The Theory of the (E) Question

Discussing about the new formalism of the EM field tensor: Does it contain a bi-vector “à la E. Cartan”? – v12

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Annex 03: double constraint on the cubes and bi-vector

The new formalism of the EM field tensor in continuous energetic contexts

Introduction

The previous exploration [01] has proposed a plausible and totally new path unifying the theory of relativity [the A. Einstein’s GTR] and the quantum theory [the De Broglie, Schrödinger, Heisenberg… QT]. It gets it via the introduction of a mathematical tool that we have called the extended (eventually exterior) product. That plausible path needs the introduction of a simplified version of the Lorentz Einstein law (LEL), lies on the Heisenberg’s uncertainty principle (HUP) for the (energy, time) pair (its simplest and quasi-naïve formulation) and on the so-called extrinsic method. It yields a specific and new expression for the (2, 0) representation of the EM field. In case of a continuous function U(t) (see [01] for explanations), that is:

\[ q \cdot [F_{\mu\nu}] = \frac{1}{2} \{ (4)G, \Gamma_{\Phi}^{\mu\nu}(4)p \cdot p, \Gamma_{\Phi}^{\mu}(4)p, (4)c \} \]

Where \( q \) is the electrical charge of a particle with a 4D kinetic momentum \( (4)p \) in some geometric environment characterized by the 4D metric \( (4)G \) and its variations; the latter are written in a condensed formalism which we have called the Levi-Civita or the Christoffel’s cube \( \nabla \Gamma \) containing the Christoffel’s symbols of the second kind. The aim of the following section is to understand the meaning of that new formalism and to discover what it learns us.

Remark 01: General considerations

The first and spontaneous remark arising from the observation of that formalism [01] is to state the existence of EM fields (the left hand term) which are directly and basically related to terms depending on dynamical properties of a mass immerged in some geometric context (a priori any one). The dependence acts in such a way that an invariant metric or a massless particle generates no EM field. That means that a massless and spin 2 graviton (the actual and presently accepted carrier of the gravitational interaction) doesn’t generate any EM field in a context where \( (4)G \) holds true. Conversely, the appearance of that kind of EM field is seemingly related to a dissymmetry which we may describe in saying that the product \( (4)G, \Gamma_{\Phi}^{\mu}(4)p \) is not comparable with its transposed; for complementary calculation please see annex 01.

Concretely, the formalism [01] suggests the existence of real circumstances for which the non-vanishing pair \( (\nabla \Gamma, (4)p) \), which is in fact equivalent to: the geometry is changing and the mass moves, generates an EM field. What is absolutely not precisely said here is: at what scale the phenomenon appears and in which way the change of geometry and the one of position of the particle at hand are correlated. So that that formalism remains a little bit mysterious if we stop our analysis here.

Remark 02: “Where is the EM-potential vector?”

The second remark is the total absence of the EM-potential vector, \( (4)A \), in that expression [01]. From a classical viewpoint [02]: p. 10, (46) this is effectively totally surprising since that potential vector is a necessary and sufficient mathematical tool to calculate the efficient EM field. This suggests, as it was and as it is very often the case in the literature, that the EM potential vector only is a tool without real existence. This doesn’t mean that we cannot reintroduce it in our discussion. And this can in fact be made in different manners depending on the real circumstances. Just for the pedagogy, we could cite the retarded potentials [03, p. 193, (63,5)] and the Stükelberg-like trick consisting in \( p \rightarrow p - (q/c) \). \( A = \pi \): the quantum kinetic momentum extending to a 4D world the approach exposed briefly in [02, p. 5, (12)].

After such a translation we get:

\[ [F_{\mu\nu}]_{\Lambda \rightarrow \Phi} = [F_{\mu\nu}]_{\Lambda} - \frac{q}{c} \{ G \cdot \Gamma_{\Phi}^{\mu}(4)A \cdot (4)p, (4)c \} \]

By definition, a potential is a static quantity: it must be attached to a set of topological points. This is why it is totally unrealisitc to imagine that some EM-potential vector could eventually be parallel transported by respect for the local coordinate system.

Nevertheless, if such absurd though could be realized, we had already the occasion to prove in [04] that the second part of the EM field [02] would vanish.

Since any motion exhibits an intrinsic duality, namely: its relativity, this is clearly suggesting that a similar effect should occur for a vector representing the local coordinate system that would be parallel transported by respect for the static EM-potential vector. This sentence exactly means nothing else but that the coordinate system is changing from a topological...
point to the next one in such a way (a) that we may believe that it is represented by a vector at the first point, say $p_1$, with $\mathbf{x} = \mathbf{x}(p_1)$, and by a second vector at the next point, say $p_2$, with $\mathbf{x} = \mathbf{x}(p_2)$, and (b) that $\mathbf{x}$ and $\mathbf{x}$ can be as customized related to each other via a connection (of course); the only difference with the viewpoint adopted in (04) is coming from the fact that the “fixed background” is now the EM-potential vector field and no more a presupposed universal coordinate system.

In the nature, where could the EM-potential vector field come from? The question receives a two steps answer: first have a look at the theoretical considerations developed in [04: Landau and Lifschitz; Field theory, German edition, p. 208, (67.4)] relating a dipole and an EM-potential vector field; second consider the recent measurements of the cosmic background polarization [06: Planck satellite collaboration].

All this is also sketching the necessity to differentiate the behavior of the kinetic momentum from the one of the EM-potential vector.

Remark 03: a constraint coming from the generalized theory of relativity (Einstein)

The relation (01) should absolutely always be put into relation with a fundamental other one [07; p. 60, (5.26)]:

$$g_{\alpha\beta,\mu} = \Gamma_{\alpha\mu}^{\nu} g_{\nu\beta} + \Gamma_{\beta\mu}^{\nu} g_{\alpha\nu}$$

From which, even if the metric is not symmetric, we may easily deduce because we are supposed to be working on R or on C which is a commutative leaf:

$$\forall \rho^\nu: g_{\alpha\rho,\mu} p^\rho = (\Gamma_{\alpha\mu}^{\nu} \cdot p^\nu) \cdot g_{\nu\beta} + g_{\alpha\nu} \cdot (\Gamma_{\beta\mu}^{\nu} \cdot p^\nu)$$

Because the Levi-Civita connection is a symmetric cube, we can also write without loss of generality:

$$\forall \rho^\nu: g_{\alpha\rho,\mu} p^\rho = (\Gamma_{\alpha\mu}^{\nu} \cdot p^\nu) \cdot g_{\nu\beta}$$

Now let us introduce colors to facilitate the progression and the understanding:

$$\forall \rho^\nu: g_{\alpha\rho,\mu} p^\rho = (\Gamma_{\alpha\mu}^{\nu} \cdot p^\nu) \cdot g_{\nu\beta}$$

Let us also recall the following evidence in our theory:

$$\forall \rho^\nu: \gamma_{\alpha\beta}^{(i)} |p|^i = \Gamma_{\alpha\mu}^{\nu} \cdot p^\nu$$

It is now extremely easy to recognize that the second term of the sum in (06) is nothing but $G \cdot \gamma_{\alpha\beta}^{(i)} |p|^i$. And the first term of (06) is $\varphi_{\alpha}^{(i)} |p|^i$. G. Putting all together:

$$[... g_{\alpha\beta,\mu} \cdot p^\rho ...] = \varphi_{\alpha}^{(i)} |p|^i \cdot G + G \cdot \gamma_{\alpha\beta}^{(i)} |p|^i$$

Let us state that the l.h.t. of (08) is nothing but the ordinary derivative of the metric tensor by respect for some scalar:

$$[... g_{\alpha\beta,\mu} \cdot p^\rho ...] = \left[\frac{d \gamma_{\alpha\beta}^{(i)}}{dx^\nu} ... \right] = \left[\frac{d^2 \gamma_{\alpha\beta}^{(i)}}{dx^\nu} ... \right] = \frac{d}{dx} G = \dot{G}$$

We can now confront (01) and (08) in the special cases of symmetric metrics:

$$G = \dot{G}$$

$$2q. [\gamma_{\alpha\beta}] \cdot \dot{G} = 2. G \cdot \gamma_{\alpha\beta}^{(i)} |p|^i$$

$$2q. [\gamma_{\alpha\beta}] \cdot \dot{G} = -2. G \cdot \gamma_{\alpha\beta}^{(i)} |p|^i$$

The interesting point here is the fact that in absence of EM field:

$$\dot{G} = 2. G \cdot \gamma_{\alpha\beta}^{(i)} |p|^i$$

Although very roughly, this is evocating the Ricci flow; for an introduction to that very complicated topic, please visit: [09].
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Remark 04: formal considerations

Formal and esthetic considerations were in fact the first elements suggesting the existence of a G. \( \Phi(\text{H}) \)-like formalism for the (2, 0) EM field tensor representation. Indeed, observing the customized formalism of the (2, 0) tensorial representation of some EM field \( \text{[03; p. 73]} \), there is no difficulty in stating that the South-East corner of it is nothing but the trivial representation \( \Phi(\text{H}) \) of some unspecified cross product:

\[
[F_{\text{[^3]}}(\text{p})] = \begin{pmatrix}
0 & E_1 & E_2 & E_3 \\
-E_1 & 0 & -H_3 & H_2 \\
-E_2 & H_3 & 0 & -H_1 \\
-E_3 & H_2 & H_1 & 0
\end{pmatrix} \begin{pmatrix}
0 & < \text{E} > \\
-|\text{E}| & 0
\end{pmatrix}
\]

The reader may discover one of my previous and scholar explorations in \( \text{[05]} \). Even if a little bit naïve, it sketches a pedagogic approach concerning that item.

Remark 05: EM fields and spinors

The relation \( \text{[01]} \) may also evoke a very precise situation for a mathematician; namely: the E. Cartan’s theory of spinors \( \text{[08; chapter IX, p. 145, (1)]} \). The EM field predicted by the theory of the (E) question in a continuous context would plausibly be an infinitesimal transformation of the metric \( \text{G} \) or of the trivial split \( \Phi(\text{H}) \) if we would be able to prove that at least one of them is the representation of a bi-vector “à la E. Cartan”.

Sub-remark 01: the trivial matrix cannot be an anti-symmetric matrix

Since the Levi-Civita cube is a symmetric cube, the following condition \( \Gamma_{\text{i} \text{j} \text{k}}^{\text{i} \text{j} \text{k}} = -\Gamma_{\text{i} \text{k} \text{j}}^{\text{i} \text{k} \text{j}} = -\Gamma_{\text{j} \text{i} \text{k}}^{\text{j} \text{i} \text{k}} = -\Gamma_{\text{k} \text{i} \text{j}}^{\text{k} \text{i} \text{j}} \) would impose a vanishing Levi-Civita cube for the coherence; which is totally meaningless in our case. This sub-remark tells us that the trivial matrix \( \Phi(\text{H}) \) cannot be an anti-symmetric one; but it doesn’t yet definitively disregard that matrix for the purpose we are scrutinizing (“is sometimes the trivial matrix \( \Phi(\text{H}) \) representing a bi-vector à la E. Cartan?”). We shall come back later on that item.

Sub-remark 02: a necessary condition of symmetry

In order to eventually be the representation of a bi-vector “à la E. Cartan”, the trivial matrix involved in \( \text{[01]} \) must be a symmetric matrix. Indeed, the formalism of \( \text{[08; chapter IX, p. 145, (1)]} \) is recovered when and if the metric and the trivial matrix involved in \( \text{[01]} \) are symmetric matrices. Because \( \Gamma_{\text{i} \text{j} \text{k}}^{\text{i} \text{j} \text{k}} = \Gamma_{\text{i} \text{k} \text{j}}^{\text{i} \text{k} \text{j}} = \Gamma_{\text{j} \text{i} \text{k}}^{\text{j} \text{i} \text{k}} = \Gamma_{\text{k} \text{i} \text{j}}^{\text{k} \text{i} \text{j}} \) is a coherent set of equalities, the eventual symmetry of the trivial matrix is totally compatible with the one of the Christoffel’s symbols of the second kind. That eventuality just reduces the total number of independent Christoffel’s symbols. The role of physics is to verify if this mathematical opportunity can be effectively realized.

Sub-remark 03: metrics and bi-vectors

The purpose of that section

Let us first explore if the metric tensor itself is representing a bi-vector “à la E. Cartan” \( \text{[08]} \). In starting this purely algebraic investigation, we ignore if that eventuality is true in general (for any metric) but we know that empty space-times are associated with Minkowskian geometry of signature (+ - - -) represented by the diverse matrices:

\[
\eta = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
|\text{0}| & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Here \( [\text{K}] \) is the matrix introduced in \( \text{[08; § 126, pp. 110-111]} \). And we also know that “allowed” metrics within the generalized theory of relativity (GTR) of A. Einstein are obtained from the latter by deformations via the action of Jacobian matrices. So that the first quest is to look for matrices representing the Minkowskian metric in the spirit explained in \( \text{[08; p. 83, § 95]} \); that is: as if that metric would be the representation of a 2-vector.

The educated reader may argue that we actually live in 2014 and that a lot of progresses have been made during the past century between the date of publication of \( \text{[08]} \) and today. That’s true. For example, just following the logical set of accumulated knowledge: 1°) a metric tensor with signature (+ - - -), thus the one revealing Minkowskian geometry, can always be constructed with the help of a Hermitian spinor \( \text{[12; pp. 71-72]} \); 2°) the Cartan-Whittacker relation establishes a one to one correspondence between a spinor and a null bi-vector \( \text{[18; p. 560]} \), it can be induced that some Hermitian spinor are in a one to one correspondence with a null bi-vector. But this argument only means that the Hermitian spinor can help us to build a metric tensor in a way given in \( \text{[12; pp. 71-72]} \). That way \( \text{[12; p. 72, (2.3.2)]} \) doesn’t match (so far our attentive observation can state it) with the formalism proposed for bi-vectors in \( \text{[08]} \). This is why we recommend the lecture of the section, even for well-educated people.
The algebraic investigation

If \( \eta \) would be such a bi-vector “à la E. Cartan”, then it would be defined by two vectors, themselves respectively represented by the matrices \( X_i \) and \( X_j \), and it would be itself represented by a matrix:

\[ \eta = \frac{1}{2} \begin{bmatrix} X_i \cdot X_j - X_j \cdot X_i \end{bmatrix} \]

If we consider that our space-time is a 4D subspace of a 5D space, then following [08; § 93, p. 81, (5)]– we should write:

\[ X_i \cdot X_j = \begin{bmatrix} x_i^0 \cdot x_i^0 + x_i^1 \cdot x_i^1 + x_i^2 \cdot x_i^2 + x_i^3 \cdot x_i^3 & x_i^0 \cdot x_i^1 - x_i^1 \cdot x_i^0 & x_i^0 \cdot x_i^2 - x_i^2 \cdot x_i^0 & x_i^0 \cdot x_i^3 - x_i^3 \cdot x_i^0 \\ x_i^1 \cdot x_i^0 - x_i^0 \cdot x_i^1 & x_i^1 \cdot x_i^1 + x_i^2 \cdot x_i^2 + x_i^3 \cdot x_i^3 & x_i^1 \cdot x_i^2 - x_i^2 \cdot x_i^1 - (x_i^2 \cdot x_i^2 - x_i^3 \cdot x_i^3) \\ x_i^2 \cdot x_i^1 - x_i^1 \cdot x_i^2 & x_i^2 \cdot x_i^2 + x_i^3 \cdot x_i^3 - x_i^1 \cdot x_i^3 & x_i^3 \cdot x_i^3 - x_i^3 \cdot x_i^1 \\ (x_i^1 \cdot x_i^1 - x_i^2 \cdot x_i^2) & x_i^2 \cdot x_i^1 \cdot x_i^2 + x_i^3 \cdot x_i^3 & x_i^3 \cdot x_i^3 - x_i^3 \cdot x_i^1 \end{bmatrix} \]

If we don’t want to involve a 5D space, it is sufficient to refer to [08; § 125, pp. 129-130]. The formalism is now a simplification of the previous one. It is obtained in writing \( x_i^0 = 0 \).

\[ (X_i, X_j) \text{ if } x_i^0 = 0 \text{ for } i = 1, 2 = \begin{bmatrix} x_i^1 \cdot x_i^1 + x_i^2 \cdot x_i^2 & 0 & 0 & x_i^1 \cdot x_i^2 - x_i^2 \cdot x_i^1 \\ 0 & x_i^2 \cdot x_i^2 - x_i^1 \cdot x_i^2 & 0 & x_i^2 \cdot x_i^2 + x_i^2 \cdot x_i^2 \\ x_i^2 \cdot x_i^1 - x_i^1 \cdot x_i^2 & 0 & 0 & x_i^2 \cdot x_i^1 + x_i^2 \cdot x_i^2 \end{bmatrix} \]

The product \( X_i \cdot X_j \) is obtained in inverting \( i \) and \( j \):

\[ (X_i, X_j) \text{ if } x_i^0 = 0 \text{ for } i = 1, 2 = \begin{bmatrix} x_i^1 \cdot x_i^1 + x_i^2 \cdot x_i^2 & 0 & 0 & x_i^1 \cdot x_i^2 - x_i^2 \cdot x_i^1 \\ 0 & x_i^2 \cdot x_i^2 - x_i^1 \cdot x_i^2 & 0 & x_i^2 \cdot x_i^2 + x_i^2 \cdot x_i^2 \\ x_i^2 \cdot x_i^1 - x_i^1 \cdot x_i^2 & 0 & 0 & x_i^2 \cdot x_i^1 + x_i^2 \cdot x_i^2 \end{bmatrix} \]

For the following matrix:

\[ \begin{bmatrix} x_i^1 \cdot x_i^1 + x_i^2 \cdot x_i^2 & 0 & 0 & x_i^1 \cdot x_i^2 - x_i^2 \cdot x_i^1 \\ 0 & x_i^2 \cdot x_i^2 - x_i^1 \cdot x_i^2 & 0 & x_i^2 \cdot x_i^2 + x_i^2 \cdot x_i^2 \\ x_i^2 \cdot x_i^1 - x_i^1 \cdot x_i^2 & 0 & 0 & x_i^2 \cdot x_i^1 + x_i^2 \cdot x_i^2 \end{bmatrix} \]
Let us examine the only potentially interesting situations corresponding to the relations (20 -0, 1, 2 and 3). Because of (20 -0, 1, 2 and 3), all elements of the diagonal are again equal. That (22) formalism appears to be "a priori" compatible with the one of η (13) if the following ad hoc conditions hold true:

\[
\begin{bmatrix}
x_{1'} \cdot x_{1'} - x_{1'} \cdot x_{1'} - (x_{1'}^0 \cdot x_{1'} - x_{1'}^0 \cdot x_{1'}) \\
0 \\
0 \\
0 \\
x_{1'} \cdot x_{1'} - x_{1'} \cdot x_{1'} - (x_{1'}^0 \cdot x_{1'} - x_{1'}^0 \cdot x_{1'}) \\
x_{1'} \cdot x_{1'} + x_{1'} \cdot x_{1'} - (x_{1'}^0 \cdot x_{1'} + x_{1'}^0 \cdot x_{1'}) \\
0 \\
x_{1'} \cdot x_{1'} + x_{1'} \cdot x_{1'} - (x_{1'}^0 \cdot x_{1'} + x_{1'}^0 \cdot x_{1'}) \\
0 \\
0 \\
x_{1'} \cdot x_{1'} + x_{1'} \cdot x_{1'} - (x_{1'}^0 \cdot x_{1'} + x_{1'}^0 \cdot x_{1'}) \\
x_{1'} \cdot x_{1'} + x_{1'} \cdot x_{1'} - (x_{1'}^0 \cdot x_{1'} + x_{1'}^0 \cdot x_{1'}) \\
0 \\
0 \\
0 \\
x_{1'} \cdot x_{1'} + x_{1'} \cdot x_{1'} - (x_{1'}^0 \cdot x_{1'} + x_{1'}^0 \cdot x_{1'})
\end{bmatrix}
\]

We should now consider two situations:

1°) either we work on R (or C); it is then easy to check that we don’t systematically obtain a diagonal matrix. But this formalism appears to be "a priori" compatible with the one of η (13) if the following ad hoc conditions hold true:

\[
\begin{bmatrix}
x_{1'} \cdot x_{1'} - x_{1'} \cdot x_{1'} - (x_{1'}^0 \cdot x_{1'} - x_{1'}^0 \cdot x_{1'}) \\
0 \\
0 \\
0 \\
x_{1'} \cdot x_{1'} - x_{1'} \cdot x_{1'} - (x_{1'}^0 \cdot x_{1'} - x_{1'}^0 \cdot x_{1'}) \\
x_{1'} \cdot x_{1'} + x_{1'} \cdot x_{1'} - (x_{1'}^0 \cdot x_{1'} + x_{1'}^0 \cdot x_{1'}) \\
0 \\
x_{1'} \cdot x_{1'} + x_{1'} \cdot x_{1'} - (x_{1'}^0 \cdot x_{1'} + x_{1'}^0 \cdot x_{1'}) \\
0 \\
0 \\
x_{1'} \cdot x_{1'} + x_{1'} \cdot x_{1'} - (x_{1'}^0 \cdot x_{1'} + x_{1'}^0 \cdot x_{1'}) \\
x_{1'} \cdot x_{1'} + x_{1'} \cdot x_{1'} - (x_{1'}^0 \cdot x_{1'} + x_{1'}^0 \cdot x_{1'}) \\
0 \\
0 \\
0 \\
x_{1'} \cdot x_{1'} + x_{1'} \cdot x_{1'} - (x_{1'}^0 \cdot x_{1'} + x_{1'}^0 \cdot x_{1'})
\end{bmatrix}
\]

2°) or, for the pedagogy, we now work with a set of anti-commutative elements \( x_i^k \) and \( x_i^l \); this means that:

\[
a \cdot b + b \cdot a = 0
\]

In which case the off-diagonal elements vanish and the matrix (20) is now:

\[
\begin{bmatrix}
x_{1'} \cdot x_{1'} + x_{1'} \cdot x_{1'} - (x_{1'}^0 \cdot x_{1'} + x_{1'}^0 \cdot x_{1'}) \\
0 \\
0 \\
0 \\
x_{1'} \cdot x_{1'} + x_{1'} \cdot x_{1'} - (x_{1'}^0 \cdot x_{1'} + x_{1'}^0 \cdot x_{1'}) \\
x_{1'} \cdot x_{1'} + x_{1'} \cdot x_{1'} - (x_{1'}^0 \cdot x_{1'} + x_{1'}^0 \cdot x_{1'}) \\
0 \\
x_{1'} \cdot x_{1'} + x_{1'} \cdot x_{1'} - (x_{1'}^0 \cdot x_{1'} + x_{1'}^0 \cdot x_{1'}) \\
0 \\
0 \\
x_{1'} \cdot x_{1'} + x_{1'} \cdot x_{1'} - (x_{1'}^0 \cdot x_{1'} + x_{1'}^0 \cdot x_{1'}) \\
x_{1'} \cdot x_{1'} + x_{1'} \cdot x_{1'} - (x_{1'}^0 \cdot x_{1'} + x_{1'}^0 \cdot x_{1'}) \\
0 \\
0 \\
0 \\
x_{1'} \cdot x_{1'} + x_{1'} \cdot x_{1'} - (x_{1'}^0 \cdot x_{1'} + x_{1'}^0 \cdot x_{1'})
\end{bmatrix}
\]

Despite the appearance, and because of (21), all elements of the diagonal are again equal. That (22) formalism is once more time not compatible with the one of η (13).

Concluding sub-remark

Let us examine the only potentially interesting situations corresponding to the relations (20-0, 1, 2 and 3). Because of (20-0 and 1) we have:

\[
\begin{bmatrix}
x_{1'} \cdot x_{1'} - x_{1'} \cdot x_{1'} = 0 \\
x_{1'} \cdot x_{1'} - x_{1'} \cdot x_{1'} = 0
\end{bmatrix}
\]
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\[ x_i^1 \cdot x_j^2 - x_i^2 \cdot x_j^1 = 0 \]

\[ x_i^1 \cdot x_j^2 - x_i^2 \cdot x_j^1 = 0 \]

Provided none of the element vanishes, these relations imply:

(24)

\[ \frac{x_j^2}{x_j^1} = \frac{x_i^1}{x_i^2} \]

\[ = \frac{x_j^1}{x_j^2} = \frac{x_i^2}{x_i^1} = \text{constant} \]

Observing attentively that result (24) allows us to say that it is equivalent to the fact that the matrix \( X_j \) is proportional to the matrix \( X_i \). In which case, the matrix (20) must vanish, reducing our attempt to nothing.

In that paragraph we have examined the hypothesis consisting to believe that the Minkowskian metric can be represented by a matrix which is itself the representation of a bi-vector “à la E. Cartan" [08]. For this we have considered all permutations on two indices. There is only one possible permutation; namely (1, 2) \( \rightarrow \) (2, 1). This is explaining the examined formalism (14) which we believe is respecting [08; § 95, p. 83].

Theorem 01
Our conclusion is actually a negative one: “There is no matrix representing a 2-vector “à la E. Cartan” which is at the same time a representation of the Minkowskian metric \( \eta \) (13)".

Complementary sub-remarks 04
If, like us and because of the preliminary argumentation developed previously in starting that section, the reader has a doubt concerning the interpretation of [08; § 95, p. 83], he may examine like us the other alternative (see annex 02) and come to the same negative conclusion.

We nevertheless emit some doubts concerning the previous theorem because of an E. Cartan’s remark that has been made in the book [08; § 125, 110, first and second eventualities]. With other words, matrices associated with bi-vectors in a 4D space \( E_4 \) have a special structure.

We also want to add some complementary sub-remarks which we believe will help to understand the historic progression surrounding the concept of spinor:

1°) the annex 02 may be seen as a soft prelude to a more general relation that students meet in studying the so-called Dirac’s matrices;

2°) the E. Cartan’s work [08; p. 44 and p. 112] and more recent works, e. g. [12; chapter 2, §§ 2.1, 2.2 and 2.3, pp. 62-76], give relatively precise indications on the relationship between the concept of spinor and the one of rotation.

3°) even the relation [08; chapter IX, p. 145, (1)] which was the starter for that section introduces some confusion in our spirit. E. Cartan himself is speaking of a bi-vector (at the end of page 145) when referring to a simple product of two matrices \( A_1 \) and \( A_2 \). This would mean that either (17) or (18) may be representations of bi-vectors; and this would perhaps facilitate our quest in a comparison with (13). This is unfortunately not the case because an attentive observation of the relations (17) and (18) shows that a direct confrontation with (13) yields (24) again. This is implying that only the square of a given matrix can perhaps exhibit the formalism which we are looking for. Unfortunately, the formalism of \( \eta \) is forbidding that eventuality. The best situation that we can hope to obtain with (17) and (18) is a fruitful comparison with the formalism [08; § 125, 110, second eventualuty]; and the latter is not directly comparable with the one of \( \eta \).

At the end of the day, despite all our efforts in trying to understand E. Cartan’s work, we are obliged to confirm the previous theorem.

Corollary 01
The EM fields of our theory in continuous energetic contexts, (01), are not equivalent to an infinitesimal transformation of the trivial matrix \( \gamma_1 \Phi^{(\mu)}(p) \) resulting from the action of the Minkowskian metric in the sense given by [08; chapter IX, § 172, (1)]. If we want to push the research into that direction, we must either (a) probably extend the domain of definition exposed in (01) and yielding (01); a first path is to reconsider the formulation of the Lorentz-Einstein law we have involved into our approach. Or (b) look for situations where the trivial matrix \( \gamma_1 \Phi^{(\mu)}(p) \) is a bi-vector without being an anti-symmetric matrix. Let us remember that the relation [08; chapter IX, § 172, (1)] is concerning simple rotations in a Euclidian geometry. This context may appear to be in contradiction with our wish to analyze a mathematical element involving the Levi-Civita cube which is, of course, the signature for a curved space. *This fact suggests that, even if we success in our quest (= we can find situations for which \( \gamma_1 \Phi^{(\mu)}(p) \) is a bi-vector “à la E. Cartan”), we then shall have to carefully reduce the validity of our conclusion to spaces where the curvature is closed to zero.
Sub-remark 05: exploring the properties of the trivial matrix.

This is the right moment to exploit the calculations of previous paragraphs. Confronting the relation (27) which is explicating the precise formalism of the elements of the trivial matrix $\gamma^p \left( \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \right)$ with the generic formalism of a matrix representing a bi-vector “à la E. Cartan”, namely (20), we are left with:

\[
\begin{bmatrix}
 x^1_1 x_2^1 + x^1_2 x_2^2 - (x^1_1 x_2^2 + x^1_2 x_2^1) & 0 & 0 & 2 (x^1_1 x_2^2 - x^1_2 x_2^1) \\
 0 & x^1_1 x_2^2 + x^1_2 x_2^1 - (x^1_1 x_2^1 + x^1_2 x_2^2) & 2 (x^1_1 x_2^2 - x^1_2 x_2^1) & 0 \\
 0 & 2 (x^1_1 x_2^2 - x^1_2 x_2^1) & x^1_1 x_2^2 + x^1_2 x_2^1 - (x^1_1 x_2^1 + x^1_2 x_2^2) & 0 \\
 2 (x^1_1 x_2^1 - x^1_2 x_2^2) & 0 & 0 & x^1_1 x_2^2 + x^1_2 x_2^1 - (x^1_1 x_2^1 + x^1_2 x_2^2)
\end{bmatrix}
\]

Let us observe the formalism attentively and state that the confrontation imposes:

\[
\Gamma^\nu_{\mu} \cdot p^\nu = \Gamma^\nu_{\mu -} \cdot p^\nu = \Gamma^\nu_{\mu +} \cdot p^\nu = \Gamma^\nu_{\mu 1} \cdot p^\nu = \Gamma^\nu_{\mu 2} \cdot p^\nu = \Gamma^\nu_{\mu 3} \cdot p^\nu = \Gamma^\nu_{\mu 4} \cdot p^\nu = \Gamma^\nu_{\mu 5} \cdot p^\nu = \Gamma^\nu_{\mu 6} \cdot p^\nu = 0
\]

\[
\begin{align*}
\Gamma^\nu_{\mu -} & = a - b = -\Gamma^\nu_{\mu +} \\
\Gamma^\nu_{\mu 1} & = c - d = -\Gamma^\nu_{\mu 2} \\
\Gamma^\nu_{\mu 3} & = 2 (x^1_1 x_2^2 - x^1_2 x_2^1) = \Gamma^\nu_{\mu 4} \cdot p^\nu \\
\Gamma^\nu_{\mu 5} & = 2 (x^1_1 x_2^2 - x^1_2 x_2^1) = \Gamma^\nu_{\mu 6} \cdot p^\nu
\end{align*}
\]

This set of conditions is associated with technical difficulties. It may be decomposed into two subsets of each eight conditions.

The first sub-set coincides with (26-1 to 8) and imposes drastic consequences. Indeed, provided we don’t want to restrict a priori the generality on the Levi-Civita cube involved in the discussion (except the facts that that cube is symmetric per definition and that we should stay in the vicinity of a flat space), we are forced to accept that the four $p^\mu$, for $\mu = 0, 1, 2, 3$, are not independent. This doesn’t really surprise us since we know that the generalized theory of relativity (GTR) imposes $p_\mu \cdot p^\mu = 0$ for massless particles (e.g.: photons, gravitons...) or the dispersion relation for non-vanishing masses.

At this stage we yet have $\{4 \times 10 = \}$ 40 different Christoffel’s symbols of the second kind and probably only three independent components for the kinetic momentum.

At a first glance, the second subset (26-13 to 20) tells us that we have at the end only six independent linear combinations of the $\Gamma^\nu_{\mu\nu} \cdot p^\nu$ type. And they all potentially depend on the 2 x 4 components of the two vectors that we are looking for. But let us examine (26-17 to 20) more attentively:
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Proposition 02: the trivial matrix is a bi-vector “à la E. Cartan”

Let us examine the first eventuality [08 § 125, p. 110, first eventuality]: 0 = \( (x^i, x^j - x^1, x^j) = (x^i, x^j - x^1, x^j) \) and \( (x^i, x^j - x^1, x^j) = (x^i, x^j - x^1, x^j) = 0 \) if a, b, c, d are correctly co-related;

B. or [08 § 125, p. 110, second eventuality]: 0 = a = b and 0 = c = d if the other terms have the ad hoc formalism.

Characterizations – eventuality A

Let us examine the first eventuality [08 § 125, p. 110, first eventuality]. We propose in fact:
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\[
\begin{bmatrix}
0 & 0 & (x_1^2, x_1^2 - x_1^2, x_1^3) \\
0 & (x_1^2, x_1^2 - x_1^2, x_1^3) & 0 \\
(x_1^2, x_1^2 - x_1^2, x_1^3) & 0 & c \\
\end{bmatrix}
\]

It is presumably yielding:

\[
\begin{align*}
\frac{x_1^2}{x_1^3} = \frac{x_1^3}{x_1^1} &= \text{constant } n \text{ and } \frac{x_1^2}{x_1^3} = \frac{x_1^3}{x_1^1} = \text{constant } n^2
\end{align*}
\]

This is of course imposing the equality of the two constants and, consequently, the relation (24) again. The two vectors are proportional and injecting this fact in (17) leaves us with a result of the E. Cartan’s theory which is known under the name of “fundamental theorem” [08; § 94, pp. 82-83]. Namely: if the unique remaining vector, (4), has now the components \((x^1, x^1, x^2, x^2, x^3)\), the Cartan’s matrix representation of its square only is the \(\left(\begin{smallmatrix} 0 & \text{id} \\ \text{id} & 0 \end{smallmatrix}\right)\). Id4 matrix where the \(\ldots_{a \rightarrow d} \) denotes the Euclidian scalar product in \(E_4\). Confronting this with our proposition (33) induces:

\[
\Gamma_{\mu \nu}^0. p^0 = \Gamma_{\mu \nu}^1. p^1 = \Gamma_{\mu \nu}^2. p^2 = \Gamma_{\mu \nu}^3. p^3 = \text{id} \Gamma_{\mu \nu}^4. \text{id}
\]

Taking a classical result of the GTR into consideration [13; § 35, 101], this is nothing but:

\[
\frac{\partial}{\partial \frac{\sqrt{g}}{\sqrt{g'}}} p^0 = \text{id} \Gamma_{\mu \nu}^4. \text{id}
\]

Following the same approach than the one made with the relation (09), this is also:

\[
m \frac{\partial}{\partial \frac{\sqrt{g}}{\sqrt{g'}}} p^0 = \text{id} \Gamma_{\mu \nu}^4. \text{id}
\]

Although the mental gymnastics is interesting, we have no idea about the meaning of the vector \(\text{id} \Gamma_{\mu \nu}^4. \text{id}\). And, in some way, we may say that the unicity of that vector is changing the nature of the quest.

Characterizations – eventuality B

The identifications

So let us now examine the other possibility [08; § 125, 110, second eventuality]. We immediately conclude that it corresponds to the case where \(a = b = c = d = 0\), thus to a null diagonal and, consequently, to a special case of the previous paragraph; precisely, this case is characterized by:

\[
|g|^{1/2} = \text{invariant}
\]

The determinant of the metric tensor (but not the metric itself) is invariant; a condition to which we must add: (26-1 to 8)

\[
\Gamma_{\mu \nu}^0. p^0 = \Gamma_{\mu \nu}^2. p^2 = \Gamma_{\mu \nu}^3. p^3 = \text{id} \Gamma_{\mu \nu}^4. p^4 = \Gamma_{\mu \nu}^4. p^4 = \text{id} \Gamma_{\mu \nu}^4. p^4 = \Gamma_{\mu \nu}^4. p^4 = \Gamma_{\mu \nu}^4. p^4 = \Gamma_{\mu \nu}^4. p^4 = 0
\]

And we must also add:

\[
\begin{align*}
(x_1^2, x_1^2 - x_1^2, x_1^3) &= \Gamma_{\mu \nu}^2. p^2 \\
(x_1^2, x_1^2 - x_1^2, x_1^3) &= \Gamma_{\mu \nu}^3. p^3 \\
(x_1^2, x_1^2 - x_1^2, x_1^3) &= \Gamma_{\mu \nu}^4. p^4 \\
(x_1^2, x_1^2 - x_1^2, x_1^3) &= \Gamma_{\mu \nu}^4. p^4
\end{align*}
\]
Remark

The relations (26-1 to 8) are all compatible with the symmetry of $\Gamma^{(4)}_{\mu\nu\alpha\beta}(p)$ – see the sub remark (a necessary condition). Concerning the relations (38-1 to 4) we can make a remark: if we add that necessary condition, these four relations are immediately reduced to only two because that condition imposes:

$$\Gamma^{\mu\nu}_{\alpha\beta} = \Gamma^{\mu\nu}_{\beta\alpha}$$

Consequently we inherit of:

$$\forall \ (4p)$$

$$(x^1 \cdot x^7 - x^2 \cdot x^6) = \Gamma^{\alpha\beta}_{\gamma\delta} p^\gamma = N = \Gamma^{\alpha\beta}_{\gamma\delta} p^\gamma = (x^1 \cdot x^7 - x^2 \cdot x^6)$$

$$(x^2 \cdot x^6 - x^1 \cdot x^7) = \Gamma^{\alpha\beta}_{\gamma\delta} p^\gamma = \Upsilon = \Gamma^{\alpha\beta}_{\gamma\delta} p^\gamma = (x^2 \cdot x^6 - x^1 \cdot x^7)$$

We stay with only two independent entities $\Upsilon$ and $N$.

**Conditions of coherence: “How many independent components for the bi-vector?”**

This being said, we have to verify if that new context doesn’t generate some incoherence between the diverse products $x^a \cdot x^b$ because of the relations (26-9 to 12). They imply here:

$$(40-1, 2, 3, 4)$$

$$a = x^1 \cdot x^7 + x^2 \cdot x^6 = 0$$

$$b = x^2 \cdot x^7 + x^1 \cdot x^6 = 0$$

$$c = x^1 \cdot x^7 + x^2 \cdot x^6 = 0$$

$$d = x^2 \cdot x^7 + x^1 \cdot x^6 = 0$$

Or more precisely:

$$\forall \ (4p)$$

$$x^2 \cdot x^7 - x^1 \cdot x^6 = 0$$

Initially, our purpose is to manipulate two any vectors $^{(4)}x_i$ and $^{(4)}x_j$ in minimizing the reduction of the generality of the Levi-Civita cube. If these two vectors are really any one, then the r.h.t. of the relations (38-1 to 4) are also any one. But we now suspect that the relations (40-1 to 4) introduce some restrictions; in extenso: we have no more 8 independent components. The next question is then: “How many independent components do we have?” Let us suppose that $^{(4)}x_i$ has the known and non-vanishing components $(x^1_i, x^2_i, x^1'_i, x^2'_i)$. The second vector is $^{(4)}x_j$ and has the unknown components $(x^1_j, x^2_j, x^1'_j, x^2'_j)$. As a first indication, because of (40-3) we get:

$$(^{(4)}x_j: (x^1_j, -x^2_j \cdot x^1_i \cdot x^1'_i, x^1'_j, x^2'_j))$$

And because of (40-1) we finally get:

$$(^{(4)}x_j: (x^1_j, -x^2_j \cdot x^1_i \cdot x^1'_i, x^1'_j, x^2'_j))$$

This is suggesting that the second and unknown vector which we are looking for has at most two independent components; namely: $x^1_j$ and $x^2'_j$. But let us observe (40-1 to 4) more attentively and state that our choice of (40-1) and (40-3) to reduce the number of independent components was arbitrary. Because of that, our conclusion is still incomplete. Of course, with (40-1) and (40-4) we have two different possibilities to rewrite $x^2_j$:

$$x^2_j = x^1_j \cdot \frac{x^1'_j}{x^1_i} = x^2_j \cdot \frac{x^1'_j}{x^1_i}$$

This is yielding:

$$x^1_j \cdot x^2_j - x^2_j \cdot x^1_j = 0$$
On the same vein, considering (40-2) and (40-3), we state that we have two different possibilities to write $x_j^2$:

$$x_j^2 = - x_j^1$$

Once again, this is yielding:

$$x_j^1, x_j^2 \cdot x_i^1 = 0$$

Confronting now (40-1) and (40-3) we may state the existence of two possibilities for $x_j^1$:

$$x_j^1 = - x_j^2, x_i^2$$

This is yielding:

$$\Gamma^0_{1, 2} = 0$$

These diverse relations are completely changing our understanding of the relations (38-1 to 4). Indeed, let us inject the results contained in (41) into the relations (38-1 to 4).

They now write:

$$x_j^1 \cdot x_j^2 + (x_j^1)^2, x_i^1 = \Gamma^0_{1, 2}, p^0$$

$$x_j^1 \cdot x_j^2 + x_j^1, x_i^1 = \Gamma^3_{1, 2, 3}, p^3 = x_j^1, (x_j^2 + x_j^1, x_i^1)$$

$$x_j^1, x_j^2 - x_j^1, x_j^2 = \Gamma^3_{1, 2, 3}, p^3 = - x_j^1, (x_j^2 + x_j^1)$$

But this is not all. A difficulty is to get a panoramic view on all possible products. Let us write:

$$T(\otimes)(x_0, x_0)$$

A special case is the transposed of that matrix:

$$T(\otimes)(x_0, x_0)$$

Let us try to simplify the matrix (45). Because of (42) and (43):

$$T(\otimes)(x_0, x_0)$$
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\[
\begin{bmatrix}
x_1, x_1^j \\
x_1^i, x_1^j \\
x_1^i, x_2^j \\
x_1^i, x_2^j \\
x_2^i, x_3^j \\
x_2^i, x_3^j \\
x_2^i, x_4^j \\
x_2^i, x_4^j \\
x_3^i, x_3^j \\
x_3^i, x_4^j \\
x_4^i, x_4^j
\end{bmatrix}
\]

And the four relations \(40-1 \text{ to } 4\) are in fact only one:

\[
\begin{align*}
a &= x_1^i, x_1^j + x_2^i, x_2^j = 0 \\
b &= x_1^i, x_1^j + x_2^i, x_2^j = 0 \\
c &= x_1^i, x_1^j + x_2^i, x_2^j = 0 \\
d &= x_1^i, x_1^j + x_2^i, x_2^j = 0
\end{align*}
\]

This is implying a new simplification:

\[
T(\otimes) \{x_i, x_j\} = \begin{bmatrix}
x_1, x_1^j \\
x_1^i, x_1^j \\
x_1^i, x_2^j \\
x_1^i, x_2^j \\
x_2^i, x_3^j \\
x_2^i, x_3^j \\
x_2^i, x_4^j \\
x_2^i, x_4^j \\
x_3^i, x_3^j \\
x_3^i, x_4^j \\
x_4^i, x_4^j
\end{bmatrix}
\]

Now let us observe \(44-1 \text{ to } 4\) again and especially \(44-2 \text{ and } 3\):

\[
\begin{align*}
x_1^i \cdot x_1^j + (x_1^j)^2 \cdot \frac{x_1^i}{x_1^j} &= \Gamma_{\mu \rho}^\varphi p^\rho \Rightarrow \xi_1^\varphi, \Gamma_{\mu \rho}^\varphi p^\rho = \xi_1^\varphi, x_1^i + (x_1^j)^2 \cdot x_1^j
\end{align*}
\]

\[
\begin{align*}
\Gamma_{\rho \sigma}^\varphi p^\rho &= x_1^i \cdot (x_1^j + x_1^j \cdot \frac{x_1^i}{x_1^j}) \Rightarrow \xi_1^\varphi, \Gamma_{\rho \sigma}^\varphi p^\rho = \xi_1^\varphi, (x_1^i, x_1^i + x_1^j, x_1^j)
\end{align*}
\]

\[
\begin{align*}
\Gamma_{\rho \sigma}^\varphi p^\rho &= - x_1^i \cdot (\frac{x_1^i}{x_1^j}, x_1^j + x_1^j) \Rightarrow \xi_1^\varphi, \Gamma_{\rho \sigma}^\varphi p^\rho = - x_1^i \cdot (x_1^i, x_1^i + x_1^j, x_1^j)
\end{align*}
\]

Let us now eliminate \(x_1^j\) in \(44-4\) with the help of \(41\):

\[
\begin{align*}
- (x_1^i, x_1^j, x_1^j + x_1^j, x_1^j) &= \Gamma_{\rho \sigma}^\varphi p^\rho \Rightarrow \xi_1^\varphi, \Gamma_{\rho \sigma}^\varphi p^\rho = - (x_1^i, x_1^i + (x_1^j)^2, x_1^j)
\end{align*}
\]

The situation is quite more complicated than initially expected. Indeed, because of Cartan’s theory and because of the use we want to make of it, we started with a priori 8 components (four for each of the two 4D-vectors we are looking for) connected via 4 relations \(38-1 \text{ to } 4\).

These relations could have been in fact understood as defining 6 planes (each pair of these 4 relations defines a plane and we can choose 6 different pairs). These planes are obviously not independent. Note that four planes in a 3D space are sufficient to build, e.g. a tetrahedral pyramid. The two supplementary planes of that theory would have told us something about the time. Note also that two bi-vectors are a sufficient tool to build a 3D tetrahedral pyramid.

Now, because of the consequences of \(40-1 \text{ to } 4\), in particular the relations \(42, 43, 48\) and \(50-1 \text{ to } 4\), we must state that supposing (a) that \(33\) hold true in case B. ([08], § 125, 110, second eventuality) our main hypothesis here) and (b) that \(46\) has the known and non-vanishing components \(x_1^i, x_1^j, x_1^j, x_1^j\) are sufficient prerequisites to discover the missing half of a second vector \(x_1^i\) of which two components would be known \(x_1^i, x_1^i, x_1^j, ...\). There are only 6 independent components in our two unknown vectors. Furthermore, these 6 components impose strict conditions to the remaining non-vanishing terms of the trivial matrix \(\otimes (\otimes\rho)\).

In writing \(33\) for the configuration B. ([08], § 125, 110, second eventuality), we had the illusion to dispose of four independent eventually non-vanishing terms in that trivial matrix. The relations \(50-1 \text{ to } 4\) are clearly indicating that at most two of them are independent and eventually non-vanishing terms.
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\[ x_1^2, \Gamma^2_{\mu \nu} \rho^\mu = x_1^2 \, x_1^\nu, x_1^\alpha (x_1^\alpha)_2 \, x_1^\nu = - x_1^2, \Gamma^2_{\mu \nu} \rho^\mu \]
\[ x_1^2, \Gamma^3_{\mu \nu} \rho^\mu = x_1^2 \, x_1^\nu, (x_1^\nu + x_1^\nu, x_1^\nu) = - x_1^2, \Gamma^3_{\mu \nu} \rho^\mu \]

Confrontation with the necessary conditions of symmetry

Note that the relations (51) have been obtained without taking attention to the consequence of the necessary conditions of symmetry (39). Let us now inject them in the discussion. Whatever the components of the two 4D vectors:

\[ x_1^2, Y = - x_1^2, N = x_1^2, x_1^\nu, x_1^\alpha (x_1^\alpha)_2 \, x_1^\nu \]
\[ x_1^2, N = - x_1^2, Y = x_1^2, (x_1^\nu + x_1^\nu, x_1^\nu) \]

Let us multiply the first identity by \( x_1^2 \):

\[ x_2^2, Y = - x_2^2, N \]
\[ (- x_1^2, Y), (x_1^2) = x_2^2, N, (- x_2^2) = - (x_1^2), N \]
\[ \downarrow \]
\[ x_2^2 = \pm x_1^2 \]

Confronting this with (43) imposes:

\[ x_1^2 = \pm x_2^2 \]

In multiplying the second identity by \( x_1^2 \) we obtain exactly the same constraint. Our first 4D vector is consequently reduced to the following formalisms:

\[ (x_1^1, x_1^2, x_1^\alpha, \pm x_1^\alpha) \]

And because of (41) our second vector is reduced to the following formalisms:

\[ (x_2^1, x_2^2, x_2^\alpha, \pm x_2^\alpha) \]

Both 4D vectors seems to be very similar in the sense that (a) three scalars are enough to define them and (b) these scalars are positioned in a similar manner. Nevertheless, looking more attentively to the details is showing that the second vector is entirely defined by the first one and two new scalars only; namely: \( x_1^1 \) and \( x_1^\alpha \). At the end, injecting (51-3) into (51-1 or 2) yields:

\[ Y = \pm N \]

The Pythagorean table of (tensor) multiplication (49) can be simplified a step further:

\[ T(\otimes)(x_1, y_1) \]

\[
\begin{bmatrix}
  x_1^1, x_1^1 & x_1^2, x_1^1 & x_1^1, x_1^\alpha & \pm x_1^2, x_1^\alpha \\
  x_1^1, x_1^2 & x_1^2, x_1^2 & x_1^1, x_1^\nu & -x_1^2, x_1^\nu \\
  x_1^1, x_1^\alpha & x_1^2, x_1^\alpha & x_1^1, x_1^\gamma & \pm x_1^2, x_1^\gamma \\
  \pm x_1^2, x_1^1 & -x_1^2, x_1^2 & \pm x_1^2, x_1^\alpha & \pm x_1^2, x_1^\gamma
\end{bmatrix}
\]
Mathematical conclusion
The relation (51 -7) introduces one more restriction and we are left with only one eventually non-systematically vanishing linear combination of the Christoffel’s symbols. This is imposing a very specific formalism for the trivial matrix of the discussion:

\[
\Gamma \Phi (p) = \begin{bmatrix}
0 & 0 & 0 & N \\
0 & 0 & Y & 0 \\
0 & Y & 0 & 0 \\
N & 0 & 0 & 0 \\
\end{bmatrix}
\text{ and } Y = \pm N
\]

Concretely this will be one of the two possible configurations:

\[
\Gamma \Phi (p) = \begin{bmatrix}
0 & 0 & 0 & N \\
0 & 0 & 0 & 0 \\
0 & 0 & N & 0 \\
N & 0 & 0 & 0 \\
\end{bmatrix}
\text{ or } \Gamma \Phi (p) = \begin{bmatrix}
0 & 0 & 0 & N \\
0 & 0 & -N & 0 \\
0 & -N & 0 & 0 \\
N & 0 & 0 & 0 \\
\end{bmatrix}
\]

For convenience, we propose to write:

\[
\begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} \Xi \end{bmatrix}, \begin{bmatrix} H \end{bmatrix}
\]

With that notation, the allowed trivial matrices for the investigation at hand are:

\[
\Gamma \Phi (p) = \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} \Xi \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} \text{ or } \Gamma \Phi (p) = \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} -H \end{bmatrix} \begin{bmatrix} -H \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix}
\]

We are presently working in an E4 = E2 x 2 space; thus v = 2. Each of the matrices [\Xi] and [H] introduced here is of degree 2^{v-1} = 2^{2-1} = 2 as expected (see footnote [08; p. 110]. Both configurations seems luckily to be compatible with the one suggested by E. Cartan [08; p. 110] for matrices associated with 2-vectors (just another name for the bi-vectors).

Constraints coming from physics
Did we have inspected all sides of the technical difficulties? No. We must also not forget another preliminary remark: that exploration holds true in contexts with a very small curvature. This is the reason why we are encouraged to test that vision more concretely with the help of the linearized version of the A. Einstein’s theory; e.g. in [16; §4, p.7, (16)]. Following the notation of that document [16], we get:

(52-1 and 2)

\[
\Gamma_{\mu 0} p^0 = (\partial_0 \phi + \partial_0 \omega_3), p^0 + (\partial_3 \omega_0 + \partial_0 \omega_3), p^3 \cdot \Psi, p^3
\]

\[
\Gamma_{\mu 3} p^0 = (\partial_3 \phi), p^0 + (\partial_3 \omega_3), p^3 \cdot \Psi, p^3
\]

From which we deduce:

(53)

\[
\frac{1}{2} (\Gamma_{\mu 3} - \Gamma_{\mu 0}) p^0 = \partial_0 \omega_3, p^3
\]

Since the trivial matrix must be symmetric, this is imposing the first concrete constraint imposed by our hypothesis:

(54)

\[
\partial_0 \omega_3, p^3 = 0
\]

This is in fact the first limpid conclusion resulting from that complicated and exhaustive confrontation. Indeed, looking at the meaning of the \(\omega_3\) term in the reference document [16; §3, p. 4, (11)] we discover that it represents one of the term of the 4D metric, namely \(h_{03}\). An expression like (54) is suggesting that that term is spatially preserved. That term plays an interesting role in considerations centered on the Thirring-Lense effect because it is directly related to a gravito-magnetism mechanism. We also have:

(55-1)

\[
\Gamma_{\mu 3} p^0 = (\partial_3 \omega_0 + \partial_3 \omega_2 - \partial_0 \omega_3), p^0 + (\partial_3 \omega_3), p^0 + 2. \partial_0 \omega_3, p^3 - 2. \partial_0 \omega_3, p^3 - 2. \partial_0 \omega_3, p^3
\]

(55-2)

\[
\text{...}
\]
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Because of (51-1), we get:

\[ (\vec{\rho}_{[1]} \omega_{12}^{\mu} + \omega_{12} - \omega_{12} \cdot \Psi), \rho^2 + (\partial_{\omega} - 2 \cdot \partial_{\omega} \cdot\partial_{\omega} - \partial_{\omega} \cdot 2 \cdot \partial_{\omega} \cdot\partial_{\omega} + 2 \cdot \partial_{\omega} \cdot\partial_{\omega} - \partial_{\omega} \cdot\partial_{\omega}), \rho^2 ) = x^2 \cdot x^2 \cdot x^2 \cdot x^2 + (x^1) \cdot x^1 \cdot x^1 \cdot x^1 = - x^2 \cdot (\partial_{\omega} - 2 \cdot \partial_{\omega} \cdot\partial_{\omega} - \partial_{\omega} \cdot\partial_{\omega} \cdot \rho^2) \]

The understanding of these equations is far from evident. Although it doesn’t represent a complete treatment, we may eventually propose identifications not depending on the components of the kinetic momentum and propose:

\[ (\vec{\rho}_{[1]} \omega_{12}^{\mu} + \omega_{12} - \omega_{12} \cdot \Psi) = - \cdot x^2 = \partial_{\omega} \]

\[ \forall k = 1, 2, 3 : (\partial_{\omega} - 2 \cdot \partial_{\omega} \cdot\partial_{\omega} - \partial_{\omega} \cdot\partial_{\omega} + 2 \cdot \partial_{\omega} \cdot\partial_{\omega} - \partial_{\omega} \cdot\partial_{\omega} \cdot (\partial_{\omega} - 2 \cdot \partial_{\omega} \cdot\partial_{\omega} - \partial_{\omega} \cdot\partial_{\omega} \cdot \rho^2) \]

Because of (51-3) this is reduced to:

\[ (\vec{\rho}_{[1]} \omega_{12}^{\mu} + \omega_{12} - \omega_{12} \cdot \Psi) = \pm \cdot \partial_{\omega} \]

\[ \forall k = 1, 2, 3 : (\partial_{\omega} - 2 \cdot \partial_{\omega} \cdot\partial_{\omega} - \partial_{\omega} \cdot\partial_{\omega} + 2 \cdot \partial_{\omega} \cdot\partial_{\omega} - \partial_{\omega} \cdot\partial_{\omega} \cdot \partial_{\omega} \cdot\partial_{\omega} \cdot \rho^2) \]

As mathematician we have actually no idea about what to do with this. We just state that our mathematical hypothesis impose precise relations between data coming from physics in the context we wanted to explore.

Additional constraints

We have already seen, with the relations (27-1, 2) to (30), that:

\[ (\Gamma_{\mu}^0 \cdot \Gamma_{\nu}^0 \cdot \Gamma_{\mu}^0 \cdot \Gamma_{\nu}^0), \rho^2 \cdot \rho^2 = (a \cdot b - c \cdot d) \]

In the specific configuration here, this is evidently reduced to:

\[ (\Gamma_{\mu}^0 \cdot \Gamma_{\nu}^0 \cdot \Gamma_{\mu}^0 \cdot \Gamma_{\nu}^0), \rho^2 \cdot \rho^2 = 0 \]

Without doing any supplementary calculations we must remark that any product of two of the 4 scalars a, b, c or d minus the product of the two remaining scalars vanishes because of the specific context we are presently studying: a = b = c = d = 0. This is suggesting that that context is compatible with a simultaneous realization of:

- \( \nabla \cdot \Phi \) being a multiplicative morphism connecting \( E_4(R) \) and \( M_4(R) \) via an extended product built on the Levi-Civita cube; note also that if \( \nabla \cdot \Phi \) is a multiplicative morphism then the components of the curvature tensor reduce to the partial derivatives of the Christoffel’s symbols of the second kind.

- The extended product built upon the Levi-Civita cube as being an associative inner product.

Theorem 02

In that second part we have explored situations realizing the ideas sketched by E. Cartan in [08; § 125, p. 110, second eventuality] and came to the (proviso because all the additional constraints have not yet been explored in extenso) conclusion that, yes, the new formalism of the EM tensor, (03), obtained with the theory of the (E) question [01] for continuous energetic contexts can be interpreted as an infinitesimal transformation of the 4D metric resulting from the action of the trivial matrix \( \nabla \cdot \Phi \) when the latter is a “bi-vector à la E. Cartan”. That eventuality is mathematically possible but associated with very strong and strict constraints exposed previously in the text. The constraints concern the components of the two vectors constituting the bi-vector and the Christoffel’s symbols of the second kind. At the end of the day, for symmetric metrics only, it seems to be a realistic ambition to write:

\[ (01-Periat) \]

\[ q = \frac{\partial^4 G}{\partial x^4 \partial y^4} \]
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Corollary 02: premonition for the existence of a plausible operation on bi-vectors
Since any (2, 0) representation of the EM tensor is a bi-vector [18; p. 560, at the beginning of § 4], the relation (01-Periat) is suggesting the formal existence of some mathematical operation characterized by the action of the representation of an Hermitian spinor, the customized metric tensor \( (4)G \), which may also be identified with a null bi-vector (see [18; p. 560]) on a bi-vector “à la E. Cartan”. Although yet at a pre-historic stage, this corollary remark should be put somewhere in the background of our head for future developments.

Generators of the Lorentz transformations and trivial splits of extended exterior products

Remark 06: the generators of the Lorentz transformations
The matrix representing the generators of the Lorentz transformations is a natural skew-symmetric (4-4) matrix with a null diagonal. It has the following generic formalism:

\[
[... M^{\mu \nu} ...] = \begin{bmatrix} 0 & <\mathbf{K}> \\ -|\mathbf{K}| & (3)[I] \end{bmatrix}
\]

Where \( (3)K : \{K^1 = M^{01}, K^2 = M^{02}, K^3 = M^{03}\} \) represents the boost and \([I]\) the rotation. This is furthermore what we have called a negative perian matrix.

Remark 07: the 4D source of a bi-3Dvector
Let us suppose that there exists an unknown 4D complex vector \( (4)X \) which is element of \((E_4(C), \triangleleft \triangledown A)\) where the symbol \( \triangleleft \triangledown A \) represents an extended “exterior” product (this means that the two indices of the cube \( \triangledown A \) have an anti-symmetric behavior; in extenso: \( A_{\lambda \mu} + A_{\mu \lambda} = 0 \)). Any extended exterior product of the following type, \( \triangleleft \triangledown A((4)X, ...) \), has a trivial split which we can write \( \triangleleft \triangledown A \Phi((4)X) \). If furthermore that trivial matrix is skew-symmetric, then: 

\[
\triangleleft \triangledown A \Phi((4)X) + \triangleleft \triangledown A \Phi^t((4)X) = 0
\]

(61)

where the subscript “t” positioned behind/after a matrix denotes the transposed of that matrix) and the following rule holds true for the components of the cube:

\[
A_{\lambda \mu} \varepsilon = - A_{\lambda \varepsilon} \mu = A_{\varepsilon \lambda} \mu
\]

We can prove that that matrix has always the formalism of a matrix representing a bi-3Dvector (Annex 03).

Theorem 03: Existence of a 4D source for the generators of the Lorentz transformations
From the three previous remarks we induce the existence of some unknown 4D complex vector \( (4)X \) and of some unknown anti-symmetric cube \( \triangleleft \triangledown A \) such that any representation of the generators of the Lorentz transformation can be written:

\[
\exists \triangleleft \triangledown A | (02) \text{ and } \exists (4)X | [... M^{\mu \nu} ...] = \triangleleft \triangledown A((4)X)
\]

Corollary 03: a first plausible formalism for the source
The relation (59) writes:

\[
M^{\mu \nu} = A_{\alpha \nu}^\alpha \cdot X^\alpha
\]

And this suggests that the formalism of the unknown source may perhaps be:

\[
X^\varepsilon = \frac{1}{2} \sum_j \sum_k A_{jk}^\varepsilon \cdot M_{jk} + \frac{1}{2} \sum_i A_{0i}^\varepsilon \cdot M_0 + \frac{1}{2} \sum_j \sum_k A_{ij0}^\varepsilon \cdot M_{ijk}
\]

The double anti-symmetry constraint on \( \triangleleft \triangledown A \) and on \([... M^{\mu \nu} ...]\) yields a reasonable proposition:

\[
X^\varepsilon = \frac{1}{2} \sum_i A_{i}^\varepsilon \cdot M^i + \sum_a A_{a}^\varepsilon \cdot K^a \text{ where } \varepsilon = 0, 1, 2, 3 \text{ whilst } j, k, a = 1, 2, 3
\]

Corollary 04: systematic existence of a pre-metric
When the relations (60) and (61) are compatible:

\[
M^{\mu \nu} = A_{\alpha \nu}^\alpha \cdot X^\alpha
\]

And this suggests that the formalism of the unknown source may perhaps be:

\[
X^\varepsilon = \frac{1}{2} \sum_i A_{i}^\varepsilon \cdot M^i + \sum_a A_{a}^\varepsilon \cdot K^a \text{ where } \varepsilon = 0, 1, 2, 3 \text{ whilst } j, k, a = 1, 2, 3
\]
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And there exists plausibly a mathematical tool insuring the lowering and the rising from the subscripts and the indices on $(E_4(C), \bigtriangleup \nabla _A)$:

\[
\frac{1}{2} \sum _\mu \sum _\nu A_{\mu \nu} \epsilon _{\alpha \nu A} = G
\]

This formalism is remembering the one introduced in [19; p. 139, (1.3)] which we propose to analyze later. The notion of pre-metric has a long history in physics. More exactly we should recall here the courageous attempts to (re)build a generalized theory of relativity [20] or the Maxwell’s laws without the pre-existence of a metric [21]. In all cases the sophisticated relations between vectors, bi-vectors and spinors play a determinant role.

Proposition 04: the kinetic momentum as the source of the generators of the Lorentz transformations?

We shall investigate the following and strange cases where the source of the generator of a Lorentz generator is an event in a phase space: \( p \). This will perhaps allow us to interpret (63) as the local 4D metric. We are in fact to propose that hypothesis because of the condition validating the theorem 02 and because of the informations contained in the annex 03.

Annexes

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Annex 01: Some useful calculations for the future

Considering the definition of the Christoffel’s symbols of the second kind [07; p. 18, (1.44)], it is easy to check that:

$$\Gamma_{\lambda\mu}^{\sigma} \cdot g_{\sigma\alpha} = g_{\upsilon\sigma} \cdot \{ \partial_{\lambda} g_{\mu\upsilon} + \partial_{\mu} g_{\lambda\upsilon} - \partial_{\upsilon} g_{\lambda\mu} \} \cdot g_{\sigma\alpha}$$

Since we are working on R (or C) which is an associative set for the multiplication:

$$\Gamma_{\lambda\mu}^{\sigma} \cdot g_{\sigma\alpha} = g_{\upsilon\sigma} \cdot \{ \partial_{\lambda} g_{\mu\upsilon} + \partial_{\mu} g_{\lambda\upsilon} - \partial_{\upsilon} g_{\lambda\mu} \} \cdot g_{\sigma\alpha} = \delta_{\upsilon}^{\alpha} \cdot \{ \partial_{\lambda} g_{\mu\upsilon} + \partial_{\mu} g_{\lambda\upsilon} - \partial_{\upsilon} g_{\lambda\mu} \} = \{ \partial_{\lambda} g_{\mu\upsilon} + \partial_{\mu} g_{\lambda\upsilon} - \partial_{\upsilon} g_{\lambda\mu} \}$$

Now let us define:

$$\omega_{\lambda\mu} = \Gamma_{\lambda\mu}^{\sigma} \cdot g_{\sigma\alpha} = \{ \partial_{\lambda} g_{\mu\upsilon} + \partial_{\mu} g_{\lambda\upsilon} - \partial_{\upsilon} g_{\lambda\mu} \}$$

We state that:

$$\omega_{\mu\lambda} = \Gamma_{\mu\lambda}^{\sigma} \cdot g_{\sigma\alpha} = \{ \partial_{\mu} g_{\lambda\upsilon} + \partial_{\lambda} g_{\mu\upsilon} - \partial_{\upsilon} g_{\mu\lambda} \}$$

$$\omega_{\lambda\alpha} = \Gamma_{\lambda\alpha}^{\sigma} \cdot g_{\sigma\upsilon} = \{ \partial_{\lambda} g_{\alpha\upsilon} + \partial_{\alpha} g_{\lambda\upsilon} - \partial_{\upsilon} g_{\lambda\alpha} \}$$

$$\omega_{\alpha\lambda} = \Gamma_{\alpha\lambda}^{\sigma} \cdot g_{\sigma\upsilon} = \{ \partial_{\alpha} g_{\lambda\upsilon} + \partial_{\lambda} g_{\alpha\upsilon} - \partial_{\upsilon} g_{\alpha\lambda} \}$$

Let us suppose that we work in a region with a symmetric metric; (10-1) holds true again and:

$$\omega_{\lambda\mu} = \omega_{\mu\lambda}$$
If we conventionally write:

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 2. \partial_\mu g_{\nu\lambda}$$

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 2. \partial_\nu g_{\mu\lambda}$$

$$\omega_{\mu\nu} + \omega_{\nu\mu} + \omega_{\lambda\mu} + \omega_{\mu\lambda} = \partial_\lambda g_{\mu\nu} + \partial_\nu g_{\lambda\mu}$$

Annex 02: in case of doubt

The formalism of all off-diagonal terms and the fact that we are working on R (or C) have a lucky consequence on the calculations; the off-diagonal terms vanish for the following matrix:

$$\frac{\partial \Phi}{\partial x_1} = \begin{bmatrix} x_1^2 & x_2 x_3 & x_3^2 \\ x_1 x_2 & x_1^2 & x_2^2 \\ x_1 x_3 & x_2 x_3 & x_3^2 \\ x_2 x_1 & x_2^2 & x_3 \\ x_3 x_1 & x_3^2 & x_1 \\ x_3 x_2 & x_3^2 & x_2 \\ \end{bmatrix}$$

Since all elements of the diagonal are equal, this formalism is unfortunately not compatible with the one of $\eta$ (13).

Annex 03: double constraint on the cubes and bi-vector

Proposition: If the components of an anti-symmetric cube are such that:

(A01-a)

then the projection of any “projectile”, say $a^{(a)}$, is the representation of a bi-vector. Let us write that projection:

(A01-b)

With an anti-symmetric cube and with (A01-a), it is easy to check that:

(A01-c)

This can be simplified with:

(A01-d)

If we conventionally write:

(A01-e)
\[ Y^1 = -x^0 + x^1 \]

This is just:

\[(A01-f)\]

\[ \nabla \Phi(a) = \begin{bmatrix} 0 & -x^1 & x^2 & -x^3 \\ x^1 & 0 & -y^2 & y^1 \\ -x^2 & y^2 & 0 & -x^1 \\ x^3 & -y^1 & x^1 & 0 \end{bmatrix} \]

which obviously represents a bi-3Dvector \((X^*, Y)\) with \(X^* : (X^1, -X^2, X^3)\) et \(Y : (Y^1, Y^2, Y^3)\). \qed