The reverse side of Helmholtz paradox:
Flow with zero Laplacian generates a constant vortex

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The reverse side of Helmholtz paradox is presented here. Helmholtz paradox itself had been formulated as follows: - in the ideal (non-viscid) newtonian fluids any vorticity should be stable (constant) during infinite time. It means that if vortex arise in non-viscid newtonian fluid, such a vortex should have the constant strength all the time.
In our note to Helmholtz paradox, we proved that Vortex with constant angular velocity of rotation could be generated only by the flow with zero Laplacian (thus, such a flow is not viscid for the case of incompressible newtonian fluids).
Thus, we could conclude in addition to Helmholtz paradox: not only any vortex should be constant in the ideal (non-viscid) newtonian fluids, but if vortex is constant it means that such a flow is turned to be ideal (unviscid) due to zero Laplacian.

Keywords: Helmholtz paradox, constant vortex, incompressible ideal flow.

1. Introduction, the Helmholtz paradox.

In accordance with [1-2], the Helmholtz paradox had been formulated as follows: - in the ideal (non-viscid) newtonian fluids any vorticity should be stable (constant) during infinite time.

It means that if vortex arise in non-viscid newtonian fluid [3], it should has the constant strength all the time: each of components of *the curl field* ***w***, a pseudovector field [4], should be equal to the proper constant. If we denote the components of velocity field ***u*** in Cartesian coordinate system as $\{U_x, U_y, U_z\}$, the assumption above yields as below

$$\begin{aligned} \frac{\partial U_z}{\partial y} - \frac{\partial U_y}{\partial z} &= a = const \, , \\ \frac{\partial U_x}{\partial z} - \frac{\partial U_z}{\partial x} &= b = const \, , \\ \frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y} &= c = const \, , \end{aligned} \tag{1.1}$$

- where ***u*** is the flow velocity, a vector field; let us also choose the Oz axis coincides to the main direction of flow propagation.

2. Flow with constant vorticity (vortex).

Let us solve the inverse problem: - first, we assume that some flow of newtonian viscid fluid [5-6] generates only the constant vortex field (1.1), - the second, we find what a flow it should be?

From the system (1.1) we could obtain the PDE-system as below

$$\begin{cases} \dfrac{\partial^2 U_z}{\partial x\,\partial y} = \dfrac{\partial^2 U_y}{\partial x\,\partial z}, \\[2ex] \dfrac{\partial^2 U_x}{\partial y\,\partial z} = \dfrac{\partial^2 U_z}{\partial y\,\partial x}, \\[2ex] \dfrac{\partial^2 U_y}{\partial z\,\partial x} = \dfrac{\partial^2 U_x}{\partial z\,\partial y}, \end{cases} \qquad (1.2)$$

- which yields the equality:

$$\frac{\partial^2 U_y}{\partial z\,\partial x} = \frac{\partial^2 U_x}{\partial z\,\partial y} = \frac{\partial^2 U_z}{\partial y\,\partial x} \qquad (1.3)$$

For any given (arbitrary) function U_z, equalities (1.2)-(1.3) exactly determine the appropriate expressions for functions U_x, U_y as below [7]

$$U_x = b\cdot z + \int_0^z \left[\frac{\partial U_z}{\partial x}\right] dz + C_1 ,$$

$$U_y = c\cdot x + \int_0^x \left[\frac{\partial U_x}{\partial y}\right] dx + C_2 , \tag{1.4}$$

- where C_1, C_2 – are the constants.

Otherwise, if we express function U_z, depending on the given (arbitrary) functions U_x or U_y, the proper expression should be as below (here C_3 – is the constant):

$$U_z = a\cdot y + \int_0^y \left[\frac{\partial U_y}{\partial z}\right] dy + C_3 .$$

The 2-nd of expressions of (1.4) could be represented in other form

$$U_y = c\cdot x + \int_0^z \left[\frac{\partial U_z}{\partial y}\right] dz + C_2 \tag{1.5}$$

3. <u>The flow with constant vortex generates a zero Laplacian.</u>

Let us present the equation of continuity [1-2] for the components (1.4)-(1.5) of velocity field $\{U_x, U_y, U_z\}$, in the form below:

$$\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z} = 0,$$

$$\int_0^z \left[\frac{\partial^2 U_z}{\partial x^2} + \frac{\partial^2 U_z}{\partial y^2} + \frac{\partial^2 U_z}{\partial z^2} \right] dz = 0,$$

$$\Rightarrow \quad \Delta U_z = 0,$$

- it means that U_z - is a harmonic function [7]. Besides, taking into consideration the expression (1.4), it means that components U_x, U_y – are harmonic functions also. For Cartesian coordinates, it means that Laplacian of the velocity field ***u*** is to be zero.

4. <u>Conclusion.</u>

Let us generalize the Helmholtz paradox (*"if any vorticity arise in non-viscid newtonian fluid, such a vortex should have the constant strength all the time"*): - we proved that Vortex with constant angular velocity of rotation could be generated only by the flow with zero Laplacian (so, such a flow is not viscid for incompressible newtonian fluid).

Thus, we could conclude in addition to Helmholtz paradox: not only any vortex should be constant in the ideal (non-viscid) newtonian fluids, but if vortex is constant it means that such a flow is turned to be ideal (unviscid) due to Laplacian which is to be zero.

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