Few interesting results regarding Poulet numbers and Egyptian fraction expansion

Marius Coman
Bucuresti, Romania
email: mariuscoman13@gmail.com

Abstract. Considering \( r \) being equal to the positive rational number\( \frac{1}{d_1 - 1} + \frac{1}{d_2 - 1} + \ldots + \frac{1}{d_n - 1} \), where \( d_1, \ldots, d_n \) are the prime factors of a Poulet number, the Egyptian fraction expansion applied to \( r \) leads to interesting results.

Note:

An Egyptian fraction is a sum of distinct unit fractions, such as \( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \ldots + \frac{1}{m} \), where the denominators \( a, b, c, \ldots, m \) are positive, distinct, integers. Every positive rational number can be represented by an Egyptian fraction.

The Egyptian fraction expansion is an algorithm due to Fibonacci for computing Egyptian fractions: the number \( \frac{x}{y} \), where \( x, y \) are positive, distinct, integers, is written as follows:

\[
x/y = \frac{1}{\text{ceiling}(y/x)} + \frac{\left((-1) \mod x\right) \cdot \text{ceiling}(y/x)}{y},
\]

where the function \( \text{ceiling}(z) \) represents the smaller integer equal to or greater than \( z \).

This algorithm is repeated to the second term of the summation above and so on until is obtained an Egyptian fraction.

Conjecture 1:

If \( r \) is equal to the positive rational number \( \frac{1}{d_1 - 1} + \frac{1}{d_2 - 1} + \ldots + \frac{1}{d_n - 1} \), where \( d_1, \ldots, d_n \) are the prime factors of a Poulet number \( P \), and \( m \) is equal to the last denominator obtained applying the Egyptian fraction expansion to \( r \), then the number \( m + 1 \) is a prime or a power of prime for an infinity of Poulet numbers.

Examples:

For \( P = 341 = 11 \cdot 31 \), we have \( r = \frac{1}{10} + \frac{1}{30} = \frac{2}{15} = \frac{1}{8} + \frac{1}{120} \); the number \( m + 1 = 120 + 1 = 121 = 11^2 \), a square of prime.
For $P = 561 = 3 \cdot 11 \cdot 17$, we have $r = 1/2 + 1/10 + 1/16 = 53/80 = 1/2 + 1/7 + 1/51 + 1/28560$; the number $m + 1 = 28560 + 1 = 28561 = 13^4$, a power of prime.

For $P = 645 = 3 \cdot 5 \cdot 43$, we have $r = 1/2 + 1/4 + 1/42 = 65/84 = 1/2 + 1/4 + 1/42$; the number $m + 1 = 42 + 1 = 43$, a prime number.

For $P = 1105 = 5 \cdot 13 \cdot 17$, we have $r = 1/4 + 1/12 + 1/16 = 19/48 = 1/3 + 1/16$; the number $m + 1 = 16 + 1 = 17$, a prime number.

For $P = 1387 = 19 \cdot 73$, we have $r = 1/18 + 1/72 = 5/72 = 1/5 + 1/27 + 1/360$; the number $m + 1 = 360 + 1 = 361 = 19^2$, a square of prime.

For $P = 1729 = 7 \cdot 13 \cdot 19$, we have $r = 1/6 + 1/12 + 1/18 = 11/36 = 1/4 + 1/18$; the number $m + 1 = 18 + 1 = 19$, a prime number.

For $P = 1905 = 3 \cdot 5 \cdot 127$, we have $r = 1/2 + 1/4 + 1/126 = 191/252 = 1/2 + 1/126$; the number $m + 1 = 126 + 1 = 127$, a prime number.

For $P = 6601 = 7 \cdot 23 \cdot 41$, we have $r = 1/6 + 1/22 + 1/40 = 313/1320 = 1/5 + 1/27 + 1/11880$; the number $m + 1 = 11880 + 1 = 11881 = 109^2$, a square of prime.

For $P = 8911 = 7 \cdot 19 \cdot 67$, we have $r = 1/6 + 1/18 + 1/66 = 47/198 = 1/5 + 1/27 + 1/2970$; the number $m + 1 = 2970 + 1 = 2971$, a prime number.

For $P = 52633 = 7 \cdot 73 \cdot 103$, we have $r = 1/6 + 1/72 + 1/102 = 233/1224 = 1/6 + 1/1224$; the number $m + 1 = 8063412364776 + 1 = 8063412364777$, a prime number.

Note:
For the first ten Carmichael numbers $C$ divisible by 7 and 19 (we don’t have a comprehensive list of Poulet numbers indexed together with their prime factors) we always obtain for the number $m + 1$ a prime or a square of prime; we have the following values for $(C, m + 1)$: (1729, 19), (8911, 2971), (63973, 2^2), (126217, 19^2), (188461, 433), (748657, 433), (825265, 1009), (997633, 577), (1050985, 23), (1773289, 1321).

Conjecture 2:
If $r$ is equal to the positive rational number $1/(d_1 - 1) + 1/(d_2 - 1) + \ldots + 1/(d_n - 1)$, where $d_1, \ldots, d_n$ are the prime factors of a Poulet number $P$, and $r$ is represented by the irreducible fraction $x/y$, where $x, y$ positive integers, then the number $y + 1$ is a prime or a power of prime for an infinity of Poulet numbers.

Examples:
(as it can be seen above)
For $P = 341$, we have $r = x/y = 2/15$; the number $y + 1 = 15 + 1 = 16 = 2^4$, a power of prime.

For $P = 561$, we have $r = x/y = 53/80$; the number $y + 1 = 80 + 1 = 81 = 3^4$, a power of prime.

For $P = 1105$, we have $r = x/y = 19/48$; the number $y + 1 = 48 + 1 = 49$, a square of prime.

For $P = 1387$, we have $r = x/y = 53/80$; the number $y + 1 = 80 + 1 = 81 = 3^4$, a power of prime.

For $P = 1729$, we have $r = x/y = 11/36$; the number $y + 1 = 36 + 1 = 37$, a prime number.

For $P = 6601$, we have $r = x/y = 313/1320$; the number $y + 1 = 1320 + 1 = 1321$, a prime number.

For $P = 8911$, we have $r = x/y = 47/198$; the number $y + 1 = 198 + 1 = 199$, a prime number.

**Note:**
As it can be seen above, the number $y$ is sometimes equal to $\text{lcm}((d_1 - 1), (d_2 - 1), \ldots, (d_n - 1))$, which is, for instance, the case of the Poulet number $1387 = 19*73$, where $y = 72 = \text{lcm}(18, 72)$, but this is not always true: this is, for instance, the case of Poulet number $341$, where $y = 15$ and $\text{lcm}(10, 30) = 30$.

**Conjecture 3:**
If $d_1, \ldots, d_n$ are the prime factors of a Poulet number $P$, then the number $\text{lcm}((d_1 - 1), (d_2 - 1), \ldots, (d_n - 1))$ is a prime or a power of prime for an infinity of Poulet numbers.