \{0, 2, 4, 6, 8, 10\} \subseteq [0, 12)\} \subseteq S_\times$ is only a subsemigroup of $S_\times$ and is not an ideal of $S_\times$.

$B_3 = \{(0, a_1, 0, a_2) \mid a_1\ a_2 \in \{0, 6\} \subseteq [0, 12)\} \subseteq S_\times$ is only a subsemigroup of $S_\times$ and is not an ideal of $S_\times$.

Thus $S_\times$ has subsemigroups which are not ideals.

Example 3.40: Let

$$S_\times = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \middle| \; a_i \in [0, 23), 1 \le i \le 7, \times_n \right\}$$

be the special interval semigroup.

$S_\times$ is of infinite order has subsemigroups and ideals.

$S_\times$ has zero divisors, units and idempotents.

Clearly $S_\times$ has infinite number of zero divisors however the number of units and idempotents are finite in number.

Let

$$P_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \; a_i \in [0, 23) \right\} \subseteq S_\times$$

be a subsemigroup as well as an ideal of $S_\times$.

Further $P_1 \cong \{[0, 23), \times\}$ is a special interval semigroup.

$$B_1 = \left\{\begin{bmatrix} a_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle|\ a_i \in \{0, 1, 2, 3, 4, \ldots, 21, 22\} \subseteq S_\times\right.$$

is only a subsemigroup and is not an ideal of $S_\times$.

$$P_2 = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle|\ a_i \in [0,23), 1 \leq i \leq 3\right\} \subseteq S_\times$$

is again an ideal of $S_\times$.

$$B_2 = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle|\ a_i \in \{0, 1, 2, 3, \ldots, 22\} \subseteq [0,23), 1 \leq i \leq 3\right\} \subseteq S_\times$$

is only a subsemigroup of finite order and is not an ideal of $S_\times$.

$$P_3 = \left\{\left.\begin{bmatrix} 0 \\ a_1 \\ 0 \\ a_2 \\ 0 \\ a_3 \\ 0 \end{bmatrix}\right| a_i \in [0,23),\ 1 \leq i \leq 3\right\} \subseteq S_\times \text{ is an ideal of } S_\times.$$

Thus we can have ideals and subsemigroups which are not ideals of $S_\times$.

Example 3.41: Let

$$S_\times = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}\right| a_i \in [0, 12),\ 1 \leq i \leq 24\right\}$$

be the special interval semigroup under the natural product $\times_n$.

$S_\times$ has subsemigroups which are not ideals.

$S_\times$ is an infinite S-semigroup.

$S_\times$ has finite number of units and idempotents, however $S_\times$ has infinite number of zero divisors.

$S_\times$ has finite number of finite subsemigroups which are not ideals of $S_\times$.

For take

$$S_\times = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ \vdots & \vdots & \vdots & \vdots \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \middle| \; a_i \in 1/2, 1/2^2, 1/2^3, \ldots, \right.$$

$$\left. 1/2^n \text{ as } n \to \infty\} \subseteq [0, 12)\} \right\} \subseteq S_\times$$

is not an ideal of $S_\times$.

Now having seen special matrix semigroups which are built using [0, n); we proceed onto give one or two examples of special interval super matrix semigroups.

Example 3.42: Let

$$S_\times = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ \hline a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ \hline a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ \hline a_{19} & a_{20} & a_{21} \\ a_{22} & a_{23} & a_{24} \\ a_{25} & a_{26} & a_{27} \\ a_{28} & a_{29} & a_{30} \\ \hline a_{31} & a_{32} & a_{33} \end{array} \right] \middle| \; a_i \in [0, 15), 1 \leq i \leq 33 \right\} \subseteq S_\times$$

be the special interval column super matrix semigroup of infinite order.

$S_\times$ has finite number of units and idempotents but infinite number of zero divisors.

$S_\times$ is a S-semigroup. $S_\times$ has number of infinite subsemigroups which are ideals as well as subsemigroups which are not ideals. $S_\times$ has finite subsemigroups which are not ideals of $S_\times$.

Example 3.43: Let
$S_\times = \{(a_1 \mid a_2\ a_3 \mid a_4\ a_5\ a_6 \mid a_7\ a_8\ a_9\ a_{10}\ a_{11}) \mid a_i \in [0, 7), 1 \leq i \leq 11\}$ be a special row super matrix of interval semigroup. $o(S_\times) = \infty$. $S_\times$ is a S-semigroup.

$S_\times$ has infinite number of zero divisors, only finite number of idempotents and units.

$S_\times$ has finite subsemigroups which are not ideals and $S_\times$ has infinite subsemigroups which are ideals.

Example 3.44: Let

$$S_\times = \left\{ \left[\begin{array}{cc|cc|c} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & \cdots & \cdots & \cdots & a_{10} \\ a_{11} & \cdots & \cdots & \cdots & a_{15} \\ \hline a_{16} & \cdots & \cdots & \cdots & a_{20} \\ a_{21} & \cdots & \cdots & \cdots & a_{25} \\ \hline a_{26} & \cdots & \cdots & \cdots & a_{30} \\ a_{31} & \cdots & \cdots & \cdots & a_{35} \\ \hline a_{36} & \cdots & \cdots & \cdots & a_{40} \end{array} \right] \middle| \; a_i \in [0, 17), 1 \leq i \leq 40 \right\} \subseteq S_\times$$

be the special interval semigroup of infinite order.

$S_\times$ has subsemigroups of finite and infinite order which are not ideals. $S_\times$ has ideals and zero divisors. $S_\times$ has finite number of units and idempotents.

Next we proceed onto study intervals of these intervals [0, n).

Example 3.45: Let $S_\times = \{[a, b] \mid a, b \in [0, 9), \times\}$ be the special interval semigroup. $S_\times$ has zero divisors, units and idempotents.

Let x = [3, 5) and y = [3, 3] $\in S_\times$. x × y = [0,0].

Let x = [2, 8] and y = [5, 8] $\in S_\times$.

x × y = [2, 8] × [5, 8] = [1, 1] so $S_\times$ has units and [1, 1] is the multiplicative identity of $S_\times$.

Let x × y = [7, 3] × [0, 6] = [0, 0] is again a zero divisor.

$S_\times$ is a semigroup of infinite order.

Suppose x = [6.3, 8.2] and y = [7.2, 5.5] $\in S_\times$.

Now x × y = [6.3, 8.2] × [7.2, 5.5] = [0.36, 0.10] $\in S_\times$.

That is why we use only natural class of intervals and the product is also a natural product.

Example 3.46: Let S = {[a, b] | a, b $\in$ [0, 13), ×} be the special interval semigroup. o(S) = ∞. S is a S-semigroup of infinite order.

S has zero divisors units and no idempotents other than [0, 1] and [1, 0] are idempotents apart from [1, 1] and [0, 0] are all trivial idempotents of $S_\times$.

$S_\times$ has no nontrivial idempotents.

x = [3, 7] $\in S_\times$ has y = [9, 2] $\in S_\times$ such that
x × y = [3, 7] × [9, 2] = [1, 1] is a unit of $S_\times$.

Every element of the form [a, b] with a, b $\in$ {1, 2, 3, 4, …, 12} has inverse.

However $S_\times$ has infinite number of elements which has no inverse. Elements of the form [a, b] with a, b $\in$ [0, 13) \ {0, 1, 2, 3, ..., 12} has no inverse and they also do not contribute to zero divisors in finite steps.

All elements in $T = \{[a, 0] \mid a \in [0, 13)\} \subseteq S_\times$ and $Q = \{[0, a] \mid a \in [0, 13)\} \subseteq S_\times$ are such that $x \times y = [0, 0]$;

for every $x \in T$ and every $y \in Q$. Thus $S_\times$ has infinite number of zero divisors.

Example 3.47: Let $S_\times = \{[a, b] \mid a, b \in [0, 24), \times\}$ be the special interval semigroup.

$S_\times$ has idempotents, zero divisors, and units, $x = [9, 1] \in S_\times$ is such that $x^2 = [9, 1] \times [9, 1] = [9, 1]$.

$x = [12, 6]$ and $y = [2, 8]$ in $S_\times$ are such that $x \times y = [12, 6] \times [2, 8] = [0, 0]$ is a zero divisor.

$S_\times$ is a S-semigroup as $P = \{[1, 1], [1, 23], [23, 1], [23, 23]\} \subseteq S_\times$ is a group of S; hence the claim.

Example 3.48: Let $S_\times = \{[a, b] \mid a, b \in [0, 6), \times\}$ be the special interval semigroup of infinite order.

$S_\times$ is a S-semigroup as $P = \{[1, 1], [1, 5], [5, 1], [5, 5]\} \subseteq S_\times$ is a group of $S_\times$.

$S_\times$ has idempotents for $x = [4, 3] \in S_\times$ is such that
$x^2 = [4, 3] \times [4, 3] = [4, 3] \in S_\times$.

Let $y = [3, 1] \in S_\times$ is such that $y^2 = y$, $y^2 = [1, 4] \in S_\times$ such that $y^2 = y$, $y^3 = [3, 4] \in S_\times$ is also an idempotent.

However $S_\times$ has only finite number of idempotents. $S_\times$ has infinite number of zero divisors.

$S_\times$ has only finite number of units.

Example 3.49: Let

$$S_\times = \left\{ \begin{bmatrix} [a_1, b_1] \\ [a_2, b_2] \\ \vdots \\ [a_8, b_8] \end{bmatrix} \middle| \; a_i, b_i \in [0, 12), 1 \leq i \leq 8, \times \right\}$$

be the special interval semigroup. $S_\times$ has infinite number of zero divisors.

$S_\times$ has idempotents. $S_\times$ has units. $S_\times$ has infinite number of subsemigroups.

$S_\times$ has finite subsemigroups also.

Let

$$T = \left\{ \begin{bmatrix} [a_1, b_1] \\ [a_2, b_2] \\ \vdots \\ [a_8, b_8] \end{bmatrix} \middle| \; a_i, b_i \in \{0, 1, 2, 3, 4, 5, 6, \ldots, 11\}, \right.$$

$$\left. 1 \leq i \leq 8 \right\} \subseteq S_\times.$$

T is a subsemigroup of $S_\times$ of finite order.

$$T_1 = \left\{ \begin{bmatrix} [a_1, b_1] \\ [a_2, b_2] \\ \vdots \\ [a_8, b_8] \end{bmatrix} \middle| \; a_i, b_i \in \{0, 3, 6, 9\} \; 1 \leq i \leq 8 \right\} \subseteq S_\times$$

is a subsemigroup of finite order.

Example 3.50: Let
$S_\times = \{([a_1, b_1], [a_2, b_2], \ldots, [a_6, b_6]) \mid a_i, b_i \in [0, 14), 1 \leq i \leq 6\}$ be the special interval semigroup. $S_\times$ has infinite number of subsemigroups.

However $S_\times$ has only finite number of finite subsemigroups. $S_\times$ is a S-semigroup. S has infinite number of zero divisors.

Example 3.51: Let

$$S_\times = \left\{\left.\begin{bmatrix} [a_1,b_1] & [a_2,b_2] \\ [a_3,b_3] & [a_4,b_4] \\ [a_5,b_5] & [a_6,b_6] \\ [a_7,b_7] & [a_8,b_8] \end{bmatrix}\right| a_i, b_i \in [0, 19), 1 \leq i \leq 8\right\}$$

be the special interval semigroup.

$S_\times$ has no non trivial idempotents except those matrices with elements [0,1] [1, 0], [1, 1] and [0, 0]. $S_\times$ has units and zero divisors.

Infact $S_\times$ is a S-semigroup. $S_\times$ has several groups but all of them are of finite order.

$S_\times$ has several subsemigroups of infinite and finite order. $S_\times$ also has ideals.

For

$$P_1 = \left\{\left.\begin{bmatrix} [a_1,b_1] & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right| a_1, b_1 \in [0, 19), \times\right\} \subseteq S_\times$$

is a subsemigroup of $S_\times$ which is also an ideal of $S_\times$. Clearly $|P_1| = \infty$.

Example 3.52: Let

$$S_\times = \left\{ \left[\begin{array}{ccc} [a_1, b_1] & [a_2, b_2] & [a_3, b_3] \\ \vdots & \vdots & \vdots \\ [a_{28}, b_{28}] & [a_{29}, b_{29}] & [a_{30}, b_{30}] \end{array} \right] \middle| \; a_i, b_i \in [0, 40),\ 1 \leq i \leq 30 \right\}$$

be a special interval semigroup.

$S_\times$ is a S-semigroup; has infinite number of zero divisors, only finite number of units and idempotents.

$S_\times$ has ideals, infinite and finite order subsemigroups.

Example 3.53: Let

$$S_\times = \left\{ \left[\begin{array}{c|ccc|c} [a_1, b_1] & [a_2, b_2] & & [a_3, b_3] & [a_4, b_4] \\ \hline [a_5, b_5] & \dots & & \dots & \dots \\ [a_9, b_9] & \dots & & \dots & \dots \\ [a_{13}, b_{13}] & \dots & & \dots & \dots \\ [a_{17}, b_{17}] & \dots & & \dots & \dots \\ \hline [a_{21}, b_{21}] & \dots & & \dots & \dots \\ [a_{25}, b_{25}] & \dots & & \dots & \dots \end{array} \right] \middle| \; a_i, b_i \in [0, 23),\ 1 \leq i \leq 28 \right\}$$

be the special interval interval semigroup of infinite order.

$S_\times$ is S-semigroup, has ideals, subsemigroups of finite and infinite order.

$S_\times$ has only finite number of units and idempotents; however has infinite number of zero divisors.

Example 3.54: Let $S_\times$ =

$$\left\{\left[\begin{array}{c|cc|ccc|c} [a_1,b_1] & [a_2,b_2] & [a_3,b_3] & [a_4,b_4] & [a_5,b_5] & [a_6,b_6] & [a_7,b_7] \\ [a_8,b_8] & [a_9,b_9] & [a_{10},b_{10}] & [a_{11},b_{11}] & [a_{12},b_{12}] & [a_{13},b_{13}] & [a_{14},b_{14}] \\ [a_{15},b_{15}] & [a_{16},b_{16}] & [a_{17},b_{17}] & [a_{18},b_{18}] & [a_{19},b_{19}] & [a_{20},b_{20}] & [a_{21},b_{21}] \end{array}\right]\right|$$

$a_i, b_i \in [0, 43), 1 \leq i \leq 21\}$

be the special interval super row matrix semigroup of infinite order.

$S_\times$ has no non trivial idempotents and the idempotent matrices in $S_\times$ has only elements from [1, 1] [0, 1] [0, 0] and [1, 0].

$S_\times$ has infinite number of zero divisors and has only finite number of units.

$S_\times$ has both infinite and finite order subsemigroups, however ideals of $S_\times$ are of infinite order.

Now having seen examples of special interval subsemigroup we now proceed onto suggest a few problems for the reader.

Problems:

1. Let S_{min} = {[0, 9), min} be the special interval semigroup under min.

   (i) Show S_{min} has infinite number of finite subsemigroups.
   (ii) Show S_{min} has infinite number of infinite subsemigroups.
   (iii) Show every pair is totally min ordered.

2. Let $S_{min} = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in [0, 12), 1 \leq i \leq 5\}$ be the special interval row matrix semigroup.

   (i) Study questions (i) to (iii) of problem (1) for this S_{min}.
   (ii) Show S_{min} has infinite number of zero divisors.

3. Let $S_{min} = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix} \middle| a_i \in [0, 19), 1 \leq i \leq 12 \right\}$

   be the special interval column matrix semigroup.

   (i) Study questions (i) to (iii) of problem (1) for this S_{min}.
   (ii) Show S_{min} has infinite number of zero divisors.
   (iii) Show S_{min} is not totally ordered with $\leq_{min}$.
   (iv) Show every subset of S_{min} can be completed into a subsemigroup.

4. Let $S_{min} = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & \ldots & \ldots & \ldots & \ldots & a_{12} \\ a_{13} & \ldots & \ldots & \ldots & \ldots & a_{18} \\ a_{19} & \ldots & \ldots & \ldots & \ldots & a_{24} \\ a_{25} & \ldots & \ldots & \ldots & \ldots & a_{30} \\ a_{31} & \ldots & \ldots & \ldots & \ldots & a_{36} \end{bmatrix} \middle| a_i \in [0, 93), \right.$

   $1 \leq i \leq 36\}$ be the special interval matrix semigroup.

   Study questions (i) to (iv) of problem (3) for this S_{min}.

5. Let $S_{min} = \left\{ \left[\begin{array}{cccc} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \end{array} \right] \middle|\; a_i \in [0, 119), 1 \leq i \leq 30 \right\}$

be the special interval semigroup.

Study questions (i) to (iv) of problem (3) for this S_{min}.

6. Let $S_{min} = \left\{ \left[\begin{array}{c|ccc|cc|c} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ \hline a_8 & \ldots & \ldots & \ldots & \ldots & \ldots & a_{14} \\ a_{15} & \ldots & \ldots & \ldots & \ldots & \ldots & a_{21} \\ a_{22} & \ldots & \ldots & \ldots & \ldots & \ldots & a_{28} \\ \hline a_{29} & \ldots & \ldots & \ldots & \ldots & \ldots & a_{35} \\ \hline a_{36} & \ldots & \ldots & \ldots & \ldots & \ldots & a_{42} \\ a_{43} & \ldots & \ldots & \ldots & \ldots & \ldots & a_{49} \end{array} \right] \middle|\; a_i \in \right.$

$[0, 105), 1 \leq i \leq 49\}$ be the special interval super matrix semigroup.

Study questions (i) to (iv) of problem (3) for this S_{min}.

7. Let $S_{min} = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ \hline a_4 & a_5 & a_6 \\ \hline a_7 & a_8 & a_9 \\ \hline a_{10} & a_{11} & a_{12} \\ \hline a_{13} & a_{14} & a_{15} \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \\ \hline a_{31} & a_{32} & a_{33} \\ a_{34} & a_{35} & a_{36} \\ a_{37} & a_{38} & a_{39} \end{array} \right] \middle|\; a_i \in [0, 437), 1 \leq i \leq 39 \right\}$ be

the special interval super column matrix semigroup under min operation.

Study questions (i) to (iv) of problem (3) for this S_{min}.

8. Let $S_{max} = \{[0, 27), \max\}$ be the special interval semigroup under max operation.

Study questions (i) to (iii) of problem (1) for this S_{max}.

9. Let $S_{max} = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| a_i \in [0, 12), 1 \leq i \leq 6 \right\}$ be the special interval semigroup.

(i) Study questions (i) to (iv) of problem (3) for this S_{max}.

(ii) Show $(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is the least element of S_{max}.

10. Let $S_{max} = \left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_8 \\ a_9 & a_{10} & \ldots & a_{16} \\ a_{17} & a_{18} & \ldots & a_{26} \\ a_{27} & a_{28} & \ldots & a_{32} \\ a_{33} & a_{34} & \ldots & a_{40} \\ a_{41} & a_{42} & \ldots & a_{48} \\ a_{49} & a_{50} & \ldots & a_{56} \\ a_{57} & a_{58} & \ldots & a_{64} \end{bmatrix} \middle| \; a_i \in [0, 27), 1 \leq i \leq 64\right\}$

be the special interval matrix semigroup under max operation.

Study questions (i) to (iii) of problem (3) for this S_{max}.

Show S_{max} has no zero divisors.

11. Let $S_\times = \{[0, 18), \times\}$ be the special interval semigroup.

(i) Find how many idempotents in $S_\times$ exist?
(ii) Find all units of $S_\times$.
(iii) Can $S_\times$ have zero divisors?
(iv) Prove $o(S_\times) = \infty$.
(v) Find finite subsemigroups of $S_\times$.
(vi) Can $S_\times$ have ideals?
(vii) Can $S_\times$ have infinite subsemigroups?
(viii) Is $S_\times$ a S-semigroup?

12. Find some special and striking features enjoyed by $S_\times$.

13. Let $S_\times = \{[0, 43), \times\}$ be a special interval semigroup.

Study questions (i) to (viii) of problem 11 for this $S_\times$.

14. Let $S_\times = \{[0, 7) \times [0, 23), \times\}$ be the special interval semigroup.

Study questions (i) to (viii) of problem (11) for this $S_\times$.

15. Let $S_\times = \{[0, p_1) \times [0, p_2) \times \ldots \times [0, p_n)$ each p_i is a distinct prime, $1 \leq i \leq n, \times\}$ be the special interval semigroup.

    Study questions (i) to (viii) of problem (11) for this $S_\times$.

16. Let $S_\times = \{(a_1, a_2, \ldots, a_{11}) \mid a_i \in [0, 12), 1 \leq i \leq 11\}$ be the special interval semigroup.

    Study questions (i) to (viii) of problem (11) for this $S_\times$.

17. Let $S_\times = \{(a_1, a_2, \ldots, a_9) \mid a_i \in [0, 19), 1 \leq i \leq 9\}$ be the special interval semigroup.

    Study questions (i) to (viii) of problem (11) for this $S_\times$.

18. Let $S_\times = \left\{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \middle| a_i \in [0, 18), 1 \leq i \leq 9, \times_n\right\}$ be the special interval semigroup.

    Study questions (i) to (viii) of problem (11) for this $S_\times$.

19. Let $S_\times = \left\{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix} \middle| a_i \in [0, 29), 1 \leq i \leq 12, \times_n\right\}$ be the special interval semigroup.

Study questions (i) to (viii) of problem (11) for this $S_\times$.

20. Let $S_\times = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{18} \end{bmatrix} \right| a_i \in [0, 9) \times [0, 15), 1 \leq i \leq 18, \times_n \right\}$ be the special interval semigroup.

Study questions (i) to (viii) of problem (11) for this $S_\times$.

21. Let $S_\times = \{(a_1, a_2, \ldots, a_{12}) \mid a_i \in [0, 3) \in [0, 11) \in [0, 23); 1 \leq i \leq 12\}$ be the special interval semigroup.

Study questions (i) to (viii) of problem (11) for this $S_\times$.

22. Let $S_\times = \left\{ \left. \begin{bmatrix} a_1 & a_2 & \cdots & a_8 \\ a_9 & a_{10} & \cdots & a_{16} \\ \vdots & \vdots & & \vdots \\ a_{57} & a_{58} & \cdots & a_{64} \end{bmatrix} \right| a_i \in [0, 43), 1 \leq i \leq 64, \times_n \right\}$ be the special interval semigroup.

Study questions (i) to (viii) of problem (11) for this $S_\times$.

23. Let $S_\times = \left\{ \left. \begin{bmatrix} a_1 & a_2 & \cdots & a_{10} \\ a_{11} & a_{12} & \cdots & a_{20} \\ a_{21} & a_{22} & \cdots & a_{30} \\ a_{31} & a_{32} & \cdots & a_{40} \\ a_{41} & a_{42} & \cdots & a_{50} \end{bmatrix} \right| a_i \in [0, 48), 1 \leq i \leq 50, \times_n \right\}$ be the special interval semigroup.

Study questions (i) to (viii) of problem (11) for this $S'_{\times}$.

24. Let

$$S_{\times} = \left\{ \left[\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \vdots & \vdots & \vdots & \vdots \\ a_{77} & a_{78} & a_{79} & a_{80} \end{array} \right] \right| a_i \in [0, 10) \times [0, 18) \times$$

$[0, 24)$, $1 \leq i \leq 80$, $\times_n\}$ be the special interval semigroup.

Study questions (i) to (viii) of problem (11) for this $S_{\times}$.

25. Let
$S_{\times} = \{(a_1\ a_2 \mid a_3 \mid a_4\ a_5\ a_6 \mid a_7\ a_8 \mid a_9) \mid a_i \in [0, 40) \times [0, 83)$;

$1 \leq i \leq 9$, $\times\}$ be the special interval semigroup.

Study questions (i) to (viii) of problem (11) for this $S_{\times}$.

26. Let

$$S_{\times} = \left\{ \left[\begin{array}{c|ccc|cc|cccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \end{array} \right] \right|$$

$a_i \in [0, 27)$, $1 \leq i \leq 20$, $\times_n\}$ be the special interval semigroup.

Study questions (i) to (viii) of problem (11) for this $S_{\times}$.

27. Let $S_\times = \left\{ \left[\begin{array}{c} a_1 \\ \hline a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ \hline a_7 \\ a_8 \\ a_9 \\ \hline a_{10} \\ a_{11} \\ \hline a_{12} \end{array} \right] \middle| \; a_i \in [0, 48), 1 \leq i \leq 12, \times_n \right\}$

be the special interval semigroup.

Study questions (i) to (viii) of problem (11) for this $S_\times$.

28. Let $S_\times = \left\{ \left[\begin{array}{cc} a_1 & a_2 \\ \hline a_3 & a_4 \\ a_5 & a_6 \\ \hline a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \\ \hline a_{13} & a_{14} \\ a_{15} & a_{16} \\ a_{17} & a_{18} \\ \hline a_{19} & a_{20} \\ a_{21} & a_{22} \\ \hline a_{23} & a_{24} \end{array} \right] \middle| \; a_i \in [0, 31) \times [0, 6), 1 \leq i \leq 24, \times_n \right\}$

be the special interval semigroup.

(i) Study questions (i) to (viii) of problem (11) for this $S_\times$.

(ii) Enumerate any of the special features enjoyed by this $S_\times$.

29. Let $S_\times = \{[a, b] \mid a, b \in [0, 29), \times\}$ be the special interval semigroup.

(i) Study all the special properties associated with this $S_\times$.
(ii) Prove $S_\times$ has infinite number of subsemigroups.
(iii) Prove $S_\times$ has finite subsemigroups.
(iv) Find the total number of finite subsemigroups in $S_\times$.
(v) Prove $S_\times$ has infinite number of zero divisors.
(vi) Prove $S_\times$ has units.
(vii) Can $S_\times$ have idempotents (if so find them)?
(viii) Find all ideals of $S_\times$.

30. Let $S_\times = \{[a, b] \mid a, b \in [0, 18) \times [0, 43), \times\}$ be the special interval semigroup.

Study questions (i) to (viii) of problem (29) for this S.

31. Let
$S_\times = \{[a_1, b_1], [a_2, b_2], [a_3, b_3] \mid a_i, b_i \in [0, 119), 1 \leq i \leq 3,$

$\times\}$ be the special interval semigroup.

Study questions (i) to (viii) of problem (29) for this S.

32. Let $S_\times = \{[a_1, b_1], [a_2, b_2], \ldots, [a_{12}, b_{12}] \mid a_i, b_i \in [0, 248),$

$1 \leq i \leq 12, \times\}$ be the special interval semigroup.

Study questions (i) to (viii) of problem (29) for this $S_\times$.

33. Let $S_\times = \{[a_1, b_1], [a_2, b_2], \ldots, [a_9, b_9] \mid a_i, b_i \in [0, 7) \times [0, 27), 1 \leq i \leq 9\}$ be the special interval semigroup.

Study questions (i) to (viii) of problem (29) for this $S_\times$.

34. Let $S_\times = \left\{\left.\begin{bmatrix} [a_1,b_1] & [a_2,b_2] & \ldots & [a_7,b_7] \\ [a_8,b_8] & [a_9,b_9] & \ldots & [a_{14},b_{14}] \end{bmatrix}\right| a_i \in [0, 33),\right.$

$1 \leq i \leq 14\}$ be the special interval semigroup.

Study questions (i) to (viii) of problem (29) for this $S_\times$.

35. Let $S_\times = \left\{\left.\begin{bmatrix} [a_1,b_1] \\ [a_2,b_2] \\ \vdots \\ [a_9,b_9] \end{bmatrix}\right| a_i \in [0, 30), 1 \leq i \leq 9, \times_n\right\}$ be the

special interval semigroup.

Study questions (i) to (viii) of problem (29) for this $S_\times$.

36. Let $S_\times = \left\{\left.\begin{bmatrix} [a_1,b_1] \\ [a_2,b_2] \\ \vdots \\ [a_{18},b_{18}] \end{bmatrix}\right| a_i \in [0, 91) \times [0, 28), 1 \leq i \leq 18,\right.$

$\times_n\}$ be the special interval semigroup.

(i) Study questions (i) to (viii) of problem (29) for this $S_\times$.
(ii) Enumerate any of the striking features of this $S_\times$.

37. Let

$$S_{\times} = \left\{ \left[\begin{array}{c} [a_1, b_1] \\ \hline [a_2, b_2] \\ [a_3, b_3] \\ \hline [a_4, b_4] \\ [a_5, b_5] \\ [a_6, b_6] \\ [a_7, b_7] \\ \hline [a_8, b_8] \\ [a_9, b_9] \\ \hline [a_{10}, b_{10}] \\ [a_{11}, b_{11}] \\ [a_{12}, b_{12}] \\ \hline [a_{13}, b_{13}] \end{array} \right] \right| a_i\ b_i \in [0, 12),\ 1 \leq i \leq 13,\ \times_n \Bigg\} \text{ be}$$

the special interval semigroup.

(i) Study questions (i) to (viii) of problem (29) for this $S_{\times}$.

(ii) Does this enjoy other special properties?

38. Let $S_{\times} = \{[a_1, b_1], [a_2, b_2] \mid [a_3, b_3] \mid [a_4\ b_4]) \mid a_i, b_i \in [0, 3)$ $\times [0, 48), 1 \leq i \leq 4\}$ be the special interval semigroup.

Study questions (i) to (viii) of problem (29) for this $S_{\times}$.

39. Let $S_\times = \left\{ \left[\begin{array}{ccc} [a_1, b_1] & [a_2, b_2] & [a_3, b_3] \\ \hline [a_4, b_4] & [a_5, b_5] & [a_6, b_6] \\ [a_7, b_7] & [a_8, b_8] & [a_9, b_9] \\ [a_{10}, b_{10}] & [a_{11}, b_{11}] & [a_{12}, b_{12}] \\ \hline [a_{13}, b_{13}] & [a_{14}, b_{14}] & [a_{15}, b_{15}] \\ [a_{17}, b_{17}] & [a_{18}, b_{18}] & [a_{19}, b_{19}] \\ [a_{20}, b_{20}] & [a_{21}, b_{21}] & [a_{22}, b_{22}] \\ [a_{23}, b_{23}] & [a_{24}, b_{24}] & [a_{25}, b_{25}] \\ \hline [a_{26}, b_{26}] & [a_{27}, b_{27}] & [a_{28}, b_{28}] \\ [a_{29}, b_{29}] & [a_{30}, b_{30}] & [a_{31}, b_{31}] \\ \hline [a_{31}, b_{31}] & [a_{32}, b_{32}] & [a_{33}, b_{33}] \end{array} \right] \right|$ $a_i\ b_i \in [0, 5)$

$\times$ [0, 12), $1 \leq i \leq 33$, $\times_n\}$ be the special interval semigroup.

Study questions (i) to (viii) of problem (29) for this $S_\times$.

40. Let $S_\times = \{[0, 3) \times [0, 22) \times [0, 17) \times [0, 40) \times [0, 256) \times [0, 27), \times\}$ be the special interval semigroup.

Study questions (i) to (viii) of problem (29) for this $S_\times$.

41. Let $S_\times = \left\{ \left[\begin{array}{cccc} [a_1, b_1] & [a_2, b_2] & \ldots & [a_6, b_6] \\ \vdots & \vdots & & \vdots \\ [a_{31}, b_{31}] & [a_{32}, b_{32}] & \ldots & [a_{36}, b_{36}] \end{array} \right] \right|$ $a_i\ b_i \in$

[0, 24), $1 \leq i \leq 36$, $\times_n\}$ be the special interval semigroup.

Study questions (i) to (viii) of problem (29) for this $S_\times$.

42. Derive some special and unique properties enjoyed by special interval semigroup under $\times$.

43. Is it ever possible to have a special interval semigroup under × which is not a S-semigroup?

44. Let $S_\times = \left\{ \begin{bmatrix} [a_1, b_1] & [a_2, b_2] & \ldots & [a_7, b_7] \\ \vdots & \vdots & & \vdots \\ [a_{29}, b_{29}] & [a_{30}, b_{30}] & \ldots & [a_{35}, b_{35}] \end{bmatrix} \right| a_i\ b_i \in$

$[0, 8) \times [0, 24) \times [0, 35)$, $1 \leq i \leq 35$, $\times_n\}$ be the special interval semigroup.

Study questions (i) to (viii) of problem (29) for this $S_\times$.

45. Suppose we define max operation on $S_\times$ of problem 44 instead of $\times_n$, can S_{max} have zero divisors?

(i) Can that S_{max} be a S-semigroup?
(ii) Can that S_{max} be a S-semigroup free from units?

46. Let $S_{max} = \left\{ \begin{bmatrix} [a_1, b_1] \\ [a_2, b_2] \\ \vdots \\ [a_9, b_9] \end{bmatrix} \right| a_i, b_i \in [0, 48)$, $1 \leq i \leq 9$, max} be

the special interval semigroup under max operation.

(i) Can S_{max} have zero divisors?
(ii) Can S_{max} have units?
(iii) Can S_{max} be a S-semigroup?
(iv) Obtain any other special feature enjoyed by S_{max} .

47. Let

$$S_{min} = \left\{ \begin{bmatrix} [a_1, b_1] \\ [a_2, b_2] \\ \vdots \\ [a_9, b_9] \end{bmatrix} \middle| \ a_i\ b_i \in [0, 19), 1 \le i \le 9, \min \right\} \text{ be}$$

the special interval semigroup be under min operation.

Study questions (i) to (iv) of problem (46) for this S_{min}.

48. Let $S_{min} = \left\{ \begin{bmatrix} [a_1, b_1] & [a_2, b_2] & \ldots & [a_7, b_7] \\ [a_8, b_8] & [a_9, b_9] & \ldots & [a_{14}, b_{14}] \\ [a_{15}, b_{15}] & [a_{16}, b_{16}] & \ldots & [a_{21}, b_{21}] \end{bmatrix} \middle| \ a_i\ b_i \in \right.$

[0, 17) × [0, 23), $1 \le i \le 21$, min} be the special interval semigroup.

(i) Study questions (i) to (viii) of problem 29 for this S_{min}.

(ii) If min is replaced by max compare them.

49. Let

$$S_{max} = \left\{ \begin{bmatrix} [a_1, b_1] & [a_2, b_2] & \ldots & [a_8, b_8] \\ \vdots & \vdots & \ldots & \vdots \\ [a_{57}, b_{57}] & [a_{58}, b_{58}] & \ldots & [a_{64}, b_{64}] \end{bmatrix} \middle| \ a_i\ b_i \in \right.$$

[0, 31) × [0, 29) × [0, 73), $1 \le i \le 64$, max} be the special interval semigroup.

(i) Study questions (i) to (viii) of problem 29 for this S_{max}.

50. Let $S_{min} = \left\{ \left[\begin{array}{cccc} [a_1, b_1] & [a_2, b_2] & [a_3, b_3] & [a_4, b_4] \\ \hline [a_5, b_5] & \ldots & \ldots & [a_8, b_8] \\ [a_9, b_9] & \ldots & \ldots & [a_{12}, b_{12}] \\ [a_{13}, b_{13}] & \ldots & \ldots & [a_{16}, b_{16}] \\ \hline [a_{17}, b_{17}] & \ldots & \ldots & [a_{20}, b_{20}] \\ [a_{21}, b_{21}] & \ldots & \ldots & [a_{24}, b_{24}] \\ [a_{25}, b_{25}] & \ldots & \ldots & [a_{28}, b_{28}] \\ [a_{29}, b_{29}] & \ldots & \ldots & [a_{32}, b_{32}] \\ \hline [a_{33}, b_{33}] & \ldots & \ldots & [a_{36}, b_{36}] \end{array} \right] \right| a_i\ b_i$

$\in [0, 53) \times [0, 83)$, $1 \leq i \leq 36$, min} be the special interval semigroup.

Study questions (i) to (viii) of problem 29 for this S_{min}.

51. Let S_{max} =

$$\left\{ \left[\begin{array}{c|ccc|cc} [a_1, b_1] & [a_2, b_2] & [a_3, b_3] & [a_4, b_4] & [a_5, b_5] & [a_6, b_6] \\ [a_7, b_7] & \ldots & \ldots & \ldots & \ldots & \ldots \\ [a_{13}, b_{13}] & \ldots & \ldots & \ldots & \ldots & \ldots \\ [a_{19}, b_{19}] & \ldots & \ldots & \ldots & \ldots & \ldots \end{array} \right] \right|$$

$a_i\ b_i \in [0, 11) \times [0, 9)$, $1 \leq i \leq 24$, max} be the special interval semigroup.

Study questions (i) to (viii) of problem 29 for this S_{max}.

52. Let S_{min} =

$$\left\{\left[\begin{array}{c|ccc|c} [a_1, b_1] & [a_2, b_2] & [a_3, b_3] & [a_4, b_4] & [a_5, b_5] \\ \hline [a_6, b_6] & \ldots & \ldots & \ldots & \ldots \\ [a_{11}, b_{11}] & \ldots & \ldots & \ldots & \ldots \\ \hline [a_{16}, b_{16}] & \ldots & \ldots & \ldots & \ldots \\ [a_{21}, b_{21}] & \ldots & \ldots & \ldots & \ldots \end{array}\right] \middle| \; a_i \, b_i \in \right.$$

[0, 19), $1 \leq i \leq 25$, min} be the special interval semigroup.

Study questions (i) to (viii) of problem 29 for this S_{min}.

53. Let $S_{max} = \left\{\begin{bmatrix} [a_1, b_1] & [a_2, b_2] & \ldots & [a_9, b_9] \\ [a_{10}, b_{10}] & [a_{11}, b_{11}] & \ldots & [a_{18}, b_{18}] \\ [a_{19}, b_{19}] & [a_{20}, b_{20}] & \ldots & [a_{27}, b_{27}] \end{bmatrix} \middle| \; a_i \, b_i \in \right.$

[0, 19), $1 \leq i \leq 27$, max} be the special interval semigroup.

(i) Study questions (i) to (viii) of problem 29 for this S_{max}.
(ii) If S_{max} is replaced by S_{min} compare them.
(iii) If S_{max} is replaced by $S_{\times}$ compare them.

54. For any special interval semigroups $S_{\times}$ and S_{min} can we define a homomorphism between them?

55. Let $S_{\times} = \{(a_1, a_2, a_3, a_4, a_5, a_6) \mid a_i \in [0, 28), 1 \leq i \leq 6, \times\}$ be the special interval semigroup.

(i) Let $\phi : S_{\times} \to S_{\times}$ be a homomorphism find ker ϕ such that ker $\phi \neq$ empty.
(ii) What is the algebraic structure enjoyed by ker ϕ?

56. Let $S_\times = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_9 \end{bmatrix} \middle| \; a_i \in [0, 43), 1 \leq i \leq 9 \right\}$ be the special interval semigroup. Let

$$S'_\times = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_6 \\ a_7 & a_8 & \ldots & a_{12} \end{bmatrix} \middle| \; a_i \in [0, 6), 1 \leq i \leq 12 \right\} \text{ be}$$

special interval semigroup.

(i) Find $\phi : S_\times \to S'_\times$ so that ker ϕ is non empty.
(ii) Study questions (i) to (viii) of problem 29 for this S and $S'_\times$.

57. Let $S_\times = \{(a_1, a_2, \ldots, a_9) \mid a_i \in [0, 43), 1 \leq i \leq 9, \times\}$ be the special interval semigroup.

$S_{max} = \{(a_1, \ldots, a_9) \mid a_i \in [0, 43), 1 \leq i \leq 9, \max\}$ be special interval semigroup under max.

(i) Find $\phi : S_{max} \to S_\times$ so that ker ϕ = empty.
(ii) Study questions (i) to (ii) of problem 56 for this S_{max} and $S_\times$.

Chapter Four

SPECIAL INTERVAL SEMIRINGS AND SPECIAL PSEUDO RINGS USING [0, n)

In this chapter we for the first time construct semirings and special pseudo rings using the continuous interval [0, n), Such study is both innovative and interesting.

These algebraic structures enjoy very many properties which are different from the semiring $R^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ or $Z^+ \cup \{0\}$ or from the ring Z_n; ($n < \infty$ ring of modulo integers) Q or Z or R.

We bring out several such distinct properties enjoyed by these new structures.

First we define semirings on [0, n) using the min and max operators.

DEFINITION 4.1: *Let $R = \{[0, n), \min, \max;\ n < \infty;\ \text{so } n \notin [0, n)\}$. $\{R, \min\}$ be a semigroup and $\{R, \max\}$ is a semigroup. The min and max operations distributes over each other. Thus R is a semiring of infinite order and is commutative. $R = \{[0, n), \min, \max\}$ is defined as the special interval semiring.*

We will first give examples of them.

Example 4.1: Let R = {[0, 20), min, max} be the special interval semiring. R has subsemirings of order 1, two, three and so on.

$P_1 = \{0, 3\}$ is a subsemiring of order two. $P_2 = \{6.3215\} \subseteq R$ is a subsemiring of order one. Every singleton set is a subsemiring of order one.

For that matter take any subset $P = \{x_1, x_2, \ldots, x_m\} \subseteq R$; $x_i \in [0, 20)$; $1 \leq i \leq m$, P in general is not a subsemiring.

Example 4.2: Let R = {[0, 120), min, max} be the special interval semiring. R is commutative and is of infinite order. Infact R is a special quasi semifield; called the special interval semifield.

R has quasi subsemifields of every order in N; N the natural numbers.

Example 4.3: Let R = {[0, 43), min, max} be a special interval semiring of infinite order which is a special quasi semifield. R has several special quasi subsemifields.

We say F = {[0, n), min, max} to be a special quasi semifield, R has only one identity for min {0, x} = 0 and max{0, x} = x. We do see 0 acts as identity with respect to max.

However F has no maximal or greatest element that is why we call F as the quasi special semifield.

Example 4.4: Let R = {[0, 27), min, max} be a special interval semiring that is quasi special interval semifield. R has infinite number of finite subsemirings and infinite number of finite subsemirings of all orders.

Infact order 1 subsemirings are infinite in number, similarly order two, order three and so on.

We can in case of semirings define both the notion of filter and ideal. For ideal we will have zero but in case of filter we will not have the greatest element as R does not contain the greatest element.

We will illustrate this situation by some examples.

Example 4.5: Let R = {[0, 12), max, min} be the special interval semiring.

Let P = {[0, 8), max, min} $\subseteq$ R be an ideal in R.

For any x, y $\in$ R we have max (x, y) $\in$ P;
further min (p, r) $\in$ P for every r $\in$ R and p $\in$ P.

Hence the claim.

R has infinite number of ideals.

It is pertinent to observe that P the ideal is not a filter of R. For if r $\in$ R and p $\in$ P, max (p, r) $\notin$ P.

Now consider T = {[a, 12); 0 < a}, T under min operation is closed for every x $\in$ R and t $\in$ T, we see max (r, t) $\in$ T as every r $\in$ R \ T is such that r < a, hence the claim.

Clearly T is not an ideal of R.

We see R has infinite number of filters.

W = {[9, 12)} $\subseteq$ R is a filter of R. M = {[3, 12)} $\subseteq$ R is also a filter of R.

We see both W and M are not ideals of R.

However we have infinite number of filters and ideals in these special interval semirings.

Example 4.6: Let R = {[0, 29), min, max} be a special interval semiring under max and min operations.

P = {[0, 20), min, max} is a subsemiring.

P is an ideal for any $p \in P$ and $r \in R \setminus \{0, 20\}$,
min $\{p, r\} = p \in P$.

However P is not a filter for if $p \in P$ and $r \in R \setminus \{0, 20\}$ max $\{r, p\} = r \notin P$.

Hence the claim.

Infact $P_t = \{[0, t); 0 < t < 28; \text{min, max}\} \subseteq R$ for infinitely many t is only an ideal of R and R has infinitely many ideals and the cardinality of each P_t is infinite.

Now consider $B_t = \{[t, 20), \text{max, min } 0 < t < 20\} \subseteq R$, B_t is a subsemiring with t as its least element.

Clearly B_t is not an ideal for if $b \in B_t$ and $r \in R \setminus [0, 20)$ we see min $\{b, r\} = r$ and is not in B_t.

However B_t is a filter as for any $x \in R$ and $b \in B_t$; max $\{x, b\} \in B_t$. B_t is a filter of infinite order. R has infinitely many such filters.

We see B_t is a filter and is not an ideal of R.

Thus R has infinite number of ideals which are not filters and infinite number of filters which are not ideals.

Example 4.7: Let
$R = \{(a_1, a_2, a_3, a_4) \mid a_i \in [0, 42), 1 \leq i \leq 4, \text{max, min}\}$ be the special interval semiring. R is commutlative. R is of infinite order.

Every singleton is a subsemiring we have $P = \{(0, 0, 0, 0), (a_1, a_2, a_3, a_4)\} \subseteq R$ to be a subsemiring for some fixed $a_1, a_2, a_3, a_4 \in [0, 42)$. Clearly P is not an ideal or filter of R.

The first important factor to observe is R is not a totally ordered set either under max or under min.

THEOREM 4.1: *Let $S = \{[0, n); 0 < n < \infty; \max, \min\}$ be a special interval semiring.*

(i) *S is of infinite cardinality and is commutative.*
(ii) *S is totally ordered both by max and min.*
(iii) *All subsemirings of the form $P_t = [0, t); \min, \max\} \subseteq R$ are ideals of R and are not filters of R which are infinite in number.*
(iv) *All subsemirings of the form $B_t = \{[a, n); 0 < a < n, \min, \max\} \subseteq R$ are filters of R and are not ideals of R and they are infinite in number and have infinite cardinality.*
(v) *R has no zero divisors but every element is an idempotent both under max and min.*
(vi) *Every proper subset T of R is a subsemiring; T may be finite or infinite.*

The proof is direct and hence left as an exercise to the reader.

Example 4.8: Let $R = \{[0, 7) \times [0, 13) \times [0, 27); \max, \min\}$ be the special interval semiring. R has zero divisors. R is not orderable by max or min.

If $x = (0.3, 5, 19.321)$.
and $y = (7, 2.4, 5.9) \in R$ then

$$\begin{aligned} \min\{x, y\} &= \{(0.3, 5, 19.321), (7, 2.4, 5.9)\} \\ &= \{(0.3, 2.4, 5.9)\} \\ \text{and } \max\{x, y\} &= \{(0.3, 5, 19.321), (7, 2.4, 5.9)\} \\ &= \{(7, 5, 19.321)\}. \end{aligned}$$

So P = {x, y} is not closed under max and min.

Suppose x = (0, 0, 16.321) and
y = (6.2134. 10.75011, 0) $\in$ R then

min {x, y} = min {(0, 0, 16.321), (6.2314, 10.75011, 0)}

= {(0, 0, 0)} … I

and

max {x, y} = max {(0, 0, 16.321), (6.2314, 10.75011, 0)}

= {(6.2134, 10.75011, 16.321)}.

I shows R has zero divisors. Infact R has infinite number of zero divisors.

Example 4.9: Let

$$R = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{bmatrix} \middle| a_i \in [0, 12); 1 \leq i \leq 9 \right\}$$

be the special interval semiring under max and min operation.

R has filters and ideals.

For take $P_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| a_1 \in [0, 12), \min, \max \right\} \subseteq R$.

P_1 is an ideal and not a filter.

$$\text{For if } x = \begin{bmatrix} 11.39 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in P_1 \text{ and } y = \begin{bmatrix} 2.3 \\ 7.5 \\ 6.2 \\ 1.5 \\ 3.7 \\ 6.3 \\ 1.6 \\ 0 \\ 10.3 \end{bmatrix} \in R;$$

$$\min\{x, y\} = \begin{bmatrix} 11.39 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ is in } P_1.$$

$$\text{However } \max\{x, y\} = \begin{bmatrix} 11.39 \\ 7.5 \\ 6.2 \\ 1.5 \\ 3.7 \\ 6.3 \\ 1.6 \\ 0 \\ 10.3 \end{bmatrix} \notin P_1.$$

Thus P_1 is only an ideal and not a filter.

Example 4.10: Let

$$R = \left\{ \left[\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{array} \right] \right| a_i \in [0, 32), 1 \leq i \leq 24, \text{min, max}\}$$

be the special interval matrix semiring.

R has several subsemirings which are ideals and are not filters.

R also has several subsemirings which are filters and not ideals.

R has infinite number of zero divisors and has no units.

We see if x, y $\in$ R then in general x $\leq_{min}$ y; y $\leq_{min}$ x and y $\leq_{max}$ x. This R is not totally orderable.

Infact R is partially orderable with respect to $\leq_{max}$ and $\leq_{min}$.

Example 4.11: Let
R = {[$a_1, a_2, \ldots, a_{10}$) | $a_i \in [0, 15)$; $1 \leq i \leq 10$; min, max} be the special interval semiring. We see R has infinite number of zero divisors and has no units.

Let x = (0, 0, 0, 4, 8, 9.1, 0, 0, 0, 7.5), and
y = (9.8, 11.31, 12.01, 0, 0, 0, 9.11, 8.5, 0.7, 0) $\in$ R, we see

min {x, y} = (0, 0, …, 0) and
max {x,y} = (9.8, 11.31, 12.01, 4, 8, 9.1, 9.11, 8.5, 0.7, 7.5) $\in$ R.

Thus R has zero divisors under min and R has several zero divisors infact infinite in number.

Example 4.12: Let

$$R = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix} \middle| a_i \in [0, 8);\ 1 \leq i \leq 12, \min, \max \right\}$$

be the special interval semiring. R has infinite number of zero divisors.

$$M_1 = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{11} & a_{12} \end{bmatrix} \middle| a_i \in [0, 4);\ 1 \leq i \leq 12, \min, \max \right\}$$

be the special interval subsemiring which is also an ideal of M_1. M_1 is not a filter of R.

$$M_2 = \left\{ \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \middle| a_i \in [0, 8);\ \min, \max \right\} \subseteq R$$

be the special interval semiring.

M_2 is an ideal of R and is not a filter.

Let

$$N_1 = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{11} & a_{12} \end{bmatrix} \middle| a_i \in [4, 8); 1 \leq i \leq 12, \min, \max\right\} \subseteq R;$$

N_1 is a filter of R but N_1 is not an ideal of R. Thus we have several interesting features enjoyed by R.

$$\text{For if } x = \begin{bmatrix} a_1 & 0 \\ a_2 & 0 \\ 0 & a_3 \\ 0 & 0 \\ \vdots & \vdots \\ a_4 & a_5 \end{bmatrix} \in R \text{ any } y \in N_1$$

we see min $\{x, y\} \notin N_1$, hence N_1 is not an ideal of R.

However for any $x \in N_1$ and $y \in R$, max $\{x, y\} \in N_1$ hence N_1 is a filter of R.

Still every element in R is an idempotent but any subset T in R is not a subsemiring however T can always be completed to a subsemiring.

If T is finite and T is only a subset T_c the completion of T is also finite and T_c is a subsemiring. If T is infinite T_c the completion is also an infinite subsemiring.

Example 4.13: Let
$R = \{(a_1, a_2, a_3) \mid a_i \in [0, 4), 1 \leq i \leq 3, \min, \max\}$ be the special interval semiring.

Let $P = \{x = (0.3, 1.4, 2.1), y = (2.1, 0.5, 1.7)\} \subseteq R$;

we see min {x, y} =

$$\min \{(0.3, 1.4, 2.1), (2.1, 0.5, 1.7)\}$$
$$= \{(0.3, 0.5, 1.7)\} \notin P.$$

$$\max \{(0.3, 1.4, 2.1), (2.1, 0.5, 1.7)\}$$
$$= \{(2.1, 1.4, 2.1)\} \notin P.$$

P is not a subsemiring however P_c the completion of P is {x, y, (0.3, 0.5, 1.7), (2.1, 1.4, 2.1)} is a subsemiring which is not an ideal or filter of R.

Likewise if A = {(0, 0, 3.2), (0.1, 0.87, 2), (3, 2.1, 0)} $\in$ R to find the completion of A.

A is not a subsemiring for min {(0, 0, 3.2), (0.1, 0.8, 2)} = {(0, 0, 2)} $\notin$ A.

$$\min \{(0, 0, 3.2), (3, 2.1, 0)\}$$
$$= \{(0, 0, 0)\} \notin A.$$

$$\min \{(0.1, 0.8, 0), (3, 2.1, 0)\}$$
$$= \{(0.1, 0.8, 0)\} \notin A.$$

$$\max \{(0, 0, 3.2), (0.1, 0.8, 2)\}$$
$$= \{(0.1, 0.8, 3.2)\} \notin A.$$

$$\max \{(0, 0, 3.2), (3.2, 1, 0)\}$$
$$= \{(3.2, 1, 3.2)\} \notin A.$$

$$\max \{(0.1, 0.8, 2), (3.2, 1, 0)\}$$
$$= \{(3.2, 1, 2)\} \notin A.$$

Thus the completion of A, A_c = {A} $\cup$ {(0, 0, 2), (0, 0, 0), (0.1, 0.8, 0), (0.1, 0.8, 3.2), (3.2, 1, 3.2), (3.2, 1, 2)} is a subsemiring.

Example 4.14: Let

$$R = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| a_i \in [0, 7); 1 \leq i \leq 4, \min, \max \right\}$$

be the special interval semiring. R is of infinite order.

R has infinite number of zero divisors every element is an idempotent and R has no units.

$$\text{Let } A = \left\{ \begin{bmatrix} 0.7 \\ 3 \\ 4.5 \\ 2.1 \end{bmatrix}, \begin{bmatrix} 6.1 \\ 2 \\ 1.5 \\ 4.7 \end{bmatrix} \right\}; \min\left\{ \begin{bmatrix} 0.7 \\ 3 \\ 4.5 \\ 2.1 \end{bmatrix}, \begin{bmatrix} 6.1 \\ 2 \\ 1.5 \\ 4.7 \end{bmatrix} \right\} = \begin{bmatrix} 0.7 \\ 2 \\ 1.5 \\ 2.1 \end{bmatrix}$$

$$\max \left\{ \begin{bmatrix} 0.7 \\ 3 \\ 4.5 \\ 2.1 \end{bmatrix}, \begin{bmatrix} 6.1 \\ 2 \\ 1.5 \\ 4.7 \end{bmatrix} \right\} = \begin{bmatrix} 6.1 \\ 3 \\ 4.5 \\ 4.7 \end{bmatrix}$$

both min and max are not in A.

So now we complete A and

$$A_c = \left\{ \begin{bmatrix} 6.1 \\ 3 \\ 4.5 \\ 4.7 \end{bmatrix}, \begin{bmatrix} 0.7 \\ 2 \\ 1.5 \\ 2.1 \end{bmatrix}, \begin{bmatrix} 0.7 \\ 2 \\ 1.5 \\ 2.1 \end{bmatrix}, \begin{bmatrix} 6.1 \\ 3 \\ 4.5 \\ 4.7 \end{bmatrix} \right\} \subseteq R \text{ is a subsemiring.}$$

Thus any finite or infinite subset of a semiring can be completed to get a subsemiring.

Example 4.15: Let

$$R = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \middle| a_i \in [0, 12); 1 \leq i \leq 6 \right\}$$

be the special interval semiring.

Let $A = \{x = \begin{bmatrix} 0 & 0.2 & 7 \\ 6.1 & 5.3 & 4.1 \end{bmatrix}$, $y = \begin{bmatrix} 2 & 0.7 & 9 \\ 3 & 4 & 8 \end{bmatrix}$ and

$z = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\} \subseteq R$ be a subset of R.

Clearly A is not closed with respect to the operation min as well as max.

$$\min \{x, y\} = \min \left\{ \begin{bmatrix} 0 & 0.2 & 7 \\ 6.1 & 5.3 & 4.1 \end{bmatrix}, \begin{bmatrix} 2 & 0.7 & 9 \\ 3 & 4 & 8 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} 0 & 0.2 & 7 \\ 3 & 4 & 4.1 \end{bmatrix} \right\} \notin A.$$

$$\min \{y, z\} = \min \left\{ \begin{bmatrix} 2 & 0.7 & 9 \\ 3 & 4 & 8 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} 1 & 0.7 & 3 \\ 3 & 4 & 6 \end{bmatrix} \right\} \notin A.$$

$$\min \{x, z\} = \min \left\{ \begin{bmatrix} 0 & 0.2 & 7 \\ 6.1 & 5.3 & 4.1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \right\}$$

$$= \left\{\begin{bmatrix} 0 & 0.2 & 3 \\ 4 & 5 & 4.1 \end{bmatrix}\right\} \notin A.$$

Consider

$$\max\{x, y\} = \max\left\{\begin{bmatrix} 0 & 0.2 & 7 \\ 6.1 & 5.3 & 4.1 \end{bmatrix}, \begin{bmatrix} 2 & 0.7 & 9 \\ 3 & 4 & 8 \end{bmatrix}\right\}$$

$$= \left\{\begin{bmatrix} 2 & 0.7 & 9 \\ 6.1 & 5.3 & 8 \end{bmatrix}\right\} \notin A.$$

$$\max\{y, z\} = \left\{\begin{bmatrix} 2 & 0.7 & 9 \\ 3 & 4 & 8 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\right\}$$

$$= \left\{\begin{bmatrix} 2 & 2 & 9 \\ 4 & 5 & 8 \end{bmatrix}\right\} \notin A.$$

$$\max\{x, z\} = \left\{\begin{bmatrix} 0 & 0.2 & 7 \\ 6.1 & 5.3 & 4.1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\right\}$$

$$= \left\{\begin{bmatrix} 1 & 2 & 7 \\ 6.1 & 5.3 & 6 \end{bmatrix}\right\} \notin A.$$

Thus the completeness of A is

$$A_c = \left\{\begin{bmatrix} 0 & 0.2 & 7 \\ 6.1 & 5.3 & 4.1 \end{bmatrix}, \begin{bmatrix} 2 & 0.7 & 9 \\ 3 & 4 & 8 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix},\right.$$

$$\begin{bmatrix} 0 & 0.2 & 7 \\ 3 & 4 & 4.1 \end{bmatrix}, \begin{bmatrix} 1 & 0.7 & 3 \\ 3 & 4 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 0.2 & 3 \\ 4 & 5 & 4.1 \end{bmatrix},$$

$$\left.\begin{bmatrix} 2 & 0.7 & 9 \\ 6.1 & 5.3 & 8 \end{bmatrix}, \begin{bmatrix} 2 & 2 & 9 \\ 4 & 5 & 8 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 7 \\ 6.1 & 5.3 & 6 \end{bmatrix}\right\} \subseteq R$$

is a subsemiring of the semiring R.

Example 4.16: Let

$$R = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{bmatrix} \middle| a_i \in [0, 15); 1 \leq i \leq 10\right\}$$

be the special interval semiring.

Consider the subset

$$A = \{x = \begin{bmatrix} 0 & 3.1 & 14.4 & 5.1 & 7 \\ 9.7 & 10.9 & 13.2 & 0 & 8.5 \end{bmatrix} \text{ and}$$

$$\left. y = \begin{bmatrix} 5 & 8.4 & 10.7 & 7.8 & 9.2 \\ 13.9 & 11.4 & 10.11 & 9.3 & 0 \end{bmatrix}\right\} \subseteq R.$$

Clearly A is not a subsemiring only a subset

$$\min\{x, y\} = \begin{bmatrix} 0 & 3.1 & 10.7 & 5.7 & 7 \\ 9.7 & 10.9 & 10.11 & 0 & 0 \end{bmatrix} \text{ and}$$

$$\max\{x, y\} = \begin{bmatrix} 5 & 8.4 & 14.4 & 7.8 & 9.2 \\ 13.9 & 11.4 & 13.2 & 9.3 & 8.5 \end{bmatrix} \text{ are not in A.}$$

But $A_c = \{x, y, \min\{x, y\}, \max\{x, y\}\} \subseteq R$ is a special interval subsemiring.

Inview of all this we have the following theorem.

THEOREM 4.2: *Let*
R = {Collection of all $m \times s$ matrix from the interval $[0, t)$, ($t < \infty$), min, max} be the special interval semiring of infinite order. Let $A \subseteq R$ be a subset of R; $A = \{x_1, x_2, \ldots, x_n\}$ is only a subset, then $A_c = \{x_1, x_2, \ldots, x_n$, min $\{x_j, x_i\}$ and max $\{x_i, x_j\}$; $i \neq j$, $1 \leq i, j \leq n\} \subseteq R$ is a subsemiring (which is the completion of A) of R.

The proof is direct and hence left as an exercise to the reader.

Note: n in A can be finite or infinite still the result is true. That is why no mention on n was made.

Example 4.17: Let

$$R = \left\{ \left[\begin{array}{ccccc} a_1 & a_6 & a_{11} & a_{16} & a_{21} \\ a_2 & \ldots & \ldots & \ldots & \ldots \\ a_3 & \ldots & \ldots & \ldots & \ldots \\ a_4 & \ldots & \ldots & \ldots & \ldots \\ a_5 & \ldots & \ldots & \ldots & \ldots \end{array} \right] \middle| \; a_i \in [0, 45), \right.$$

$$1 \leq i \leq 25, \text{max, min}\}$$

be the special interval semigroup.

$$B = \left\{ x = \left[\begin{array}{cc|ccc} 0.5 & 0 & 0 & 0 & 0 \\ 0.9 & 12 & 0 & 0 & 0 \\ \hline 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0 & 44 & 0 \\ 0 & 0 & 0 & 0 & 42.7 \end{array} \right], y = \left[\begin{array}{cc|ccc} 0.2 & 4 & 0 & 0 & 0 \\ 7 & 8 & 0 & 0 & 0 \\ \hline 0 & 0 & 11 & 0 & 0 \\ 0 & 0 & 0 & 15 & 0 \\ 0 & 0 & 0 & 0 & 29 \end{array} \right] \right\} \subseteq R$$

is such that; B is only a subset

$$\min\{x, y\} = \left[\begin{array}{cc|ccc} 0.2 & 0 & 0 & 0 & 0 \\ 0.9 & 8 & 0 & 0 & 0 \\ \hline 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0 & 15 & 0 \\ 0 & 0 & 0 & 0 & 29 \end{array}\right] \text{ and}$$

$$\max\{x, y\} = \left[\begin{array}{cc|ccc} 0.5 & 0 & 0 & 0 & 0 \\ 7 & 12 & 0 & 0 & 0 \\ \hline 0 & 0 & 11 & 0 & 0 \\ 0 & 0 & 0 & 44 & 0 \\ 0 & 0 & 0 & 0 & 42.7 \end{array}\right] \text{ are not in B.}$$

Now we complete B as
$B_c = \{x, y, \min\{x, y\}, \max\{x, y\}\} \subseteq R$ is a subsemiring of R.

Now let R be a special interval matrix semiring or special interval super matrix semiring still we can complete the subset to the subsemiring.

Now we proceed onto study the special pseudo interval ring or special interval pseudo ring.

Let [0, n) be a continuous interval. We define addition modulo n as follows:

If $x, y \in [0, n)$ then if $x + y = t$ with $t > n$ then we put
$x + y \equiv (t–n)$ if $x + y = t = n$ then
$x + y = 0$ if $x + y = t$ and $t < n$ then $x + y = t$.

Thus {[0, n), +} is an abelian group with respect to '+' and '0' acts as the additive identity.

Suppose we have [0, 12) is the given interval define + on the interval [0, 12) as follows.

If $x = 6.73$ and $y = 10.927$ are in $[0, 12)$ then
$x + y = 6.73 + 10.927 = 17.657 \pmod{12} = 5.657 \in [0, 12)$.

Let $x = 6.05$ and $y = 5.95 \in [0, 12)$, then
$x + y = 6.05 + 5.95 = 12.00 = 0 \pmod{12}$.

Thus 6.05 is the additive inverse of 5.95 and vice versa.

Let $x = 0.3125$ and $y = 3.10312 \in [0, 12)$.
Now $x + y = 0.3125 + 3.10312 = 3.41562 \in [0, 12)$.

Thus $\{[0, 12), +\}$ is an additive abelian group of infinite order.

Now on $[0, n)$ we define product if $x \times y = t$ then if $t < 12$ take $x \times y$ as the product if $t > 12$ then take $x \times y = t - 12 \in [0, 12]$ and $x \times y = 0$ if and only if one of x or y is zero.

Take $x = 0.31$ and $y = 5 \in [0, 12)$; $x \times y = 1.55 \in [0, 12)$.

Take $x = 11$ and $y = 11.5 \in [0, 12)$ then

$$\begin{aligned} x \times y &= 11 \times 11.5 = 12.65 \pmod{12} \\ &= 12.65 - 12 = 0.65 \in [0, 12). \end{aligned}$$

Thus $\{[0, 12), \times\}$ under product is a semigroup and $1 \in [0, 12)$ acts as the multiplicative identity.

However this semigroup has zero divisors even if n is a prime.

For take $[0, 6)$ and let $x = 2$ and $y = 3 \in [0, 6)$ we see $x \times y = 2 \times 3 = 6 \pmod{6} = 0 \pmod{6}$ hence is a zero divisor.

Suppose $[0, 7)$ is the interval under consideration, we see for no pair $x, y \in [0, 7) \setminus \{0\}$, $x \times y = 0$.

We will now claim $\{[0, n), \times, +\}$ is not a ring as $(a + b)\, c \neq ab + bc$ in general for all $a, b, c \in [0, n)$.

Hence we define R = {[0, n), +, ×} as a special pseudo interval ring.

We will give examples and describe the special properties enjoyed by them.

Example 4.18: Let R = {[0, 10), +, ×} be the pseudo ring of special interval [0, 10). Let x = 9 and y = 6.2 ∈ [0, 10].

$$x \times y = 9 \times 6.2 = 55.8 \pmod{10} = 5.8 \in R.$$
$$x + y = 9 + 6.2 = 15.2$$
$$= 5.2 \in R.$$

Suppose x = 5 and y = 2 ∈ R then x × y = 10 (mod 10) = 0 hence R has zero divisors.

Let x = 5 and y = 8 ∈ [0, 10).

x × y = 5 × 8 ≡ 40 (mod 10) = 0 is a zero divisor in R = {[0, 10), ×, +}.

However R = {[0, 17), ×, +} has zero divisors but has non trivial units for take x = 16 we see x × x = 162 ≡ 1 (mod 17) is a unit in R.

Let x = 2 and y = 9 ∈ R then x × y = 2 × 9 ≡ 18 (mod 17) = 1 is a unit in [0, 17).

We see however large n may be in [0, n) (n < ∞) then R = {[0, n), +, ×} has only finite number of units infact only (n–2) of the elements in R alone are units that too for any finite prime number n.

Example 4.19: Let R = {[0, 23), +, ×} be a special pseudo interval ring. R has zero divisors.

We have 21 units in R. R has no idempotents. R has subrings viz. P_1 = {0, 1, 2, …, 22} as well special pseudo subrings.

$P_2 = \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 22, 22.5\} \subseteq R$ is not a subring of finite order.

Example 4.20: Let $R = \{[0, 24), +, \times\}$ be the special pseudo interval ring of infinite order. $P_1 = \{0, 2, 4, 6, 8, 10, 12, \ldots, 22\} \subseteq R$ is again a special interval subring.

$P_2 = \{0, 4, 8, 12, \ldots, 20\} \subseteq R$ is again a special interval subring. $P_3 = \{0, 8, 16\} \subseteq R$ is again a special interval subring. All the subrings of R are not ideals.

$P_4 = \{0, 1, 2, \ldots, 23\} \subseteq R$ is a special interval subring which is not an ideal.

$P_5 = \{0, 12\} \subseteq R$ is a subring.

$P_6 = \{0, 0.5, 1, 1.5, 2, 2.5, \ldots, 23, 23.5\} \subseteq R$; is not a subring.

$P_7 = \{0, 0.1, 0.2, \ldots, 23.9\} \subseteq R$ is a not subring of R. R has several subrings of very many different orders.

None of these subrings are ideals of R. R has zero divisors.

For $x = 2$ and $y = 12 \in R$ is such that $x \times y = 2 \times 12 = 0$.

Let $x = 3$ and $y = 8 \in R$ is such that $x \times y = 3 \times 8 = 0 \pmod{24}$.

Let $x = 6$ and $y = 4 \in R$ is such that $x \times y = 6 \times 4 = 0 \pmod{24}$. R has only finite number of zero divisors.

R has finite number of idempotents.

For $x = 9 \in R$ is such that $x^2 = x = 9$ is an idempotent. $x = 7 \in R$ is a unit as $x^2 = 1 \pmod{24}$, $5 = y \in R$ is again a unit as $5^2 = 1 \pmod{24}$ $y = 16 \in R$ is such that $16^2 \equiv 16 \pmod{24}$. Consider $x = 23 \in R$ is such that $23^2 \equiv 1 \pmod{24}$.

Example 4.21: Let R = {[0, 11), +, ×} be a special interval pseudo ring. R has zero divisors. R is only an infinite pseudo interval ring. R has no idempotents however R has 9 elements which are units of R.

P = {0, 1, 2, 3, …, 10} ⊆ R is a subring of R.

Example 4.22: Let R = {[0, 4), +, ×} be the special pseudo interval ring of infinite order; R is not a pseudo integral domain; for x = 2 ∈ R is such that $x^2 = 0$ (mod 4).

y = 3 ∈ R is a unit as $3^2 \equiv 1$ (mod 4) which is the only unit of R. P = {0, 1, 2, 3} ⊆ R is a subring of R and P ≅ Z_4.

T = {0, 2} ⊆ R is again a finite subring of R.

Apart from this we are unaware of any other finite subring. For if we try to use 0.1, 0.01, 0.001, 0.0001, …, 0.2, 0.02, 0.004, 0.0016, and so on and the inverses 0.9, 0.99, 0.999, 0.9999 and so on thus it can be only countably infinite.

Example 4.23: Let R = {[0, 15), +, ×} be a special pseudo interval ring. R has finite number of zero divisors. Finite number of units and finite number of idempotents.

x = 10 ∈ R is such that 10 × 10 ≡ 10 (mod 15) y = 4 ∈ R is such that $y^2 = 1$ (mod 15) is a unit in R. x = 6 ∈ R is such that $x^2 = 6^2 = 6$ (mod 15) is an idempotent.

x = 11 ∈ R is such that $x^2 = 11^2 = 1$ (mod 15); y = 14 ∈ R is such that $y^2 = 14^2 = 1$ (mod 15). Thus we have found the units, and idempotents of R.

We now work out the zero divisors of R.

y = 3 and x = 5 ∈ R are such that x × y = 10 × 6 = 0 (mod 15), x = 10 and y = 9 is such that x × y = 10 × 9 = 0 (mod 15), x = 12 and y = 10 is such that x × y = 0 (mod 15); x = 6 and y = 5 is such that x × y = 30 (mod 15) = 0 (mod 15).

R has finite number of zero divisors.

Example 4.24: Let R = {[0, 26), +, ×} be a special pseudo interval ring. $x = 2$ and $y = 13 \in R$ is such that $x \times y = 2 \times 13 = 0 \pmod{26}$

$x = 4$ and $y = 13 \in R$ is such that $x \times y = 4 \times 13 = 0 \pmod{26}$. $13 \in R$ is such that $13^2 \equiv 13 \pmod{26}$.

$14 \in R$ is such that $14 \times 14 = 14 \pmod{26}$ so 13 and 14 are two idempotent of R.

$x = 25 \in R$ is such that $x^2 = 1 \pmod{26}$. R has units, zero divisors and idempotents but all of them are only finite in number.

R has subrings of finite order given by $H_1 = \{0, 13\} \subseteq R$ and $H_2 = \{0, 2, 4, 6, \ldots, 24\} \subseteq R$ are subrings of R of finite order.

Next we build more pseudo interval rings using these special interval pseudo rings.

Example 4.25: Let R = {[0, 10) × [0, 19), +, ×} be the product of two special interval pseudo ring. R is again a special interval pseudo ring.

We see R has infinite number of zero divisors, finite number of units and idempotents.

R is of infinite order and R is commutative.

$x = (5, 0) \in R$ is an idempotent $y_1 = (5, 1)$, $y_2 = (6, 0)$ and $y_3 = (6, 1)$ are all idempotents of R.

$x = (9, 3)$ and $y = (9, 13)$ in R is such that
$x \times y = (9, 3) \times (9, 13) = (1, 1) \in R$ is the unit of R.

$x = (0, 0.3315)$ and $y = (0.21301, 0) \in R$ are such that

$x \times y = (0, 0)$ is a zero divisor.

Infact R has infinite number of zero divisors but only finite number of units and idempotents.

R contains one subset which is a pseudo integral domain. R also contains a finite subset which is a field. R has two pseudo ideals. It is left for the reader to find whether R has more pseudo ideals.

Example 4.26: Let R = $\{[0, 7) \times [0, 11) \times [0, 43), +, \times\}$ be the special interval pseudo ring of infinite order. R has infinite number of zero divisors.

R has three subsets viz. $V_1 = \{(\{0\} \times [0, 11) \times \{0\}\} \subseteq R$ which is a pseudo interval integral domain.

$V_2 = \{[0, 7) \times \{0\} \times \{0\}\} \subseteq R$ and
$V_3 = \{\{0\} \times \{0\} \times [0, 43)\} \subseteq R$, are three pseudo integral domains.
$V_4 = \{\{0\} \times [0, 11) \times [0, 43)\} \subseteq R$,
$V_5 = \{[0, 7) \times [0, 11) \times \{0\}\} \subseteq R$

and $V_6 = \{[0, 7) \times \{0\} \times [0, 43)\} \subseteq R$ are not pseudo integral domains but infinite order pseudo subrings and pseudo subrings which are also pseudo ideals of R.

V_1, V_2 and V_3 are not pseudo ideals of R. R has no idempotents only elements of the form (0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (0, 1, 1), (1, 0, 1), (1, 1, 1) are the only idempotents of R. However R has several units.

$x = (6, 10, 42) \in R$ is such that $x^2 = (1, 1, 1)$.
$x = (3, 5, 3)$ and $y = (5, 9, 29) \in R$ are such that
$x \times y = (1, 1, 1)$ is a unit.

R has pseudo subrings which are not pseudo ideals of infinite order also. R has pseudo interval subrings whih are not subrings.

For $T_1 = \{[0, 7) \times \{0\} \times \{0\}\}$ is a pseudo interval subring which is a pseudo ideal of R.

$T_2 = \{\{0\} \times \{[0, 11)\} \times [0, 43)\} \subseteq R$ is a pseudo interval subring which is a pseudo ideal of R.

Both T_1 and T_2 are of infinite order.

Example 4.27: Let
$R = \{[0, 6) \times [0, 12) \times [0, 15) \times [0, 21), +, \times\}$ be the special interval pseudo ring.

R has infinite number of zero divisors. R has no subset which is a pseudo integral domain. R has units and idempotents.

Further R has finite subrings which are not ideals.

R has subsets of infinite order which are pseudo ideals.

R has subsets of infinite order which are not pseudo ideals but subrings or pseudo subrings.

Let
$T_1 = \{\{0, 2, 4\} \times \{0, 6\} \times \{0, 3, 6, 9, 12\} \times \{0, 7, 14\}\} \subseteq R$ be a subring of finite order and is not an ideal of R.

$T_2 = \{[0, 6) \times \{0\} \times \{0\} \times \{0\}\} \subseteq R$ is a pseudo ideal of R; however $T_3 = \{[0, 6) \times \{0\} \times \{0\} \times \{0\}\} \subseteq R$ is a pseudo ideal of R only a pseudo interval subring of infinite order.

$T_4 = [0, 6) \times [0, 12) \times \{0\} \times \{0\}\} \subseteq R$ is a pseudo ideal of R; however $T_5 = \{[0, 6) \times [0, 12) \times \{Z_{15}\} \times \{0\}\} \subseteq R$ is only a pseudo interval subring.

Thus we have commutative special interval pseudo rings of infinite order.

Now we proceed onto describe the notion of special pseudo interval matrix ring by some examples.

Example 4.28: Let
R = {(a_1, a_2, a_3, a_4, a_5, a_6) | $a_i \in [0, 53)$, $1 \leq i \leq 6$, +, ×} be the special interval pseudo ring of infinite order.

R is commutative R has zero divisors and units.

We have only finite number of units, however has infinite number of zero divisors and finite number of idempotents like (a_1, …, a_6) where $a_i \in \{0, 1\}$; $1 \leq i \leq 6$.

Let x = (52, 1, 27, 18, 6, 9) ∈ R we see y = (52, 1, 2, 3, 9, 6) ∈ R is such that x × y = (1, 1, 1, 1, 1, 1) is a unit of R.

T_1 = {(a_1, 0, 0, 0, 0, 0) | $a_1 \in [0, 53)$} ⊆ R is a pseudo interval subring as well as an ideal of infinite cardinality.

T_2 = {(a_1, 0, 0, 0, 0, 0) | $a_1 \in \{0, 1, 2, 3, 4, 5, \ldots, 52\} \subseteq [0, 53)$} ⊆ R is a subring of finite order and is not a pseudo ideal of R.

T_3 = {(a_1, a_2, 0, 0, 0, 0) | $a_1, a_2 \in [0, 53)$, +, ×} ⊆ R is a pseudo interval subring as well as an pseudo ideal of R.

We see T_1 and T_2 are not pseudo integral domains.

T_4 = {(0, a_1, 0, 0, 0, 0) | $a_1 \in [0, 53)$, +, ×} ⊆ R is again a pseudo interval ideal of infinite order of S.

T_4 = ⟨x = (0, 1, 0, 0, 0, 0)⟩ that is generated by x is not a pseudo ideal.

Likewise T_3 = {⟨(1, 1, 0, 0, 0, 0)⟩ = y⟩ is generated by y is not a pseudo ideal of R.

T_5 = {(a_1, a_2, a_3, 0, 0, 0) | $a_1 \in [0, 53)$, $a_2, a_3 \in \{0, 1, 2, 3, 4, \ldots, 52\} \subseteq [0, 53)$} is only a pseudo interval subring of infinite order and is not a pseudo ideal of R.

Example 4.29: Let

$$R = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \middle| \; a_i \in [0, 16), 1 \leq i \leq 9, +, \times_n \right\}$$

be a special interval pseudo ring of infinite order.

Clearly R is a commutative pseudo ring with infinite number of zero divisors.

R does not contain any pseudo subring which is a pseudo integral domain. R has pseudo ideals which are principal.

For take

$$B_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_1, a_2, a_3 \in [0, 16) \right\} \subseteq R$$

is a pseudo subring as well as a pseudo ideal of R.

M_1 generated by

$$\left\{ \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\rangle \right\}$$

is only a subring of finite order.

Let

$$B_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_i \in [0, 16); 1 \leq i \leq 4, +, \times_n \right\} \subseteq R;$$

B_2 is an infinite pseudo interval subring which is also a pseudo ideal of R.

M_2 generated by $\left\langle \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ is only a subring of finite order.

Let

$$B_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \end{bmatrix} \middle| \; a_1 \in [0, 16), a_2 \in \{0, 1, 2, 3, 4, \ldots, 16\}, +, \times_n \right\} \subseteq R$$

be a pseudo subring of infinite order but B_3 is not a pseudo ideal only a pseudo subring.

Thus R has subrings of infinite order which are pseudo subrings and are not pseudo ideals of R.

R has units, zero divisors and idempotents.

Let

$$A_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_3 \end{bmatrix} \middle| \; a_i \in [0, 16); 1 \leq i \leq 3, +, \times_n \right\} \subseteq R \text{ and}$$

$$A_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ 0 \end{bmatrix} \middle| \; a_i \in [0, 16); 1 \leq i \leq 5 \right\} \subseteq R$$

be two pseudo interval subrings of infinite order which are also pseudo interval ideals of R.

Clearly every $x \in A_1$ is such that for every $y \in A_2$ we have

$$x \times_n y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Infact

$$A_1 \times A_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus R has infinite number of zero divisors as $|A_1| = \infty$ and $|A_2| = \infty$.

Apart from this also R has infinite number of zero divisors.

Let $x = \begin{bmatrix} 3 \\ 11 \\ 13 \\ 5 \\ 7 \\ 9 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $y = \begin{bmatrix} 11 \\ 3 \\ 5 \\ 13 \\ 7 \\ 9 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in R$ are such that

$$x \times_n y = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ which is the unit of R.}$$

Thus R has finite number of units.

$$x = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \in R \text{ is such that } x \times_n x = x.$$

R has several idempotents of this form.

Example 4.30: Let
$R = \{(a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mid a_i \in [0, 30), 1 \leq i \leq 7, +, \times\}$ be a special interval pseudo ring of infinite order.

R is commutative. R has infinite number of zero divisors. R has no pseudo subring which is a pseudo integral domain. R has pseudo interval subrings which are pseudo ideals.

For take
$B = \{(a_1, a_2, 0, 0, 0, 0, 0) \mid a_i \in [0, 30), 1 \leq i \leq 2\} \subseteq R$ is a pseudo interval subring as well as pseudo interval ideal of R of infinite order.

$B_2 = \{(0, 0, a_1, a_2, 0, 0, 0) \mid a_1 \in [0, 30), a_2 \in \{0, 1, 2, 3, \ldots, 29\}, +, \times\} \subseteq R$ is a pseudo interval subring of R of infinite order. B_2 is not a pseudo ideal of R.

$B_3 = \{(a_1, a_2, a_3, a_4, a_5, a_7) \mid a_i \in \{0, 2, 4, 6, 8, \ldots, 28\} \subseteq [0, 30), 1 \leq i \leq 6\} \subseteq R$ is a pseudo interval subring of R of finite order and is not a pseudo ideal of R.

$B_4 = \{(a_1, a_2, \ldots, a_7) \mid a_i \in \{0, 10, 20\} \subseteq \{[0, 30)\}, +, \times, 1 \leq i \leq 7\} \subseteq R$ is a subring of finite order which is not a pseudo ideal of R.

$B_5 = \{(a_1, 0, a_2, 0, a_3, 0, a_4) \mid a_i \in \{0, 5, 10, 15, 20, 25\} \subseteq [0, 30), 1 \leq i \leq 4, +, \times\} \subseteq R$ is a subring of finite order and is not a pseudo ideal of R.

We see R has idempotents.

Let $x = (6, 10, 1, 0, 15, 16, 6) \in R$ is such that $x^2 = x$.

Thus R has non trivial idempotents.

However the number of idempotents in R is finite.

Example 4.31: Let

$$R = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \middle| \; a_i \in [0, 13), 1 \leq i \leq 7\right\}$$

be the special interval pseudo ring under $\times_n$ and +. R is of infinite order and R is commutative. R has infinite number of zero divisors.

R has units. $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ acts as the unit or identity element of

R with respect to $\times_n$.

Let $x = \begin{bmatrix} 7 \\ 9 \\ 3 \\ 1 \\ 10 \\ 4 \\ 1 \end{bmatrix}$ and $y = \begin{bmatrix} 7 \\ 3 \\ 9 \\ 1 \\ 4 \\ 10 \\ 1 \end{bmatrix} \in R$;

clearly $x \times_n y = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is a unit or x is the inverse of y and vice versa.

However R has only finite number of inverses, that is finite number of units.

Example 4.32: Let

$$R = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \, a_i \in [0, 24), 1 \le i \le 16, +, \times \right\}$$

be the special interval pseudo ring. R is non commutative as '×' is the usual matrix multiplication.

We have zero divisors, units and idempotents in R.

R has pseudo interval subrings as well as pseudo ideals.

$A = \left\{ \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| \, a_1 \in [0, 24) \right\} \subseteq R$ is not a pseudo ideal of infinite order a pseudo subring of R.

Thus R in this case is a non commutative pseudo interval ring.

$$\text{Let } A = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 0 & 4 & 4 \\ 0 & 4 & 4 & 0 \\ 4 & 4 & 4 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 & 0 & 6 & 0 \\ 0 & 6 & 0 & 6 \\ 6 & 6 & 0 & 0 \\ 0 & 0 & 6 & 6 \end{bmatrix} \in R$$

$$\text{we see } A \times B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\text{Let } A = \begin{bmatrix} 0.312 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 7.30101 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in R.$$

$$A \times B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is a zero divisors of R. R has infinite number of zero divisors.

Example 4.33: Let

$$R = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \, a_i \in [0, 19), 1 \leq i \leq 9, +, \times \right\}$$

be the non commutative special interval pseudo ring under the usual matrix product.

$$T = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ is the inverse of } S = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ as}$$

$$T \times S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ identity of R with respect to the usual}$$

product $\times$.

$$M = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \middle| \, a_i \in [0, 19), 1 \leq i \leq 3 \right\} \subseteq R$$

is a pseudo subring of R and pseudo ideal of R.

R has several pseudo subrings which are not pseudo ideals.

$$N = \left\{\left.\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \right| a_i \in \{0, 1, 2, 3, 4, 5, \ldots, 18\};\right.$$

$$\left. 1 \leq i \leq 3\right\} \subseteq R$$

is a subring of finite order and is not a pseudo ideal of R. Infact N is a commutative subring.

Example 4.34: Let

$$R = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right| a_i \in [0, 12), 1 \leq i \leq 9, +, \times_n\right\}$$

be the special interval pseudo ring.

R has infinite cardinality. R is commutative. R has infinite number zero divisors.

R has pseudo ideals and all pseudo ideals are both right and left.

R has idempotents $x = \begin{bmatrix} 0 & 4 & 0 \\ 4 & 9 & 4 \\ 9 & 4 & 1 \end{bmatrix} \in R$ is such that

$x \times_n x = x$. R has only finite number of idemponents.

R has units and $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ is the unit element or identity

with respect to $\times_n$ in R.

$$P_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1 \in [0, 12) \right\} \subseteq R.$$

P_1 is a pseudo subring as well as a pseudo ideal of infinite order.

$$P_2 = \left\{ \begin{bmatrix} 0 & a_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1 \in [0, 12) \right\} \subseteq R$$

is a pseudo subring as well as a pseudo ideal of R of infinite order.

$$\text{Let } P_3 = \left\{ \begin{bmatrix} a_1 & 0 & a_2 \\ 0 & a_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1, a_2, a_3 \in [0, 12) \right\} \subseteq R$$

is a pseudo subring as well as a pseudo ideal of R of infinite order.

$$P_4 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_i \in [0, 12), 1 \leq i \leq 5 \right\} \subseteq R$$

is a pseudo subring as well as a pseudo ideal of R of infinite order.

Let

$$A = \left\{ \begin{bmatrix} a_1 & a_2 & 0 \\ 0 & 0 & a_3 \\ 0 & 0 & a_4 \end{bmatrix} \middle| \; a_i \in \{0, 1, 2, 3, 4, 5, \ldots, 11, 1 \leq i \leq 4\} \right\} \subseteq R$$

be a subring and is not a pseudo ideal of R. $|A| < \infty$.

$$B = \left\{ \left. \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & a_3 & 0 \\ a_4 & a_5 & a_6 \end{bmatrix} \right| a_i \in \{0, 1, 2, 3, 4, 5, \ldots, 11\}, \right.$$
$$\left. 1 \leq i \leq 6 \right\} \subseteq R$$

is only a subring of finite order and B is not a pseudo ideal of R.

Example 4.35: Let

$$R = \left\{ \left. \begin{bmatrix} a_1 & a_2 & \ldots & a_5 \\ a_6 & a_7 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{15} \\ a_{16} & a_{17} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{25} \end{bmatrix} \right| a_i \in [0, 15), 1 \leq i \leq 25, +, \times_n \right\}$$

be the special interval a pseudo ring of infinite order which is commutative.

$$A = \left\{ \left. \begin{bmatrix} a_1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} \right| a_1 \in [0, 15), +, \times_n \right\} \subseteq R$$

be the special interval pseudo subring of infinite order which is also commutative and is a pseudo ideal of R.

$$B = \left\{ \left. \begin{bmatrix} a_1 & a_2 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_3 & a_4 & a_5 & a_6 & a_7 \end{bmatrix} \right| a_i \in [0, 15), 1 \leq i \leq 7, +, \times_n \right\}$$

be the special interval pseudo subring of R as well as the pseudo ideal of R of infinite order.

Let

$$M = \left\{ \left[\begin{matrix} a_1 & a_2 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & ... & 0 \end{matrix} \right] \middle| \; a_1, a_2 \in \{0, 1, 2, 3, 4, \ldots, 14\}, +, \times_n \right\} \subseteq R$$

be the special interval pseudo subring and M is of finite order and M is not a pseudo ideal of R.

$$N = \left\{ \left[\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \right] \middle| \; a_i \in \{0, 3, 6, 9, 12\} \subseteq [0, 15); 1 \leq i \leq 15, +, \times_n \right\} \subseteq R$$

is a pseudo subring of finite order and is not a pseudo ideal of R.

R has several subrings of finite order and none of them are ideals of R.

R has several interval pseudo subrings of finite order and none of them are pseudo ideals of R.

Example 4.36: Let

$$R = \left\{ \left[\begin{matrix} a_1 & a_2 & ... & a_7 \\ a_8 & a_9 & ... & a_{14} \\ a_{15} & a_{16} & ... & a_{21} \end{matrix} \right] \middle| \; a_i \in [0, 200), 1 \leq i \leq 21, +, \times_n \right\}$$

be the special interval pseudo ring.

R is of infinite order and commutative R has infinite number of pseudo subrings are pseudo ideals.

Let

$$A = \left\{\left.\begin{bmatrix} a_1 & a_2 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & 0 & ... & 0 \end{bmatrix}\right| a_1 \in [0, 200), a_2 \in \{0, 10, 20, 30, \ldots, 190\}, +, \times_n\right\} \subseteq R$$

is a pseudo subring of infinite order and is not a pseudo ideal of R.

Let

$$B = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 & 0 & ... & 0 \\ 0 & 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & 0 & ... & 0 \end{bmatrix}\right| a_i \in [0, 200), 1 \le i \le 3, +, \times_n\right\}$$

$\subseteq$ R is a pseudo subring which is also a pseudo ideal of B.

Clearly B is of infinite order.

$$C = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 & 0 & ... & 0 \\ 0 & 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & 0 & ... & 0 \end{bmatrix}\right| a_i \in \{0, 25, 50, 75, 100, 125, 150, 175\} \subseteq [0, 200), 1 \le i \le 3, +, \times_n\right\} \subseteq R$$

is a subring of R are of finite order.

However C is not a pseudo ideal of R. Thus R has subrings of finite order which are not pseudo ideals of R.

Example 4.37: Let

$$R = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{array} \right] \middle| \; a_i \in [0, 131), 1 \leq i \leq 30, +, \times_n \right\}$$

be the special interval pseudo ring of R of infinite order which is commutative.

R has pseudo subrings of infinite order which are pseudo ideals and R has also pseudo subrings of infinite order which are not pseudo ideals of R.

R has finite pseudo subrings which are not pseudo ideals. R has infinite number of zero divisors, has only finite number of idempotents and finite number of units.

$$A = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{array} \right] \middle| \; a_1, a_2, a_3 \in [0, 131), +, \times_n \right\} \subseteq R$$

be the pseudo subring which is a pseudo ideal of R of infinite order.

$$B = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{array} \right] \middle| \; a_1 \in [0, 131), a_2, a_3 \in [0, 1, 2, 3, 4, \ldots, 130], +, \times_n \right\} \subseteq R$$

be the pseudo subring which is not a pseudo ideal of infinite order.

B is not an ideal of R.

$$C = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_1, a_2, a_3 \in \{0, 1, 2, 3, \ldots, 130\}, +, \times_n \right\} \subseteq R$$

be the subring which is not a pseudo ideal of R and is of finite order.

$$D = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \\ a_4 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a_i \in [0, 1, 2, \ldots, 130\}, \; 1 \leq i \leq 4, +, \times_n \right\} \subseteq R$$

be the pseudo subring of R which is not a pseudo ideal of R.

Example 4.38: Let

$$R = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in [0, 48), 1 \leq i \leq 12, +, \times_n \right\}$$

be the special interval pseudo ring of infinite order and R is commutative.

R has infinite number zero divisors but only finite number of units and idempotents. R has only finite number of subrings and none of them is a pseudo ideal.

R has pseudo subrings of infinite order which are not pseudo ideals as well as infinite order pseudo subrings which are pseudo ideals.

Let

$$M_1 = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in [0, 4, 8, 12, 16, 20, 24, \ldots, 40, 44\} \subseteq [0, 48), 1 \leq i \leq 12\} \subseteq R$$

be the subring of finite order. M_1 is not a pseudo ideal of R.

Let

$$N_1 = \left\{ \begin{bmatrix} a_1 & 0 \\ a_3 & 0 \\ \vdots & \vdots \\ a_6 & 0 \end{bmatrix} \middle| \; a_i \in [0, 48), 1 \leq i \leq 6, +, \times_n\right\} \subseteq R$$

be a pseudo subring of R. N_1 is of infinite cardinality. N_1 is also a pseudo ideal of R.

Let

$$T_1 = \left\{ \begin{bmatrix} a_1 & 0 \\ a_2 & 0 \\ a_3 & 0 \\ a_4 & 0 \\ a_5 & 0 \\ a_6 & 0 \end{bmatrix} \middle| \; a_1, a_2, a_3 \in [0, 48), a_4, a_5, a_6 \in \{0, 12, 24, 36\} \subseteq [0, 48), +, \times_n\right\}$$

be the pseudo subring of R. $|T_1| = \infty$; but T_1 is not a pseudo ideal of R only a subring.

Thus we have pseudo subrings of infinite cardinality which are not pseudo ideals of R.

Example 4.39: Let

$$R = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ \hline a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ \hline a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \end{array}\right] \middle| \; a_i \in [0, 41), 1 \leq i \leq 21, +, \times_n \right\}$$

be the special interval pseudo ring super column matrices. $|R| = \infty$.

R has infinite number of zero divisors and only finite number of idempotents and only finite number of units and idempotents. R has only finite number of subrings of finite order.

R has infinite pseudo subrings which are ideals as well as infinite pseudo subrings which are not pseudo ideals of R.

$$M_1 = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \middle| \; a_i \in [0, 41), 1 \leq i \leq 3, +, \times_n \right\} \subseteq R$$

is a special interval pseudo subring which is a pseudo ideal of R. M_1 is also of infinite order.

$$M_2 = \left\{ \left[\begin{array}{ccc} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \right| a_1, a_2, a_3 \in [0, 41), a_3, a_5, a_6 \in$$

$$\{0, 1, 2, 3, \ldots, 40\}, +, \times_n\} \subseteq R$$

is a pseudo subring of R and is of infinite order M_2 is not a pseudo ideal of R.

$$\text{For if } A = \left[\begin{array}{ccc} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 7.9 & 3.1 & 0 \\ 8 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \in M_2 \text{ and}$$

$$B = \left[\begin{array}{ccc} 9 & 8 & 3.1 \\ \hline 4.3 & 3.7 & 19.1 \\ 40.1 & 4.7 & 8.19 \\ \hline 0 & 40 & 0 \\ 3.1 & 2.4 & 0.14 \\ 0 & 0 & 0 \\ 0.75 & 0.95 & 1.98 \end{array}\right] \in R.$$

$$\text{We find } A \times B = \begin{bmatrix} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 12.4 & 0 \\ 24.8 & 2.4 & 0.70 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \notin M_2 \text{ as } 24.8,$$

2.4 and 0.70 $\notin$ {0, 1, 2, 3, 4, 5, 6, …, 40} $\subseteq$ [0, 41).

Hence M_2 is of infinite order only a pseudo subring and not a pseudo ideal of R.

$$\text{Let } N_1 = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ \hline 0 & a_2 & 0 \\ 0 & 0 & a_3 \\ \hline a_4 & 0 & 0 \\ 0 & a_5 & 0 \\ 0 & 0 & a_6 \\ a_7 & 0 & 0 \end{bmatrix} \middle| \; a_i \in [0, 1, 2, \ldots, 40\} \subseteq [0, 41), \right.$$

$$1 \leq i \leq 7, +, \times_n\} \subseteq R$$

is only a subring of finite order and is not a pseudo ideal of R. Let

$$A = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ \hline a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ \hline a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \end{bmatrix} \middle| \; a_i \in \{1, 2, 3, \ldots, 40\} \subseteq [0, 41), \right.$$

$$1 \leq i \leq 21, +, \times_n\} \subseteq R,$$

is such that for every x ∈ A there exist a unique y in A such that

$$x \times_n y = \left[\begin{array}{ccc} 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right] \text{ the unit of R.}$$

All units of R in totality be the subset A. Infact A is not a subring. A is a subgroup of R under $\times_n$. A is not closed under +.

Example 4.40: Let
R = $\{(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) \mid a_i \in [0, 15) \times [0, 21); 1 \leq i \leq 8\}$ be the special interval pseudo ring. R has infinite number of zero divisors.

I = ((1, 1), (1, 1), (1, 1), …, (1, 1)) is the multiplicative identity of R and Q = ((0, 0), (0, 0), (0, 0), …, (0, 0)) is the additive identity of R.

A = $\{(a_1, 0), (a_2, 0), \ldots, (a_8, 0)\}$ and B = $\{(0, b_1), (0, b_2), \ldots, (0, b_8)\} \in$ R is such that A × B = ((0, 0), (0, 0), (0, 0), …, (0, 0))}.

A = $\{(a_1, \ldots, a_8) \mid a_i \in [0, 15), \times \{0\}, 1 \leq i \leq 8\} \subseteq$ R is an infinite pseudo subring as well as pseudo ideal of R.

B = $\{(a_1, a_2, \ldots, a_8) \mid a_i \in \{0\} \times [0, 21), 1 \leq i \leq 8\} \subseteq$ R is an infinite pseudo subring as well as pseudo ideal of R.

We see A × B = {((0, 0), (0, 0), (0, 0), …, (0, 0))}

Thus R has infinite number of zero divisors.

Let M = $\{(a_1, a_2, \ldots, a_8) \mid a_i \in \{0, 5, 10\} \cup \{0, 7, 14\}; 1 \leq i \leq 8, +, \times\} \subseteq R$ be a pseudo subring of R of finite order and is not a pseudo ideal of R.

$N_1 = \{(a_1, \ldots, a_8) \mid a_i \in \{0, 5, 10\} \times [0, 21); 1 \leq i \leq 8\} \subseteq R$ is a pseudo subring of infinite order.

However N_1 is not an ideal of R.

Example 4.41: Let R = $\{(a_1, a_2, a_3, a_4) \mid a_i \in [0, 12) \times [0, 7) \times [0, 11); 1 \leq i \leq 4\}$ be the special interval pseudo ring under the operation + and ×. R is of infinite order. R has infinite number of zero divisors, finite number of units and idempotents.

The additive identity of R is
a = ((0, 0, 0), (0, 0, 0) (0, 0, 0) (0, 0, 0)) and the multiplicative identity of R is I = ((1, 1, 1), (1, 1, 1), (1, 1, 1), (1, 1, 1)).

We see x = {(5, 5, 7), (7, 2, 1), (11, 3, 5), (1, 4, 6))} ∈ R has y = {((5, 3, 8), (7, 4, 1), (11, 5, 9), (1, 2, 2))} ∈ R to be the unique inverse of x; for
x × y = {((1, 1, 1), (1, 1, 1), (1, 1, 1), (1, 1, 1))} ∈ R. It is easily verified R has only finite number of units.

x = {((4, 1, 1), (9, 1, 0), (0, 1, 1), (1, 1, 1))} ∈ R is such that $x^2 = x$ thus x in R is an idempotent of R.

R has only finite number of idempotents in it.

Example 4.42: Let

$$R = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_{10} \end{bmatrix} \middle| \; a_i \in [0, 40) \times [0, 31); 1 \leq i \leq 10, +, \times_n \right\}$$

be the special interval pseudo ring of infinite order. R is commutative. R has infinite number of zero divisors.

R has ideals, R has finite pseudo subrings none of which are pseudo ideals.

R has also infinite pseudo subrings which are not ideals. R has only finite number of units and idempotents.

$$I = \left\{\begin{bmatrix}(1,1)\\(1,1)\\(1,1)\\\vdots\\(1,1)\end{bmatrix}\right\} \in R \text{ is the multiplicative identity of R.}$$

$$(0) = \left\{\begin{bmatrix}(0,0)\\(0,0)\\(0,0)\\\vdots\\(0,0)\end{bmatrix}\right\} \in R \text{ is the additive identity of R.}$$

$$\text{Let } A = \left\{\begin{bmatrix}(a_1,0)\\(a_2,0)\\\vdots\\(a_{10},0)\end{bmatrix} \middle|\ a_i \in [0, 40); 1 \leq i \leq 10, +, \times_n\right\} \subseteq R$$

is a pseudo ideal of R and is of infinite order.

$$\text{Let } B = \left\{\begin{bmatrix}(0,b_1)\\(0,b_2)\\\vdots\\(0,b_{10})\end{bmatrix} \middle|\ b_i \in [0, 31); 1 \leq i \leq 10, +, \times_n\right\} \subseteq R$$

is a pseudo ideal of R and is of infinite order.

$$A \times B = \left\{\begin{bmatrix} (0,0) \\ (0,0) \\ (0,0) \\ \vdots \\ (0,0) \end{bmatrix}\right\}$$ that is for every $a \in A$ and every

$b \in B$ we have $A \times B = \left\{\begin{bmatrix} (0,0) \\ (0,0) \\ (0,0) \\ \vdots \\ (0,0) \end{bmatrix}\right\}$.

Let $M_1 = \left\{\begin{bmatrix} (a_1,a_2) \\ (a_3,0) \\ (a_4,a_5) \\ (0,0) \\ \vdots \\ (0,0) \end{bmatrix} \middle|\ a_1, a_4, a_3 \in [0, 40) \text{ and } a_2, a_5 \in [0, 31); +, \times_n\right\} \subseteq R$

be a pseudo subring of infinite order and M_1 also a pseudo ideal of R.

Let

$N_1 = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_{10} \end{bmatrix} \middle|\ a_i \in \{0, 10, 20, 30\} \times [0, 31); 1 \leq i \leq 10, +, \times_n\right\} \subseteq R$

be a pseudo subring of R of infinite order.

N_1 is not an ideal of R.

$$\text{Let } P_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_{10} \end{bmatrix} \middle| \; a \in \{0, 4, 8, 12, 16, 20, 24, 28, 32, 36\} \times \right.$$

$$\{0, 1, 2, 3, 4, 5, 6, \ldots, 30\}; 1 \leq i \leq 10, +, \times_n\} \subseteq R$$

is a subring of R of finite order. Clearly R is not a pseudo ideal of R. Let

$$x = \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 0 \\ 0 \\ a_1 \\ a_2 \\ \vdots \\ a_8 \end{bmatrix} \in R \text{ we see } x \times_n y = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in R$$

is a zero divisor in R.

Example 4.43: Let

$$R = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \; a_i \in [0, 12) \times [0, 9) \times [0, 17); 1 \leq i \leq 4 +, \times_n \right\}$$

be the commutative special interval pseudo ring of infinite order.

The additive identity of R is $(0) = \begin{bmatrix} (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) \end{bmatrix}$.

The multiplicative identity of R is I = $\begin{bmatrix} (1,1,1) & (1,1,1) \\ (1,1,1) & (1,1,1) \end{bmatrix}$

under the natural product $\times_n$ of matrices.

$$x = \begin{bmatrix} (5,8,9) & (11,5,7) \\ (1,5,6) & (1,1,1) \end{bmatrix} \text{ and}$$

$$y = \begin{bmatrix} (5,8,9) & (11,2,5) \\ (1,2,16) & (1,1,1) \end{bmatrix} \in R \text{ is such that}$$

$$x \times y = \begin{bmatrix} (1,1,1) & (1,1,1) \\ (1,1,1) & (1,1,1) \end{bmatrix}.$$

$$\text{Let } x = \begin{bmatrix} (6,0,0) & (4,3,2) \\ (8,6,7) & (6,3,5) \end{bmatrix} \text{ and } y = \begin{bmatrix} (2,8,12) & (3,3,0) \\ (3,3,0) & (6,3,0) \end{bmatrix}$$

we see $x \times_n y = \begin{bmatrix} (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) \end{bmatrix}$ is the zero divisor of R.

R has infinite number of zero divisors, but only finite number of units and idempotents.

$$M = \left\{ \begin{bmatrix} (a_1,b_1,c_1) & (0,0,0) \\ (0,0,0) & (0,0,0) \end{bmatrix} \middle| (a_1, b_1, c_1) \in ([0, 12) \times [0, 9) \right.$$

$\times$ [0, 17))} $\subseteq$ R is a pseudo subring as well as a pseudo ideal of R of infinite order.

We have also pseudo subrings of infinite order which are not pseudo ideals.

$$\text{For take } N = \left\{ \begin{bmatrix} (a_1,b_1,c_1) & (0,0,0) \\ (a_2,0,0) & (0,0,a_3) \end{bmatrix} \middle| (a_1, b_1, c_1) \in ([0, 12) \right.$$

$\times$ [0, 9) $\times$ [0, 17)), $a_2 \in \{0, 2, 4, 6, 8, 10\} \times \{0\} \times \{0\}$, $a_3 \in \{0\} \times \{0\} \times \{0, 1, 2, \ldots, 16\}\} \subseteq R$ is a pseudo subring of infinite order and is not a pseudo ideal of R.

We have seen special types of special interval pseudo matrix rings.

Now we proceed onto study group pseudo rings using these special interval pseudo rings.

Example 4.44: Let RG = $\left\{\sum_{i=0}^{n} a_i g_i \right.$ n finite where G = {g = 1, $g_1, \ldots, g_n$} and $a_i \in R = \{[0, 25); 0 \leq i \leq n\}$ be the group interval pseudo ring which will be known as the special interval group pseudo ring as R is a special interval pseudo ring.

RG has zero divisors, RG has torsion elements as R $\subseteq$ RG and G $\subseteq$ RG (1 $\in$ G and 1 $\in$ R). RG has pseudo subrings, pseudo ideals and idempotents and units.

RG will be non commutative and if G is non commutative and if G is commutlative RG will be commutative.

Example 4.45: Let R = {[0, 12), +, $\times$} be the special interval pseudo ring and G = {S_3 the symmetric group of degree three}. RS_3 = RG be the special interval group pseudo ring of the group S_3 over the special interval pseudo ring R.

We see $S_3 \subseteq RS_3$ as 1 $\in$R and R $\subseteq RS_3$ as 1 = $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ of S_3 is the identity of S_3.

Let x = 3 + 6 $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and y = 4 $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ +

$$8 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} + 4 \in RS_3.$$

We see $x \times y = [3 + 6 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}] \times [4 + 4 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} +$

$$8 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}]$$

$$= 12 + 24 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} + 12 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} +$$

$$24 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} + 24 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} +$$

$$48 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = 0 \ (\text{mod } 12).$$

Thus RS_3 has zero divisors.

Let $x = 4 + 9 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} + 9 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \in RS_3$

We find $x^2 = (4 + 9 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} + 9 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}) \times$

$$(4 + 9 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} + 9 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix})$$

$$= 4 + 9 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} + 9 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

$$= 4 + 9 + 9 = 10.$$

Thus we have elements x in $RS_3 \setminus R$ which are such that $x^2 \in R$.

Consider $y = 6 + 6 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \in RS_3$

$$y^2 = \left(6 + 6\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\right) \left(6 + 6 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\right) = 0;$$

thus y is a nilpotent element of order two.

$$\text{Let } x = 11 + 5 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \in RS_3;$$

$$x^2 = \left(11 + 5 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\right) \times \left(11 + 5 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\right)$$

$$= 1 + 55 \times 2 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} + 1)$$

$$= 2 + 2 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = 2 \left[1 + \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\right]$$

$$= 2x.$$

This is also a special condition for in reals other than 2 cannot be like this.

$$x = 11 + 5 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \text{ and}$$

$$y = 10 + 5 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \in RS_3.$$

We find $x \times y = \left(11 + 5 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\right) \times \left(10 + 5 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\right)$

$$= 110 + 50 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} + 55 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} + 1$$

$$= 3 + 9 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

Let $x = 11 + 5 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ and $y = 11 + 7 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$.

$$x \times y = \left[11 + 5 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\right] \left[11 + 7 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\right]$$

$$= 121 + 55 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} + 77 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} + 11 \pmod{12}$$

$$= 0.$$

Thus this gives a zero divisor. The study of RS_3 paves way to special properties like elements whose square is two times it and so on.

RS_3 is a non commutative infinite special interval pseudo group ring.

We have several interesting properties like substructures and so on.

Let $S = RP_1$ where $P_1 = \left\{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\right\}$ be a subgroup of S_3, we see S is a commutative pseudo subring of RS_3 which is not a pseudo ideal of RS_3.

Let $P_4 = \left\{\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}\right\} \subseteq R$ is a normal subgroup. RP_4 is also a pseudo subring and is a pseudo ideal of RP_4.

We see RS_3 has non commutative pseudo subrings and all subrings are only commutative pseudo subrings. This is a special type of pseudo ring which has non commutative pseudo subrings also for take TS_3 where T = {0, 1, 2, 3, 4, …, 11} is the ring of modulo integers Z_{12}.

We see TS_3 is of finite order and is a non commutative subring. This subring has zero divisors, units and idempotents.

Example 4.46: Let RS_4 where R = {[0, 19), +, ×} be the special interval pseudo ring and S_4 be the symmetric group of degree four.

RS_4 has pseudo subrings which are both commutative and non commutative.

Take PA_4 where A_4 is the alternative subgroup of S_4. RA_4 is non commutative pseudo subring of RS_4.

Consider RP where $P = \left\langle \begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 3 & 4 & 1\end{pmatrix} \right\rangle$ be the subgroup of S_4.

RP is a commutative pseudo subring of infinite order. This has both finite and infinite pseudo subrings. Let S = TP_1 where T = {0, 1, 2, …, 17, 18} ⊆[0, 19);

$P_1 = \left\{\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 3 & 4 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 4 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 3 & 4\end{pmatrix}\right\} \subseteq S_4$; S = TP_1 is a pseudo subring which is commutative and is of finite order. RS_4 has several subrings of both finite and infinite order. RS_4 is non commutative and has units.

Example 4.47: Let B = $RD_{2,7}$ be the special interval group pseudo ring of the group $D_{2,7}$ over the special interval pseudo ring R = {[0, 10), +, ×}.

We see B has zero divisors, units and idempotents. B has commutative pseudo subrings as well as non commutative pseudo subrings. B has also pseudo ideals. However B is a non commutative pseudo ring.

Example 4.48: Let
R = ($S_3 \times D_{2,7} \times A_4$) = B where R = {[0, 10) ×[0, 31) × [0, 48); +, ×} be the special interval pseudo ring, be the group pseudo ring.

B has several zero divisors units and idempotents. B has pseudo ideals. B is non commutative and of infinite order.

We study RG = B when R is an infinite pseudo integral domain like R = {[0, p); p a prime, ×, +} and G any group. This study will be interesting.

Now we introduce special interval pseudo polynomial rings.

Example 4.49: Let R[x] = $\left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 35); +, \times \right\}$ be the special interval pseudo polynomial ring where R is the special pseudo interval ring viz. R = [0, 35), +, ×}.

R[x] has zero divisor.
Let p(x) = 7 + 21x + $14x^2$ and q (x) = 5 + $10x^3$ ∈ R[x]

$p(x)\, q(x) = (7 + 21x + 14x^2) \times (5 + 10x^3)$
$= 35 + 105x + 70x^2 + 70x^3 + 210x + 140x^2 \pmod{35}$
$= 0.$

Thus R[x] has zero divisors, as R is not a pseudo integral domain.

Example 4.50: Let R[x] = $\left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 41); +, \times\right\}$ be the special interval pseudo polynomial ring over the special interval pseudo ring R = {[0, 41), +, ×}.

Since R is a interval ring R[x] has zero divisor.

How to solve equations in R[x]? We cannot use the formula to solve the quadratic equations with real coefficients.

Let p(x) = $6x^2 + 19x + 34 \in$ R[x];
now p(x) = $6x^2 + 19x + 34$
= (3x + 40) (2x + 7) = 0.
Hence 3x + 40 = 0.
and 2x + 7 = 0

Now 3x = 1, x = 14.
2x + 7 = 0 implies 2x = 34;
x = 34 × 21 (mod 41). (As $2^{-1} = 21$)

Thus x = 17 and x = 14 are the two roots of the equation $6x^2 + 19x + 34 = 0$.

However if the coefficients of the polynomials are decimals we work for the roots in the following way.

Suppose p(x) = $14.775x^2 + 25119x + 6.834$
= (2.01 + 5.91x) (3.4 + 2.5x) = 0
Thus 5.91 x + 2.01 = 0 and
2.5 x + 3.4 = 0.

As these elements have no inverse we take 5.91x = 38.99; 2.5x = 37.6

Interested reader can study how to solve these equations.
p(x) = $x^3 + 5.62x^2 + 9.124x + 4.416$
= (x + 3.2) (x + 0.92) (x + 1.5) = 0.

$x + 0.92 = 0$ and
$x + 1.5 = 0.$

This gives x = 37.8, 40.08 and 39.5.

However solving these equations is as hard as solving any equation in reals, here the special interval pseudo ring is infinite but it should be worked with modulo p if [0, p) is the interval used.

However if $R = \{[0, p), +, \times\}$ is taken as the pseudo interval ring we cannot make use of inverses as inverses do not exist so the question of making any polynomial p(x) into monic cannot be done. So every polynomial in R[x] cannot be made into a monic polynomial.

However if p(x) is a polynomial $p'(x) \in R[x]$ $p'(x)$ the derivative of p(x) with respect to x, for the coefficients are always take modulo p, where p is used in the interval [0, p) . If p is not a prime the differentiation behaves in a different way.

Example 4.51: Let $R[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 6), +, \times\right\}$ be the special interval pseudo polynomial ring.

Let $p(x) = 0.73x^6 + 2x^3 + 3x + 5 \in R[x]$.

The derivative of p(x) is

$$\frac{dp(x)}{dx} = 0.73 \times 6x^5 + 2.3x^2 + 3 \pmod 6$$

$= 3$ a constant.

This is the unique property enjoyed by these special interval polynomials.

The differentiation is performed in a unique way.

We can also integrate in a similar way.

Thus polynomials in special interval pseudo ring is an interesting study for solving equations are difficult.

Example 4.52: Let R[x] = $\left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 16), +, \times\right\}$ be the special interval pseudo polynomial ring using the special interval R = {[0, 16), +, ×}. R[x] has zero divisors, and a finite number of units. R[x] has pseudo subrings, pseudo ideals and subrings.

For P = $\left\{\sum_{i=0}^{\infty} a_i x^{2i} \,\middle|\, a_i \in [0, 16), +, \times\right\} \subseteq R[x]$ is a polynomial pseudo subring of R[x] of infinite order; if $a_i \in \{0, 1, 2, \ldots, 15\}$ then also P is an infinite pseudo subring which is not a pseudo ideal.

We suggest the following problems for this chapter.

Problems:

1. Obtain some special features enjoyed by
   R = {[0, n), min, max} the special interval semiring.

2. Prove R = {[0, 27), min, max} has infinite number of subsemirings of order n, $0 < n < \infty$.

3. Prove R = {[0, 143), min, max} has infinite number of subsemirings of infinite order.

4. Prove R has no zero divisors and every element is an idempotent both with respect to min as well as max.

5. Let R = {[0, 129), min, max} be the special interval semiring.

   Prove R has infinite number of ideals which are not filters.

6. Let R = {[0, 24), min, max} be a special interval semiring.

   Prove R has infinite number of filters which are not ideals of R.

7. Can $P \subseteq R$ = {[0, 25), min, max} where R is a semiring have P to be both an ideal and filter of R?

8. Let R = {[0, 48), min, max} be the special interval semiring.

   (i) Find all subsemirings which are of infinite order (Is it infinite collection?)
   (ii) Can R have ideals which are filters?
   (iii) Find some special features related with R.

9. Let R = $\{(a_1, a_2, \ldots, a_7) \mid a_i \in [0, 24), 1 \leq i \leq 7$, min, max$\}$ be a special interval semiring.

   (i) Show R has zero divisors.
   (ii) Prove R has infinite number of idempotents with respect to both min and max.
   (iii) Prove R has infinite number of subsemirings.
   (iv) Can R have infinite number of ideals?
   (v) Find all semirings which are not ideals.
   (vi) Can R have filters which are ideals?
   (vii) Find all filters which are ideals and vice versa (if any).

10. Let S = $\{(a_1, a_2, \ldots, a_9) \mid a_i \in [0, 29), 1 \leq i \leq 9$, max, min$\}$ be the special interval semiring.

    Study questions (i) to (vii) of problem (9) for this S.

11. Let $R_1 = \left\{ \left[\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_{12} \end{array}\right] \right|$ $a_i \in [0, 10); 1 \leq i \leq 12, \max, \min\}$

be the special interval semiring.

Study questions (i) to (vii) of problem (9) for this R_1.

12. Let $R = \left\{ \left[\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_9 \end{array}\right] \right|$ $a_i \in [0, 9) \times [0, 19); 1 \leq i \leq 9, \max, \min\}$

be the special interval semiring.

(i) Study questions (i) to (vii) of problem (9) for this R.
(ii) Compare this R with R_1 of problem 11.

13. Let $R = \left\{ \left[\begin{array}{ccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_8 & \cdots & \cdots & \cdots & \cdots & \cdots & a_{14} \\ a_{15} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{21} \\ a_{22} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{28} \\ a_{29} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{35} \end{array}\right] \right|$ $a_i \in$

$[0, 143); 1 \leq i \leq 35, \max, \min\}$ be the special interval semiring.

(i) Study questions (i) to (vii) of problem (9) for this R.

14. Let $R_1 = \left\{ \begin{bmatrix} a_1 & a_2 & ... & a_{12} \\ a_{13} & a_{14} & ... & a_{24} \\ a_{25} & a_{26} & ... & a_{36} \end{bmatrix} \middle| \; a_i \in [0, 13) \times [0, 119); \right.$

$1 \leq i \leq 36$, max, min} be the special interval semiring.

(i) Study questions (i) to (vii) of problem (9) for this R.
(ii) Compare R_1 with R of problem 13.

15. Let $R = \left\{ \begin{bmatrix} a_1 & a_2 & ... & a_5 \\ a_6 & a_7 & ... & a_{10} \\ a_{11} & a_{12} & ... & a_{15} \\ a_{16} & a_{17} & ... & a_{20} \\ a_{21} & a_{22} & ... & a_{25} \end{bmatrix} \middle| \; a_i \in [0, 125); 1 \leq i \leq 25, \right.$

max, min} be the special interval semiring.

Study questions (i) to (vii) of problem (9) for this R.

16. Let $R_1 = \left\{ \begin{bmatrix} a_1 & a_2 & ... & a_7 \\ a_8 & a_9 & ... & a_{14} \\ a_{15} & a_{16} & ... & a_{21} \\ a_{22} & a_{23} & ... & a_{28} \\ a_{29} & a_{30} & ... & a_{35} \\ a_{36} & a_{37} & ... & a_{42} \\ a_{43} & a_{44} & ... & a_{49} \end{bmatrix} \middle| \; a_i \in [0, 10) \times [0, 26); \right.$

$1 \leq i \leq 4$, max, min} be the special interval semiring.

(i) Study questions (i) to (vii) of problem (9) for this R.

(ii) Compare R_1 with R of problem 15.

17. Let S = {[0, 13) × [0, 24) × [0, 53) × [0, 128); min, max} be the semiring.

    Study questions (i) to (vii) of problem (9) for this S.

18. Let R_t = {(a_1 | a_2 a_3 | a_4 a_5 | a_6 a_7 a_8 a_9 | a_{10}) | a_i ∈[0, 4) × [0, 41); 1 ≤ i ≤ 10; max, min} be the special interval super matrix semiring.

    Study questions (i) to (vii) of problem (9) for this R_t.

19. Let M = $\left\{\left.\left[\begin{array}{c} a_1 \\ \hline a_2 \\ a_3 \\ a_4 \\ a_5 \\ \hline a_6 \\ a_7 \\ a_8 \\ \hline a_9 \\ a_{10} \\ \hline a_{11} \end{array}\right]\right| a_i \in [0, 8) \times [0, 11) \times [0, 101);\right.$ 1 ≤ i ≤11, max, min} be the special interval super column matrix semiring.

    Study questions (i) to (vii) of problem (9) for this S.

    Compare M with R_t of problem 18.

20. Let $S = \left\{ \left[\begin{array}{c|cc|c|cc|c} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_8 & \ldots & \ldots & \ldots & \ldots & \ldots & a_{14} \\ a_{15} & \ldots & \ldots & \ldots & \ldots & \ldots & a_{21} \end{array} \right] \middle| \; a_i \in [0, 23) \right.$

$\times$ [0, 14); $1 \leq i \leq 21$, max, min} be the special interval super row matrix semiring.

Study questions (i) to (vii) of problem (9) for this S.

21. Let $M = \left\{ \left[\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ \hline a_5 & \ldots & \ldots & a_8 \\ a_9 & \ldots & \ldots & a_{12} \\ a_{13} & \ldots & \ldots & a_{16} \\ a_{17} & \ldots & \ldots & a_{20} \\ \hline a_{21} & \ldots & \ldots & a_{24} \\ a_{25} & \ldots & \ldots & a_{28} \\ \hline a_{29} & \ldots & \ldots & a_{32} \\ a_{33} & \ldots & \ldots & a_{36} \\ a_{37} & \ldots & \ldots & a_{40} \\ \hline a_{41} & \ldots & \ldots & a_{44} \end{array} \right] \middle| \; a_i \in [0, 15) \times [0, 24) \times \right.$

[0, 9); $1 \leq i \leq 44$, max, min} be the special interval super column matrix semiring.

Study questions (i) to (vii) of problem (9) for this M.

22. Let N = $\left\{ \left[\begin{array}{c|ccc|ccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ \hline a_8 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{15} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{22} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{29} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline a_{36} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{42} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array} \right] \right| a_i \in [0, 25)$

$\times$ [0, 36); $1 \leq i \leq 49$, max, min} be the special interval super column matrix semiring.

Study questions (i) to (vii) of problem (9) for this M.

23. Let T = $\left\{ \left[\begin{array}{ccc|c|cc|cc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ a_9 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{17} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline a_{25} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{33} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline a_{41} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array} \right] \right| a_i \in$

[0, 48); $1 \leq i \leq 48$, max, min} be the special interval super column matrix semiring.

Study questions (i) to (vii) of problem (9) for this T.

24. Let S = $\left\{ \left[\begin{array}{cccc} a_1 & a_2 & \cdots & a_{15} \\ a_{16} & a_{17} & \cdots & a_{30} \\ a_{31} & a_{32} & \cdots & a_{45} \\ a_{46} & a_{47} & \cdots & a_{60} \end{array} \right] \right| a_i \in [0, 32) \times [0, 32) \times$

[0, 32); $1 \leq i \leq 60$, max, min} be the special interval super column matrix semiring.

Study questions (i) to (vii) of problem (9) for this S.

25. Can R = {[0, 29), +, ×}, the special interval pseudo ring have non trivial pseudo ideals?

26. Can R = {[0, 24), +, ×}; the special interval pseudo ring have pseudo ideals?

27. Can R = {[0, 125), +, ×}; the special interval pseudo ring have pseudo subrings of infinite order.

28. Can R = {[0, 127), +, ×}; the special interval pseudo ring have infinite pseudo subrings which are not pseudo ideals?

29. Study the special and distinct properties enjoyed by special interval pseudo rings.

30. Compare special interval pseudo rings with special interval semirings for any interval [0, n).

31. Let R = {[0, 23), +, ×} be the special interval pseudo ring.

    (i) Can R have finite subrings?
    (ii) How many finite subrings R contains?
    (iii) Can R have infinite number of infinite pseudo subrings?
    (iv) Can R have units?
    (v) Can R contain infinite number of units?
    (vi) Can R have idempotents?
    (vii) Can R have zero divisors?
    (viii) Can R have ideals?

32. Let R_1 = {[0, 25), +, ×} be a special interval pseudo ring.

    (i) Study questions (i) to (viii) of problem 31 for this R_1.
    (ii) Compare R_1 with R of problem 31.

33. Let $R = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in [0, 11); 1 \le i \le 3, +, \times \right\}$

be the special interval pseudo ring under usual matrix product.

(i) Prove R is commutative.
(ii) Prove R has infinite number of zero divisors.
(iii) Find at least 5 left zero divisors which are not right zero divisors.
(iv) Find atleast 4 right zero divisors which are not left zero divisors.
(v) Find idempotents of R.
(vi) Find left pseudo ideals which are not right pseudo ideals of R and vice versa.
(vii) Can R have finite subrings?
(viii) Is it possible for R to have finite pseudo ideals?

34. Let $R_1 = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in [0, 28), 1 \le i \le 5, +, \times\}$ be the special interval pseudo ring.

Study questions (i) to (viii) of problem 33 for this R_1.

35. Let $R_2 = \{(a_1, a_2, \ldots, a_9) \mid a_i \in [0, 32) \times [0, 48), 1 \le i \le 9, +, \times\}$ be the special interval row matrix pseudo ring.

(i) Study questions (i) to (viii) of problem 33 for this R_2.
(ii) Compare R_1 of problem 34 with this R_2.

36. Let $R = \{[0, 43), +, \times\}$ be a special interval pseudo ring.

Study questions (i) to (viii) of problem 33 for this R.

37. Let $M = \{[0, 20) \times [0, 53), +, \times\}$ be the special pseudo interval ring.

Study questions (i) to (viii) of problem 33 for this M.

38. Let N = {[0, 12) × [0, 28) × [0, 35), +, ×} be a special interval pseudo ring.

Study questions (i) to (viii) of problem 33 for this N.

39. Let T = {[0, 7) × [0, 19) × [0, 23) × [0, 43), +, ×} be a special interval pseudo ring.

Study questions (i) to (viii) of problem 33 for this T.

40. Let $P = \left\{ \left[\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{15} \end{array} \right] \middle| \; a_i \in [0, 42); 1 \le i \le 15, +, \times_n \right\}$ be the special interval pseudo ring.

Study questions (i) to (viii) of problem 33 for this P.

41. Let $M = \left\{ \left[\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{10} \end{array} \right] \middle| \; a_i \in [0, 31); 1 \le i \le 10, +, \times_n \right\}$ be the special interval pseudo ring.

Study questions (i) to (viii) of problem 33 for this M.

Compare P of problem 40 with this M.

42. Let $M = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{18} \end{bmatrix} \middle| a_i \in [0, 30) \times [0, 48); 1 \leq i \leq 18, +, \times_n\right\}$

be the special interval pseudo ring.

(i) Study questions (i) to (viii) of problem 33 for this L.

43. Let $D = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_9 \end{bmatrix} \middle| a_i \in [0, 29) \times [0, 61); 1 \leq i \leq 9, +, \times_n\right\}$ be

the special interval pseudo ring.

Study questions (i) to (viii) of problem 33 for this S.

44. Let $M = \left\{\begin{bmatrix} a_1 & a_2 & \dots & a_9 \\ a_{10} & a_{11} & \dots & a_{18} \\ a_{19} & a_{20} & \dots & a_{27} \\ a_{28} & a_{29} & \dots & a_{36} \\ a_{37} & a_{38} & \dots & a_{45} \\ a_{46} & a_{47} & \dots & a_{54} \\ a_{55} & a_{56} & \dots & a_{63} \end{bmatrix} \middle| a_i \in [0, 24); 1 \leq i \leq 63, +,\right.$

$\left.\times_n\right\}$ be the special interval pseudo ring.

Study questions (i) to (viii) of problem 33 for this M.

45. Let $T = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{24} \\ a_{25} & a_{26} & \ldots & a_{36} \end{bmatrix} \middle| \, a_i \in [0, 23) \times [0, 24); \right.$

$1 \leq i \leq 36, +, \times\}$ be a special interval pseudo ring.

(i) Study questions (i) to (viii) of problem 33 for this T.

(ii) Compare when in T; $\{+, \times\}$ is replaced by {min, max} so that the resulting algebraic structure is a semiring.

46. Let $S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & \ldots & \ldots & \ldots & \ldots & a_{18} \\ a_{19} & \ldots & \ldots & \ldots & \ldots & a_{24} \\ a_{25} & \ldots & \ldots & \ldots & \ldots & a_{30} \\ a_{31} & \ldots & \ldots & \ldots & \ldots & a_{36} \end{bmatrix} \middle| \, a_i \in [0, 36) \times \right.$

$[0, 41)$; $1 \leq i \leq 36, +, \times\}$ be a special interval pseudo ring under the usual matrix product.

(i) Prove S is non commutative.

(ii) Study questions (i) to (viii) of problem 33 for this S.

(iii) Find some right pseudo ideals of S which are not left pseudo ideals of S.

(iv) Find some right zero divisors of S which are not left zero divisors of S.

47. Let $W = \left\{ \left[\begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & \dots & \dots & \dots & a_{15} \\ a_{16} & \dots & \dots & \dots & a_{20} \\ a_{21} & \dots & \dots & \dots & a_{25} \end{array}\right] \middle| \; a_i \in [0, 48);\right.$

$1 \leq i \leq 25, +, \times\}$ be the special interval pseudo ring under the usual matrix product.

Study questions (i) to (viii) of problem 33 for this W.

Study questions (i) to (iv) of problem 46 for this W.

48. Let V = $\{(a_1\ a_2\ a_3\ a_4 \mid a_5\ a_6 \mid a_7\ a_8 \mid a_9) \mid a_i \in [0, 3) \times [0, 12) \times [0, 44);\ 1 \leq i \leq 9, + \times\}$ be the special interval super matrix pseudo ring.

Study questions (i) to (viii) of problem 33 for this V.

49. Let

$$P = \left\{ \left[\begin{array}{c|cc|ccc|cc|cc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & a_{20} \\ a_{21} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & a_{30} \\ a_{31} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & a_{40} \end{array}\right] \middle| \; a_i \right.$$

$\in [0, 43);\ 1 \leq i \leq 40, +, \times_n\}$ be the special interval super row matrix pseudo ring.

(i) Study questions (i) to (viii) of problem 33 for this P.

(ii) If [0, 43) is replaced by the interval [0, 30) what are the special features associated with that P.

50. Let $L = \left\{ \left[\begin{array}{c} a_1 \\ \hline a_2 \\ a_3 \\ \hline a_4 \\ a_5 \\ a_6 \\ a_7 \\ \hline a_8 \\ a_9 \\ a_{10} \\ \hline a_{11} \\ a_{12} \\ \hline a_{13} \end{array}\right] \middle| \; a_i \in [0, 36) \times [0, 48); 1 \leq i \leq 13, +, \times_n \right\}$

be the special interval column super matrix pseudo ring.

Study questions (i) to (viii) of problem 33 for this L.

51. Let $M = \left\{ \left[\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ \hline a_4 \\ a_5 \\ a_6 \\ \hline a_7 \\ a_8 \\ a_9 \\ a_{10} \\ \hline a_{11} \end{array}\right] \middle| \; a_i \in [0, 43) \times [0, 29) \times [0, 61); \right.$

$\left. 1 \leq i \leq 11, +, \times_n \right\}$ be the special pseudo interval super column matrix ring.

(i) Study questions (i) to (viii) of problem 33 for this M.

(ii) Compare this M with L of problem 50.

52. Let P = $\left\{\left[\begin{array}{cc} a_1 & a_2 \\ a_3 & a_4 \\ \hline a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ \hline a_{11} & a_{12} \\ a_{13} & a_{14} \\ \hline a_{15} & a_{16} \\ a_{17} & a_{18} \\ a_{19} & a_{20} \\ \hline a_{21} & a_{22} \end{array}\right] \middle| \; a_i \in [0, 38) \times [0, 54); 1 \leq i \leq 22, +,\right.$

$\times_n\}$ be the special interval super column matrix pseudo ring.

Study questions (i) to (viii) of problem 33 for this M.

53. Let D = $\left\{\left[\begin{array}{c|ccc|cc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ \hline a_7 & \dots & \dots & \dots & \dots & a_{12} \\ a_{13} & \dots & \dots & \dots & \dots & a_{18} \\ a_{19} & \dots & \dots & \dots & \dots & a_{24} \\ \hline a_{25} & \dots & \dots & \dots & \dots & a_{30} \\ a_{31} & \dots & \dots & \dots & \dots & a_{36} \end{array}\right] \middle| \; a_i \in [0, 21) \times\right.$

$[0, 48) \times [0, 32)$; $1 \leq i \leq 36$, $+, \times\}$ be the special interval super matrix pseudo ring.

Study questions (i) to (viii) of problem 33 for this D.

54. Let $R = \left\{ \left[\begin{array}{c|cc|c|ccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_8 & ... & ... & ... & ... & ... & a_{14} \\ a_{15} & ... & ... & ... & ... & ... & a_{21} \end{array} \right] \middle| \; a_i \in [0, 48) \right.$

$\times$ [0, 27); $1 \leq i \leq 21$, +, $\times$} be the special interval super matrix pseudo ring.

Study questions (i) to (viii) of problem 33 for this R.

55. Let M = {RG where R = {[0, 13), +, $\times$}, G = S_4} be the special interval group pseudo ring.

(i) Prove M is non commutative.
(ii) Find idempotents of any in M.
(iii) Find zero divisors of M.
(iv) Find units of M.
(v) Can M have pseudo subrings which are not pseudo ideals?
(vi) Can M have finite ideals?
(vii) Characterize those subrings which are not ideals.
(viii) Find any other property associated with this M.
(ix) Can M be a pseudo principal ideal domain? (prove your claim).

56. Let $RD_{2,7}$ be the special interval group pseudo ring of $D_{2,7}$ over the ring R = {[0, 49), $\times$, +}.

Study questions (i) to (ix) of problem 55 for this $RD_{2,7}$.

57. Let $RD_{2,11}$ be the special interval group pseudo ring of $D_{2,11}$ over the special interval pseudo ring R = {[0, 9) $\times$ [0, 24), +, $\times$}.

Study questions (i) to (ix) of problem 55 for this $RD_{2,11}$.

58. Let R($D_{2,8}$ $\times$ S(3)) be the special interval group semigroup pseudo ring of the group semigroup $D_{2,8}$ $\times$ S(3) over the special interval pseudo ring R = {[0, 29), +, $\times$}.

Study questions (i) to (ix) of problem 55 for this $R(D_{2,8} \times S(3))$.

59. Let RS(5) be the special interval semigroup pseudo ring where $R = \{[0, 11) \times [0, 23) \times [0, 28), +, \times\}$ and S(5) is the symmetric semigroup.

    Study questions (i) to (ix) of problem 55 for this RS(5).

60. Let RZ_{24} where $R = \{[0, 24) \times [0, 17), +, \times\}$ be the special interval pseudo ring and Z_{24} be the semigroup under $\times$, be the special interval semigroup ring of the semiring $(Z_{24}, \times)$ over the special interval pseudo ring R.

    Study questions (i) to (ix) of problem 55 for this RZ_{24}.

61. Let $M = R(S_7 \times D_{2,12})$ where $R = \{[0, 18), \times, +\}$ be the special interval group pseudo ring of $S_7 \times D_{2,12}$ over R.

    Study questions (i) to (ix) of problem 55 for this M.

62. Let $B = R(S(10) \times S_{12} \times D_{2,11})$ where $R = \{[0, 7) \times [0, 20) \times [0, 48), +, \times\}$ be the special interval pseudo ring. B a special interval semigroup pseudo ring.

    Study questions (i) to (ix) of problem 55 for this B.

63. Let $RD_{2,12}$ be the special interval group pseudo ring of the group $D_{2,12}$ over the special interval pseudo ring $R = \{[0, 12) \times [0, 4), +, \times\}$.

    Study questions (i) to (ix) of problem 55 for this B.

64. Let RS(7) be the special interval semigroup pseudo ring of the symmetric semigroup S(7) over the special interval pseudo ring $R = \{[0, 7) \times [0, 12) \times [0, 15), +, \times\}$.

    (i) Prove RS(7) has zero divisors.

(ii) Study questions (i) to (ix) of problem 55 for this RS(7).

65. Let $R(S_5 \times D_{2,9})$ be the special interval group pseudo ring where $R = \{[0, 5) \times [0, 9), +, \times\}$ be the interval pseudo ring.

Study questions (i) to (ix) of problem 55 for this $R(S_5 \times D_{2,9})$.

66. Let $B = RS_4$ be the special interval group pseudo ring where $R = \{[0, p);$ p any prime$\}$ be the special interval pseudo ring.

Study questions (i) to (ix) of problem 55 for this B.

67. Let $R[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 13), +, \times\right\}$ be the special polynomial interval pseudo ring.

(i) Can R[x] have zero divisors?
(ii) Can R[x] have units?
(iii) Can R[x] have pseudo ideals?
(iv) Can R[x] have finite pseudo subrings?
(v) Solve $p(x) = 8x^3 + 4x + 3 = 0$.
(vi) Is the solution to every polynomial in p(x) has a unique set of roots?
(vii) Characterize those $p(x) \in R[x]$ whose derivatives are constants.

68. Let $R[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 12), +, \times\right\}$ be the special polynomial interval pseudo ring.

Study questions (i) to (vii) of problem 67 for this R[x].

69. Let $R_1 [x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 13) \times [0, 24), +, \times \right\}$ be the special polynomial interval pseudo ring.

Study questions (i) to (vii) of problem 67 for this $R_1 [x]$.

Compare $R_1[x]$ with $R[x]$ in problem 68.

70. Let $R[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 12) \times [0, 25) \times [0, 37), +, \times \right\}$ be the special polynomial interval pseudo ring.

Study questions (i) to (vii) of problem 67 for this R[x].

71. Let $R[x_1, x_2] = \{\sum a_{ij} x_1^i x_2^j$; $0 \leq i, j \leq \infty$, +, × where R = [0, 39) × [0, 81)} be the special interval pseudo polynomial ring.

(i) Study questions (i) to (vii) of problem 67 for this $R[x_1, x_2]$.
(ii) Compare $R[x_1, x_2]$ with R[x] in problem 70.

72. Let $R[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [0, 31) \times [0, 23) \times [0, 43), +, \times \right\}$ be the special polynomial interval pseudo polynomial ring.

Study questions (i) to (vii) of problem 67 for this R[x].

73. Find a C-program for finding roots of polynomials in special interval polynomial pseudo ring.

74. Find some innovative application of these new polynomial pseudo rings.

75. Find some special features enjoyed by these pseudo rings.

FURTHER READING

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INDEX

ABOUT THE AUTHORS

Dr.W.B.Vasantha Kandasamy is a Professor in the Department of Mathematics, Indian Institute of Technology Madras, Chennai. In the past decade she has guided 13 Ph.D. scholars in the different fields of non-associative algebras, algebraic coding theory, transportation theory, fuzzy groups, and applications of fuzzy theory of the problems faced in chemical industries and cement industries. She has to her credit 646 research papers. She has guided over 100 M.Sc. and M.Tech. projects. She has worked in collaboration projects with the Indian Space Research Organization and with the Tamil Nadu State AIDS Control Society. She is presently working on a research project funded by the Board of Research in Nuclear Sciences, Government of India. This is her 90th book.

On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.

She can be contacted at vasanthakandasamy@gmail.com
Web Site: http://mat.iitm.ac.in/home/wbv/public_html/
or http://www.vasantha.in

Dr. Florentin Smarandache is a Professor of Mathematics at the University of New Mexico in USA. He published over 75 books and 200 articles and notes in mathematics, physics, philosophy, psychology, rebus, literature. In mathematics his research is in number theory, non-Euclidean geometry, synthetic geometry, algebraic structures, statistics, neutrosophic logic and set (generalizations of fuzzy logic and set respectively), neutrosophic probability (generalization of classical and imprecise probability). Also, small contributions to nuclear and particle physics, information fusion, neutrosophy (a generalization of dialectics), law of sensations and stimuli, etc. He got the 2010 Telesio-Galilei Academy of Science Gold Medal, Adjunct Professor (equivalent to Doctor Honoris Causa) of Beijing Jiaotong University in 2011, and 2011 Romanian Academy Award for Technical Science (the highest in the country). Dr. W. B. Vasantha Kandasamy and Dr. Florentin Smarandache got the 2012 New Mexico-Arizona and 2011 New Mexico Book Award for Algebraic Structures. He can be contacted at smarand@unm.edu

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