

*The reverse side of Helmholtz paradox:*  
**Flow with zero Laplacian generates a constant vortex**

**Sergey V. Ershkov**

Institute for Time Nature Explorations,  
M.V. Lomonosov's Moscow State University,  
Leninskie gory, 1-12, Moscow 119991, Russia

e-mail: [sergej-ershkov@yandex.ru](mailto:sergej-ershkov@yandex.ru)

The reverse side of Helmholtz paradox is presented here. Helmholtz paradox itself had been formulated as follows: - in the ideal (non-viscid) newtonian fluids any vorticity should be stable (constant) during infinite time. It means that if vortex arise in non-viscid newtonian fluid, such a vortex should have the constant strength all the time.

In our note to Helmholtz paradox, we proved that Vortex with constant angular velocity of rotation could be generated only by the flow with zero Laplacian (thus, such a flow is not viscid for the case of incompressible newtonian fluids).

Thus, we could conclude in addition to Helmholtz paradox: not only any vortex should be constant in the ideal (non-viscid) newtonian fluids, but if vortex is constant it means that such a flow is turned to be ideal (unviscid) due to zero Laplacian.

**Keywords:** Helmholtz paradox, constant vortex, incompressible ideal flow.

## 1. Introduction, the Helmholtz paradox.

In accordance with [1-2], the Helmholtz paradox had been formulated as follows: - in the ideal (non-viscid) newtonian fluids any vorticity should be stable (constant) during infinite time.

It means that if vortex arise in non-viscid newtonian fluid [3], it should has the constant strength all the time: each of components of *the curl field*  $\mathbf{w}$ , a pseudovector field [4], should be equal to the proper constant. If we denote the components of velocity field  $\mathbf{u}$  in Cartesian coordinate system as  $\{U_x, U_y, U_z\}$ , the assumption above yields as below

$$\begin{aligned}\frac{\partial U_z}{\partial y} - \frac{\partial U_y}{\partial z} &= a = \text{const} , \\ \frac{\partial U_x}{\partial z} - \frac{\partial U_z}{\partial x} &= b = \text{const} , \\ \frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y} &= c = \text{const} ,\end{aligned}\tag{1.1}$$

- where  $\mathbf{u}$  is the flow velocity, a vector field; let us also choose the Oz axis coincides to the main direction of flow propagation.

## 2. Flow with constant vorticity (vortex).

Let us solve the inverse problem: - first, we assume that some flow of newtonian viscid fluid [5-6] generates only the constant vortex field (1.1), - the second, we find what a flow it should be?

From the system (1.1) we could obtain the PDE-system as below

$$\left\{ \begin{array}{l} \frac{\partial^2 U_z}{\partial x \partial y} = \frac{\partial^2 U_y}{\partial x \partial z}, \\ \frac{\partial^2 U_x}{\partial y \partial z} = \frac{\partial^2 U_z}{\partial y \partial x}, \\ \frac{\partial^2 U_y}{\partial z \partial x} = \frac{\partial^2 U_x}{\partial z \partial y}, \end{array} \right. \quad (1.2)$$

- which yields the equality:

$$\frac{\partial^2 U_y}{\partial z \partial x} = \frac{\partial^2 U_x}{\partial z \partial y} = \frac{\partial^2 U_z}{\partial y \partial x} \quad (1.3)$$

For any given (arbitrary) function  $U_z$ , equalities (1.2)-(1.3) exactly determine the appropriate expressions for functions  $U_x, U_y$  as below [7]

$$\begin{aligned}
 U_x &= b \cdot z + \int_0^z \left[ \frac{\partial U_z}{\partial x} \right] dz + C_1, \\
 U_y &= c \cdot x + \int_0^x \left[ \frac{\partial U_x}{\partial y} \right] dx + C_2,
 \end{aligned}
 \tag{1.4}$$

- where  $C_1, C_2$  – are the constants.

Otherwise, if we express function  $U_z$ , depending on the given (arbitrary) functions  $U_x$  or  $U_y$ , the proper expression should be as below (here  $C_3$  – is the constant):

$$U_z = a \cdot y + \int_0^y \left[ \frac{\partial U_y}{\partial z} \right] dy + C_3.$$

The 2-nd of expressions of (1.4) could be represented in other form

$$U_y = c \cdot x + \int_0^x \left( \int_0^z \left[ \frac{\partial^2 U_z}{\partial x \partial y} \right] dz \right) dx + C_2
 \tag{1.5}$$

### **3. The flow with constant vortex generates a zero Laplacian.**

Let us present the equation of continuity for the components (1.4)-(1.5) of velocity field  $\{U_x, U_y, U_z\}$ , in the form below [1-2]

$$\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z} = 0,$$

$$\int_0^z \left[ \frac{\partial^2 U_z}{\partial x^2} + \frac{\partial^2 U_z}{\partial y^2} + \frac{\partial^2 U_z}{\partial z^2} \right] dz = 0,$$

$$\Rightarrow \Delta U_z = 0,$$

- it means that  $U_z$  - is a harmonic function [7]. Besides, taking into consideration the expression (1.4), it means that components  $U_x, U_y$  – are harmonic functions also.

#### **4. Conclusion.**

Let us generalize the Helmholtz paradox (“*if any vorticity arise in non-viscid newtonian fluid, such a vortex should have the constant strength all the time*”): - we proved that Vortex with constant angular velocity of rotation could be generated only by the flow with zero Laplacian (so, such a flow is not viscid for incompressible newtonian fluid).

Thus, we could conclude in addition to Helmholtz paradox: not only any vortex should be constant in the ideal (non-viscid) newtonian fluids, but if vortex is constant it means that such a flow is turned to be ideal (unviscid) due to Laplacian which is to be zero.

## References:

- [1]. Ladyzhenskaya, O.A. (1969), *The Mathematical Theory of viscous Incompressible Flow* (2nd ed.).
- [2]. Landau, L. D.; Lifshitz, E. M. (1987), *Fluid mechanics, Course of Theoretical Physics 6* (2nd revised ed.), Pergamon Press, ISBN 0-08-033932-8.
- [3]. Lighthill, M. J. (1986), *An Informal Introduction to Theoretical Fluid Mechanics*, Oxford University Press, ISBN 0-19-853630-5.
- [4]. Saffman, P. G. (1995), *Vortex Dynamics*, Cambridge University Press, ISBN 0-521-42058-X.
- [5]. Batchelor, G. K. (1967, reprinted in 2000), *An Introduction to Fluid Dynamics*, Cambridge University Press.
- [6]. Kundu, P; Cohen, I (2002), *Fluid Mechanics*, 2nd edition, Academic Press.
- [7]. Kamke E. (1971), *Hand-book for Ordinary Differential Eq.* Moscow: Science.