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The Divergence-Free Effective Action for a Scalar Field Theory

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Abstract

We give detailed computations and results of the application of our divergence-free framework for quantum field theory to a scalar field model with quartic coupling. Computations up to two loops are illustrated, and expressions for the renormalized mass and coupling parameters are obtained. It is shown how it is possible to obtain a divergence-free effective action. Moreover, the ambiguous logarithmic mass contributions of conventional renormalization theory are fixed perturbatively in favour of a well-defined vacuum that is consistent with flat-space perturbative quantization.

1 Introduction

Let us consider the scalar field system with the Lagrangian density

$$\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{1}$$

The corresponding action in momentum space gives:

$$-\delta(r+s)\frac{1}{2}\phi(r)\phi(s)\Delta(r) - \delta(r+s+t+u)\frac{\lambda}{4!}\phi(r)\phi(s)\phi(t)\phi(u) \tag{2}$$

where we suppress integrations over momenta r, s, t, u, the δ functions are 4-dimensional, and we have $\Delta(p) = -p^2 + m^2$.

We shall compute contributions in the effective action to two loops and vacuum contributions to three loops. We shall follow our divergence-free scheme[1]. To that end, the functional derivatives are:

$$W(r) = -\delta(r+s)\phi(s)\Delta(r) - \delta(r+s+t+u)\frac{\lambda}{3!}\phi(s)\phi(t)\phi(u) \tag{3}$$

$$W(r,s) = -\delta(r+s)\Delta(r) - \delta(r+s+t+u)\frac{\lambda}{2}\phi(t)\phi(u) \tag{4}$$

$$W(r,s,t) = -\delta(r+s+t+u)\lambda\phi(u) \tag{5}$$

$$W(r,s,t,u) = -\delta(r+s+t+u)\lambda \tag{6}$$

2 One-Loop Contributions

One-loop contributions are described by the expression

$$\frac{i}{2}\mathrm{tr}\ln W_{ij} \tag{7}$$

where W_{ij} denotes the bilinear kernel in momentum space; we shall write $W = \Delta + Y$, where Y is the field dependent part. In our case, we have in momentum space,

$$\Delta(r,s) = \delta(r+s)\Delta(p) \qquad Y(r,s) = \delta(r+s+t+u)\frac{\lambda}{2}\phi(t)\phi(u) \tag{8}$$

Notice that an overall minus in the definition of both Δ and Y is insignificant.

According to our scheme, the corresponding regularized (or divergence-free) one-loop contribution takes the form

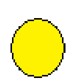
$$-\frac{i}{2}\varrho_\epsilon\,\mathrm{tr}\left\{\frac{1}{\epsilon}\frac{1}{(\Delta+Y)^\epsilon}\right\} \tag{9}$$

where ϵ is a limiting parameter and ϱ_ϵ denotes the operator $(\frac{\partial}{\partial\epsilon})\epsilon$. In our approach, this operator is applied and the limit $\epsilon \to 0$ is taken *after the integration over loop momentum* is performed.

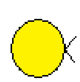
We must expand the above expression with respect to the field insertions represented by Y to obtain the formal series:

$$-\frac{i}{2}\varrho_\epsilon\,\mathrm{tr}\left\{\begin{array}{l} \frac{1}{\epsilon}\frac{1}{\Delta^\epsilon} - \frac{1}{\Delta^{1+\epsilon}}Y + \frac{1}{2}\frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)}\underbrace{\frac{1}{\Delta}Y\frac{1}{\Delta}}_{2+\epsilon}Y \\ \\ -\frac{1}{3}\frac{\Gamma(3+\epsilon)}{\Gamma(1+\epsilon)}\underbrace{\frac{1}{\Delta}Y\frac{1}{\Delta}Y\frac{1}{\Delta}}_{3+\epsilon}Y + \cdots \end{array}\right\} \tag{10}$$

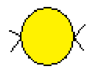
This formal series must be combined with a rule of combining the momentum space propagators (shown underbraced and indicating combining exponent) using Feynman parameters.[1] The above series corresponds to a virtual scalar field loop with successive numbers of external (effective) field insertions. With mini diagrams shown these are:

vacuum contribution: 

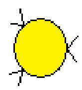
$$-\frac{i}{2}\varrho_\epsilon\,\mathrm{tr}\left\{\frac{1}{\epsilon}\frac{1}{\Delta^\epsilon}\right\} \tag{11}$$

bilinear contribution: 

$$\frac{i}{2}\varrho_\epsilon\,\mathrm{tr}\left\{\frac{1}{\Delta^{1+\epsilon}}Y\right\} \tag{12}$$

quartic contribution: 

$$-\frac{i}{2}\varrho_\epsilon \,\mathrm{tr}\left\{\frac{1}{2}\frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)}\underbrace{\frac{1}{\Delta}Y\frac{1}{\Delta}}_{2+\epsilon}Y\right\} \tag{13}$$

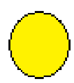
hexilinear contribution: 

$$\frac{i}{2}\varrho_\epsilon \,\mathrm{tr}\left\{\frac{1}{3}\frac{\Gamma(3+\epsilon)}{\Gamma(1+\epsilon)}\underbrace{\frac{1}{\Delta}Y\frac{1}{\Delta}Y\frac{1}{\Delta}}_{3+\epsilon}Y\right\} \tag{14}$$

In the followings, we shall show how to utilize the above expressions for our scalar field model by translating to momentum space.

2.1 Vacuum Contribution

The expression for the one-loop vacuum contribution is



$$-\frac{i}{2}\varrho_\epsilon \,\mathrm{tr}\left\{\frac{1}{\epsilon}\frac{1}{\Delta^\epsilon}\right\} \tag{15}$$

In momentum space this gives

$$-\frac{i}{2}\varrho_\epsilon \frac{1}{\epsilon}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{(-p^2+m^2)^\epsilon} \tag{16}$$

Transforming to a Euclidean momentum integral $p_0 \to ip_4$, we have[1]

$$\frac{1}{2}\varrho_\epsilon \frac{1}{\epsilon}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{(p^2+m^2)^\epsilon} \tag{17}$$

which integrates to give

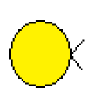
$$\frac{1}{32\pi^2}\varrho_\epsilon \frac{\Gamma(-2+\epsilon)}{\Gamma(1+\epsilon)}(m^2)^{2-\epsilon} \tag{18}$$

Simplifying the gamma functions, and applying ϱ_ϵ, we obtain:

$$\frac{1}{32\pi^2}m^4\varrho_\epsilon \frac{(m^2)^{-\epsilon}}{\epsilon(-1+\epsilon)(-2+\epsilon)} = \frac{1}{64\pi^2}m^4\left\{\frac{3}{2}-\ln(m^2)\right\} \tag{19}$$

2.2 Bilinear Contribution

Our expression for the one-loop contribution which is bilinear in the effective fields is given by



$$\frac{i}{2}\varrho_\epsilon \,\mathrm{tr}\left\{\frac{1}{\Delta^{1+\epsilon}}Y\right\} \tag{20}$$

[1]It should be wellknown that the transformation from Minkowskian momenta to Euclidean counterparts is equivalent to Feynman's $i\epsilon$ prescription for propagators.

Comparing with our scalar field bilinear kernel, we have in momentum space a contribution of the form $\frac{1}{2}\phi^2\{\cdots\}$, where the momentum-independent coefficient is given by

$$\frac{\lambda}{2} i \varrho_\epsilon \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{(-p^2+m^2)^{1+\epsilon}} \tag{21}$$

For the corresponding Euclidean loop integral, we have

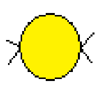
$$-\frac{\lambda}{2} \varrho_\epsilon \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{(p^2+m^2)^{1+\epsilon}} \tag{22}$$

Integrating and simplifying, we get

$$-\frac{\lambda}{32\pi^2} \varrho_\epsilon \frac{\Gamma(-1+\epsilon)}{\Gamma(1+\epsilon)} (m^2)^{1-\epsilon} = -\frac{\lambda}{32\pi^2} m^2 \varrho_\epsilon \frac{(m^2)^{-\epsilon}}{\epsilon(-1+\epsilon)} = \frac{\lambda}{32\pi^2} m^2 \left\{1 - \ln(m^2)\right\} \tag{23}$$

2.3 Quartic Contribution

The expression for the one-loop contribution which is quartic in the external fields is given by



$$-\frac{i}{4} \varrho_\epsilon \, \mathrm{tr} \left\{ \frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)} \underbrace{\frac{1}{\Delta} Y \frac{1}{\Delta}}_{2+\epsilon} Y \right\} \tag{24}$$

Correspondingly, we have in momentum space the contribution $\frac{1}{8}\phi^2(r)\phi^2(-r)\{\cdots\}$, where the brackets contain

$$-\frac{\lambda^2}{2} i \varrho_\epsilon \frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{\{(1-x)(-p^2+m^2) + x[-(p+r)^2+m^2]\}^{2+\epsilon}} \tag{25}$$

where according to our operator expansion rule, the momentum-space propagators are combined using a Feynman parameter x, with an exponent $(2+\epsilon)$. The argument of the combined propagator simplifies to

$$-p^2 - 2xp\cdot r - xr^2 + m^2 = -(p+xr)^2 - x(1-x)r^2 + m^2$$

Making a shift in the integration momentum $p \to p - xr$, and going to Euclidean loop momentum, obtain

$$\frac{\lambda^2}{2} \varrho_\epsilon \frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{\{p^2 - x(1-x)r^2 + m^2\}^{2+\epsilon}} \tag{26}$$

Integrating over p, we obtain

$$\frac{\lambda^2}{32\pi^2} \varrho_\epsilon \frac{1}{\epsilon} \int_0^1 \mathrm{d}x \{m^2 - x(1-x)r^2\}^{-\epsilon} \tag{27}$$

Executing ϱ_ϵ, we obtain

$$-\frac{\lambda^2}{32\pi^2}\int_0^1 \mathrm{d}x\,\ln\{m^2 - x(1-x)r^2\} \tag{28}$$

We may expand this with respect to r^2, and evaluate the parametric integral. For instance, to order r^2, we have

$$-\frac{\lambda^2}{32\pi^2}\int_0^1 \mathrm{d}x\left\{\ln(m^2) - x(1-x)\frac{r^2}{m^2} + \cdots\right\} = -\frac{\lambda^2}{32\pi^2}\left\{\ln(m^2) - \frac{1}{6}\frac{r^2}{m^2} + \cdots\right\} \tag{29}$$

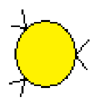
It is the first term which corrects the coupling constant in one loop.

Alternatively, doing the parametric integral, we get the expression

$$\frac{\lambda^2}{16\pi^2}\lambda^2\{1 - \frac{1}{2}\ln(m^2) - \frac{\sqrt{4m^2-r^2}\,\tan^{-1}(\sqrt{r^2}/\sqrt{4m^2-r^2})}{\sqrt{r^2}}\} \tag{30}$$

2.4 Hexilinear Contribution

We now compute the one-loop contribution to the effective ϕ^6 term. This comes from



$$\frac{i}{6}\varrho_\epsilon\,\mathrm{tr}\left\{\frac{\Gamma(3+\epsilon)}{\Gamma(1+\epsilon)}\underbrace{\frac{1}{\Delta}Y\frac{1}{\Delta}Y\frac{1}{\Delta}}_{3+\epsilon}Y\right\} \tag{31}$$

In momentum space this gives a term of the form

$$\frac{1}{3!}\frac{1}{8}\phi^2(r)\phi^2(s)\phi^2(-r-s)\{\cdots\}$$

where the brackets are given by

$$\lambda^3 i\varrho_\epsilon\frac{\Gamma(3+\epsilon)}{\Gamma(1+\epsilon)}\int_0^1 \mathrm{d}x\int_0^1 \mathrm{d}y\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{\{(1-y)[(1-x)\Delta(p)+x\Delta(p+r)]+y\Delta(p+r+s)\}^{3+\epsilon}} \tag{32}$$

The argument of the combined propagator simplifies as follows:

$$(1-y)(-p^2-2xp\cdot r - xr^2 + m^2) + y\{-(p+r+s)^2+m^2\} =$$
$$-p^2 - 2(x+y-xy)p\cdot r - (x+y-xy)r^2 - 2yp\cdot s - ys^2 + m^2 =$$
$$-\{p+(x+y-xy)r+ys\}^2 - (x+y-xy)(1-x-y+xy)r^2 - y(1-y)s^2 - 2y(1-x-y+xy)r\cdot s + m^2$$

Suppose we are interested with the ϕ^6 contribution without momentum dependence. Making the shift $p \to p-(x+y-xy)r - ys$, we are left with the Euclidean integral

$$-\lambda^3\varrho_\epsilon\frac{\Gamma(3+\epsilon)}{\Gamma(1+\epsilon)}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{(p^2+m^2)^{3+\epsilon}} = -\frac{\lambda^3}{16\pi^2m^2} \tag{33}$$

Notice that the above integral does not need regularization. We can dispense with ϱ_ϵ, put $\epsilon = 0$, and evaluate the Euclidean integral normally, for it is convergent. However, it does not hurt to leave the universal prescription in place, for it gives the same answer.

3 Two-Loop Contributions

The two loop contributions to the effective action are described by the expressions:[2]

$$-\frac{1}{8}W_{ijkl}W_{ij}^{-1}W_{kl}^{-1}+\frac{1}{12}W_{ijk}W_{lmn}W_{il}^{-1}W_{jm}^{-1}W_{kn}^{-1} \tag{34}$$

Here, W_{ijk} and W_{ijkl} are the trilinear and the quadrilinear effective vertices, while W_{ij}^{-1} is the effective propagator. Our regularization here consists of replacing each effective propagator by a limiting counterpart:

$$W_{ij}^{-1}\rightarrow \varrho_\epsilon\, W_{ij}^{-(1+\epsilon)} \tag{35}$$

Hence the regularized (divergence-free) counterpart of the above two-loop expression is[1]

$$\left\{\begin{array}{l} -\frac{1}{8}\varrho_a\,\varrho_b\,W_{ijkl}W_{ij}^{-(1+a)}W_{kl}^{-(1+b)} \\ \\ +\frac{1}{12}\varrho_a\,\varrho_b\,\varrho_c\,W_{ijk}W_{lmn}W_{il}^{-(1+a)}W_{jm}^{-(1+b)}W_{kn}^{-(1+c)} \end{array}\right. \tag{36}$$

Notice that *each effective propagator has its own limiting parameter and procedure.* These limits would be applied *after all loop integrals are done with.* We can now expand the above expressions to any desired order in the implicit effective fields. We write for the bilinear kernel $W_{ij}=(\Delta+Y)_{ij}$, where Δ is field independent and Y is the field dependent counterpart. In our scalar field theory, the trilinear kernel W_{ijk} is field dependent and the quadrilinear kernel W_{ijkl} is field independent. Notice that while Y is of order 2 in the scalar field, W_{ijk} is only of order 1.

The regularized effective propagator expands like[1]

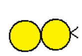
$$\frac{1}{(\Delta+Y)^{1+\epsilon}}=\frac{1}{\Delta^{1+\epsilon}}-\frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)}\underbrace{\frac{1}{\Delta}Y\frac{1}{\Delta}}_{2+\epsilon}+\frac{1}{2}\frac{\Gamma(3+\epsilon)}{\Gamma(1+\epsilon)}\underbrace{\frac{1}{\Delta}Y\frac{1}{\Delta}Y\frac{1}{\Delta}}_{3+\epsilon}+\cdots \tag{37}$$

where, again, the meaning associated with the above terms is that the momentum-space propagators are understood to be combined using Feynman parameters with a total power equal to their number plus ϵ (argument of the associated upper gamma function). Hence our 2-loop contributions expand as follows (mini diagrams shown):

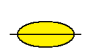
vacuum contribution: 

$$-\frac{1}{8}\varrho_a\,\varrho_b\,W_{ijkl}\Delta_{ij}^{-(1+a)}\Delta_{kl}^{-(1+b)} \tag{38}$$

bilinear contribution:



$$\frac{1}{4}\varrho_a\,\varrho_b\,W_{ijkl}\Delta_{ij}^{-(1+a)}\frac{\Gamma(2+b)}{\Gamma(1+b)}\left(\underbrace{\frac{1}{\Delta}Y\frac{1}{\Delta}}_{2+b}\right)_{kl}+$$

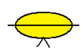


$$\frac{1}{12}\varrho_a\,\varrho_b\,\varrho_c\,W_{ijk}W_{lmn}\Delta_{il}^{-(1+a)}\Delta_{jm}^{-(1+b)}\Delta_{kn}^{-(1+c)} \tag{39}$$

quartic contribution:

$$-\tfrac{1}{8}\varrho_a\,\varrho_b\,W_{ijkl}\tfrac{\Gamma(2+a)}{\Gamma(1+a)}\left(\underbrace{\frac{1}{\Delta}Y\frac{1}{\Delta}}_{2+a}\right)_{ij}\tfrac{\Gamma(2+b)}{\Gamma(1+b)}\left(\underbrace{\frac{1}{\Delta}Y\frac{1}{\Delta}}_{2+b}\right)_{kl}$$

$$-\tfrac{1}{8}\varrho_a\,\varrho_b\,W_{ijkl}\Delta_{ij}^{-(1+a)}\tfrac{\Gamma(3+b)}{\Gamma(1+b)}\left(\underbrace{\frac{1}{\Delta}Y\frac{1}{\Delta}Y\frac{1}{\Delta}}_{3+b}\right)_{kl} \tag{40}$$



$$-\tfrac{1}{4}\varrho_a\,\varrho_b\,\varrho_c\,W_{ijk}W_{lmn}\Delta_{il}^{-(1+a)}\Delta_{jm}^{-(1+b)}\tfrac{\Gamma(2+c)}{\Gamma(1+c)}\left(\underbrace{\frac{1}{\Delta}Y\frac{1}{\Delta}}_{2+c}\right)_{kn}$$

In the followings, we shall adapt the above contributions to our scalar field system, translate them to momentum space integrals, and compute them. The 2-loop quartic contribution is of order λ^3, hence would be irrelevant for our present purposes.

3.1 Vacuum Contribution

In our scalar field system, the 2-loop vacuum contribution

$$-\frac{1}{8}\varrho_a\,\varrho_b\,W_{ijkl}\Delta_{ij}^{-(1+a)}\Delta_{kl}^{-(1+b)} \tag{41}$$

translates to the momentum space expression

$$\frac{\lambda}{8}\varrho_a\,\varrho_b\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{(-p^2+m^2)^{1+a}}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{(-p^2+m^2)^{1+b}} \tag{42}$$

This gives the expression with Euclidean momentum space expression

$$-\frac{\lambda}{8}\left\{\varrho_\epsilon\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{(p^2+m^2)^{1+\epsilon}}\right\}^2 \tag{43}$$

Now we have

$$\varrho_\epsilon\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{(p^2+m^2)^{1+\epsilon}}=\frac{m^2}{16\pi^2}\varrho_\epsilon\frac{(m^2)^{-\epsilon}}{\epsilon(-1+\epsilon)}=-\frac{m^2}{16\pi^2}\{1-\ln(m^2)\} \tag{44}$$

which leads to the following two-loop vacuum contribution

$$-\lambda\frac{m^4}{8}\left(\frac{1}{16\pi^2}\right)^2\{1-\ln(m^2)\}^2 \tag{45}$$

3.2 Bilinear Contribution

The first bilinear contribution

$$\frac{1}{4}\varrho_a\,\varrho_b\,W_{ijkl}\Delta_{ij}^{-(1+a)}\frac{\Gamma(2+b)}{\Gamma(1+b)}\left(\underbrace{\frac{1}{\Delta}Y\frac{1}{\Delta}}_{2+b}\right)_{kl} \tag{46}$$

gives, for the coefficient of $\frac{1}{2}\phi^2$, the momentum space expression

$$-\frac{\lambda^2}{4}\left\{\varrho_a\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{(-p^2+m^2)^{1+a}}\right\}\left\{\varrho_b\frac{\Gamma(2+b)}{\Gamma(1+b)}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{(-q^2+m^2)^{2+b}}\right\} \tag{47}$$

Transforming to Euclidean loop momenta, we get

$$\frac{\lambda^2}{4}\left\{\varrho_a\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{(p^2+m^2)^{1+a}}\right\}\left\{\varrho_b\frac{\Gamma(2+b)}{\Gamma(1+b)}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{(q^2+m^2)^{2+b}}\right\} \tag{48}$$

Now we have the followings:

$$\varrho_a\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{(p^2+m^2)^{1+a}}=\frac{m^2}{16\pi^2}\varrho_a\frac{(m^2)^{-a}}{a(-1+a)}=-\frac{m^2}{16\pi^2}\left\{1-\ln(m^2)\right\} \tag{49}$$

$$\varrho_b\frac{\Gamma(2+b)}{\Gamma(1+b)}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{(q^2+m^2)^{2+b}}=\frac{1}{16\pi^2}\varrho_b\frac{(m^2)^{-b}}{b}=-\frac{1}{16\pi^2}\ln(m^2) \tag{50}$$

Hence we have the following result for the first 2-loop bilinear contribution:

$$\frac{\lambda^2}{4}\left(\frac{1}{16\pi^2}\right)^2 m^2\left\{1-\ln(m^2)\right\}\ln(m^2) \tag{51}$$

The second bilinear contribution

$$\frac{1}{12}\varrho_a\,\varrho_b\,\varrho_c\,W_{ijk}W_{lmn}\Delta_{il}^{-(1+a)}\Delta_{jm}^{-(1+b)}\Delta_{kn}^{-(1+c)} \tag{52}$$

gives, for the coefficient of $\frac{1}{2}\phi(r)\phi(-r)$, the momentum space expression

$$-\frac{\lambda^2}{6}\varrho_a\,\varrho_b\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{(-p^2+m^2)^{1+a}}\frac{1}{(-q^2+m^2)^{1+b}}\frac{1}{-(p+q+r)^2+m^2} \tag{53}$$

The third limiting parameter c and associated procedure ϱ_c are eliminated because in a 2-loop Feynman integral computation, just two parameters are enough to guarantee freedom from divergences; the third one is dummy.

We first perform the q integral. For that purpose, we combine the involved propagators using a Feynman parameter x. The argument of the resulting propagator becomes

$$\left\{\begin{array}{l}(1-x)(-q^2+m^2)+x\{-(p+q+r)^2+m^2\}= \\ \\ -q^2-2xq\cdot(p+r)-x(p+r)^2+m^2= \\ \\ -\{q+x(p+r)\}^2-x(1-x)(p+r)^2+m^2\end{array}\right.$$

The q integral becomes

$$\varrho_b \frac{\Gamma(2+b)}{\Gamma(1+b)} \int_0^1 \mathrm{d}x\,(1-x)^b \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{\{-[q+x(p+r)]^2 - x(1-x)(p+r)^2 + m^2\}^{2+b}} \tag{54}$$

Making the shift $q \to q - x(q+r)$, and going to Euclidean loop momentum we have

$$i\varrho_b \frac{\Gamma(2+b)}{\Gamma(1+b)} \int_0^1 \mathrm{d}x\,(1-x)^b \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{\{q^2 - x(1-x)(p+r)^2 + m^2\}^{2+b}} \tag{55}$$

Integrating over q we obtain

$$\frac{i}{16\pi^2} \varrho_b \frac{\Gamma(b)}{\Gamma(1+b)} \int_0^1 \mathrm{d}x\,(1-x)^b \{-x(1-x)(p+r)^2 + m^2\}^{-b} \tag{56}$$

Incorporating this in our full expresion, we have

$$-\frac{\lambda^2}{6} \frac{i}{16\pi^2} \varrho_a\,\varrho_b \frac{\Gamma(b)}{\Gamma(1+b)} \int_0^1 \mathrm{d}x\,(1-x)^b \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{(-p^2+m^2)^{1+a}} \frac{1}{\{-x(1-x)(p+r)^2 + m^2\}^b} \tag{57}$$

Before performing the integration over p, we must combine the two propagators using a Feynman parameter y, where the argument of the resulting propagator becomes

$$\begin{cases} (1-y)(-p^2+m^2) + y\{-x(1-x)(p+r)^2 + m^2\} = \\ -\{1-y+yx(1-x)\}p^2 - 2yx(1-x)p\cdot r - yx(1-x)r^2 + m^2 = \\ -\{1-y+yx(1-x)\}\left\{p + \frac{yx(1-x)}{1-y+yx(1-x)} r\right\}^2 - \frac{x(1-x)y(1-y)}{1-y+yx(1-x)} r^2 + m^2 \end{cases}$$

Making the shift

$$p \to p - \frac{yx(1-x)}{1-y+yx(1-x)} r$$

then rescaling $p \to p/\sqrt{1-y+yx(1-x}$, and going to Euclidean momentum p, we obtain

$$\begin{cases} \frac{\lambda^2}{6}\left(\frac{1}{16\pi^2}\right) \varrho_a\,\varrho_b \frac{\Gamma(1+a+b)}{\Gamma(1+a)\Gamma(1+b)} \times \\ \int_0^1 \mathrm{d}x\,(1-x)^b \int_0^1 \mathrm{d}y\,(1-y)^a y^{b-1} \frac{1}{\{1-y+yx(1-x)\}^2} \times \\ \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{\left\{p^2 - \frac{x(1-x)y(1-y)}{1-y+yx(1-x)} r^2 + m^2\right\}^{1+a+b}} \end{cases} \tag{58}$$

Now integrating over p, we obtain

$$\begin{cases} \frac{\lambda^2}{6}\left(\frac{1}{16\pi^2}\right)^2 \varrho_a\,\varrho_b \frac{\Gamma(-1+a+b)}{\Gamma(1+a)\Gamma(1+b)} \times \\ \int_0^1 \mathrm{d}x\,(1-x)^b \int_0^1 \mathrm{d}y\,(1-y)^a y^{b-1} \frac{1}{\{1-y+yx(1-x)\}^2} \times \\ \left\{m^2 - \frac{x(1-x)y(1-y)}{1-y+yx(1-x)} r^2\right\}^{1-a-b} \end{cases} \tag{59}$$

The above may be expanded with respect to the external momentum r. Such expansion corresponds to the order of derivatives in the coordinate-space effective action. To order r^2, we have

$$\left\{\begin{array}{l} \frac{\lambda^2}{6}\left(\frac{1}{16\pi^2}\right)^2 \varrho_a\,\varrho_b\,\frac{\Gamma(-1+a+b)}{\Gamma(1+a)\Gamma(1+b)}(m^2)^{1-a-b}\times \\ \int_0^1 \mathrm{d}x\,(1-x)^b \int_0^1 \mathrm{d}y\,(1-y)^a y^{b-1}\frac{1}{\{1-y+yx(1-x)\}^2}\times \\ \left\{1-(1-a-b)\frac{x(1-x)y(1-y)}{1-y+yx(1-x)}\left(\frac{r^2}{m^2}\right)\right\} \end{array}\right. \tag{60}$$

This gives for the *momentum-independent term*,

$$\left\{\begin{array}{l} \frac{\lambda^2 m^2}{6}\left(\frac{1}{16\pi^2}\right)^2 \varrho_a\,\varrho_b\,\frac{\Gamma(-1+a+b)}{\Gamma(1+a)\Gamma(1+b)}(m^2)^{-a-b}\times \\ \int_0^1 \mathrm{d}x\,(1-x)^b \int_0^1 \mathrm{d}y\,(1-y)^a y^{b-1}\frac{1}{\{1-y+yx(1-x)\}^2} \end{array}\right. \tag{61}$$

The parametric integrals may be evaluated in the form of a series by expanding $1/\{1-y+yx(1-x)\}^2$ with respect to $yx(1-x)/(1-y)$. For instance we have for the parametric integral:

$$\left\{\begin{array}{l} \int_0^1 \mathrm{d}x\,(1-x)^b \int_0^1 \mathrm{d}y\,(1-y)^{a-2} y^{b-1}\left\{1-\frac{2yx(1-x)}{1-y}+\cdots\right\}\approx \\ \int_0^1 \mathrm{d}x\,(1-x)^b \int_0^1 \mathrm{d}y\,(1-y)^{a-2} y^{b-1} - 2\int_0^1 \mathrm{d}x\, x(1-x)^{1+b}\int_0^1 \mathrm{d}y\,(1-y)^{a-3}y^b = \\ \frac{\Gamma(1+b)\Gamma(-1+a)\Gamma(b)}{\Gamma(2+b)\Gamma(-1+a+b)} - 2\frac{\Gamma(2+b)\Gamma(-2+a)\Gamma(1+b)}{\Gamma(4+b)\Gamma(-1+a+b)} \end{array}\right. \tag{62}$$

Hence, we have the following (approximate) result for the momentum-independent 2-loop bilinear contribution:

$$\left\{\begin{array}{l} \frac{\lambda^2 m^2}{6}\left(\frac{1}{16\pi^2}\right)^2 \varrho_a\,\varrho_b\,(m^2)^{-a-b}\frac{\Gamma(-1+a+b)}{\Gamma(1+a)\Gamma(1+b)}\times \\ \left\{\frac{\Gamma(1+b)\Gamma(-1+a)\Gamma(b)}{\Gamma(2+b)\Gamma(-1+a+b)} - 2\frac{\Gamma(2+b)\Gamma(-2+a)\Gamma(1+b)}{\Gamma(4+b)\Gamma(-1+a+b)}\right\} \end{array}\right. \tag{63}$$

Simplifying and executing the limiting operations $\varrho_a\,\varrho_b$, we obtain

$$\frac{\lambda^2 m^2}{6}\left(\frac{1}{16\pi^2}\right)^2\left\{\frac{3}{4}+\frac{1}{6}\ln(m^2)-\ln^2(m^2)\right\} \tag{64}$$

On the other hand, the *momentum-dependent term* of order r^2 is

$$\left\{\begin{array}{l} \frac{\lambda^2}{6}\left(\frac{1}{16\pi^2}\right)^2 \varrho_a\,\varrho_b\,\frac{\Gamma(a+b)}{\Gamma(1+a)\Gamma(1+b)}(m^2)^{-a-b}r^2\times \\ \int_0^1 \mathrm{d}x\,x(1-x)^{1+b} \int_0^1 \mathrm{d}y\,(1-y)^{1+a} y^{b}\frac{1}{\{1-y+yx(1-x)\}^3} \end{array}\right. \tag{65}$$

Here we can dispense with one of the limiting parameters, say b, and apply the limiting operation ϱ_a to obtain immediately,

$$\frac{\lambda^2}{6}\left(\frac{1}{16\pi^2}\right)^2 r^2 \int_0^1 \mathrm{d}x\, x(1-x) \int_0^1 \mathrm{d}y\, \frac{(1-y)}{\{1-y+yx(1-x)\}^3}\{\ln(1-y)-\ln(m^2)\} \tag{66}$$

The "approximated" numerical answer is

$$-\frac{\lambda^2}{6}\left(\frac{1}{16\pi^2}\right)^2 r^2\{0.8+0.5\ln(m^2)\} \tag{67}$$

4 Two-Loop Renormalization

Collecting the above results for one- and two-loop contrubutions, and adding these to the classical action, we obtain

$$\left\{\begin{array}{l} \frac{m^4}{64\pi^2}\left\{\frac{3}{2}-\ln(m^2)-\frac{\lambda}{32\pi^2}\{1-\ln(m^2)\}^2\right\} \\ +\left\{1-\frac{\lambda^2}{6\times 256\pi^4}\{0.8+0.5\ln(m^2)\}\right\}\frac{1}{2}(\partial\phi)^2 \\ -\left\{1-\frac{\lambda}{32\pi^2}\{1-\ln(m^2)\}-\frac{\lambda^2}{512\pi^4}\{1-\ln^2(m^2)\}\right\}\frac{1}{2}m^2\phi^2 \\ -\left\{1+\frac{\lambda}{32\pi^2}\ln(m^2)\right\}\frac{\lambda}{4!}\phi^4 \end{array}\right. \tag{68}$$

The 1st line corresponds to the vacuum contribution (cosmological term in gravitational framework). The 2nd line corresponds to the corrected kinetic term. The 3rd line corresponds to the corrected mass term. The 4th line corresponds to the corrected ϕ^4 coupling.

Notice that the constant vacuum contribution must be set equal to zero, since *to be consistent with our flat space quantization*, any generated cosmological term must be made to vanish; this would serve to fix the scale of $\ln(m^2)$, and render the theory unambiguous.

Hence solving the series equation (iteratively in λ)

$$\frac{3}{2}-\ln(m^2)-\frac{\lambda}{32\pi^2}\{1-\ln(m^2)\}^2+\cdots=0 \tag{69}$$

for $\ln(m^2)$, we obtain

$$\ln(m^2)=\frac{3}{2}+\frac{\lambda}{128\pi^2}+\cdots \tag{70}$$

To the desired order in λ, the effective action becomes independent of $\ln(m^2)$. The coefficient of the kinetic term $\frac{1}{2}(\partial\phi)^2$ becomes

$$\left\{1-\frac{\lambda^2}{6}\frac{1}{256\pi^4}(0.8+\frac{3}{4})\right\} \tag{71}$$

Hence we must rescale ϕ such as

$$\phi \to \left\{1 + \frac{1}{2}\frac{\lambda^2}{6}\frac{1}{256\pi^4}(0.8 + \frac{3}{4})\right\}\phi \tag{72}$$

The resulting coefficients of ϕ^2 and ϕ^4 would define the respective renormalized mass and coupling constants in terms of their bare counterparts; inverting for m and λ would give the expressions that should be used in order to express any process in terms of the physical mass and coupling constant.

5 Discussion

Whereas a preceding article[1] has laid down the foundations of our effective action scheme for divergence-free quantum field theory, the present article, dealing with a standard scalar field theory, gives ample evidence that our techniques are easy to implement in practice, and that the divergence-free framework simplifies the handling of the renormalization program. Other articles would deal with the more interesting applications, to quantum electrodynamics,[3] gauge theories,[4] and quantum gravity.[5]

References

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