On Global Solution of Incompressible Navier-Stokes equations

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Abstract

The fluid equations, named after Claude-Louis Navier and George Gabriel Stokes, describe the motion of fluid substances. These equations arise from applying Newton's second law to fluid motion, together with the assumption that the stress in the fluid is the sum of a diffusing viscous term (proportional to the gradient of velocity) and a pressure term - hence describing viscous flow. Due to specific of NS equations they could be transformed to full/partial inhomogeneous parabolic differential equations: differential equations in respect of space variables and the full differential equation in respect of time variable and time dependent inhomogeneous part. Finally, orthogonal polynomials as the partial solutions of obtained Helmholtz equations were used for derivation of analytical solution of incompressible fluid equations in 1D, 2D and 3D space for rectangular boundary. Solution in 2D and 3D space for any shaped boundary was expressed in term of 2D and 3D global solution of Helmholtz equation accordantly.

1 Introduction

In physics, the fluid equations, named after Claude-Louis Navier and George Gabriel Stokes, describe fluid substances motion. These equations arise from applying Newton's second law to fluid motion, together with the assumption that the stress in the fluid is the sum of a diffusing viscous term (proportional to the gradient of velocity) and a pressure term - hence describing viscous flow. Equations were introduced in 1822 by the French engineer Claude Louis Marie Henri Navier [1] and successively re-obtained, by different arguments, by a several authors including Augustin-Louis Cauchy in 1823 [2], Simeon Denis Poisson in 1829, Adhemar Jean Claude Barre de Saint-Venant in 1837, and, finally, George Gabriel Stokes in 1845 [3]. Detailed and thorough analysis of the history of the fluid equations could be found in by Olivier Darrigol [4]. The invention of the digital computer led to many changes. John von Neumann, one of the CFD founding fathers, predicted already in 1946 that automatic computing machines' would replace the analytic solution of simplified flow equations by a numerical' solution of the full nonlinear flow equations for arbitrary geometries. Von Neumann suggested that this numerical approach would even make experimental fluid dynamics obsolete. Von Neumann's prediction did not fully come true, in the sense that both analytic theoretical and experimental research still coexist with CFD. Crucial properties of CFD methods such as consistency, stability and convergence need mathematical study [5].

Aims of this article are to propose new approach for solution of incompressible fluid equations. The article has three basic parts: first part explains how to solve NS in one dimension, second part extend solution to two-dimensional space and, finally, third part summarize with three-dimensional space.

2 Parabolic formulation of equations

Incompressible fluid equations are expressed as follow

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \quad (\mathbf{v}\cdot\nabla)\mathbf{v}\right) - \mu\Delta\mathbf{v} + \nabla p \quad = \quad f \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) \quad = \quad 0 \tag{2}$$

where equation (2) for incompressible flow reduces to $\frac{d\rho}{dt} = 0$ or $\rho = const$ due to $\nabla \cdot \mathbf{v} = 0$. Equations of fluid motion (1) could be expressed in full time derivative replacing covariant time derivative by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \quad (\mathbf{v} \cdot \nabla) \tag{3}$$

So, we obtain

$$\frac{d\mathbf{v}}{dt} - a^2 \Delta \mathbf{v} = \frac{1}{\rho}(-\nabla p + f) \tag{4}$$

3 inhomogeneous parabolic like equation for full time derivative, where $a = \sqrt{\mu/\rho}$. Tensor of the inner pressure of fluid for existing solution of velocities could be found by using of equations

$$\mathbf{p}^i_{in} = -p\mathbf{e}_i + f_i \mathbf{e}_i + \mu \nabla_i (\sum_{i=1}^{3} v^i \mathbf{e}_i) \tag{5}$$

where $\mathbf{e}_i$ are eigenvectors of corresponding coordinate system.

3 One dimensional inhomogeneous solution

Consider the initial-boundary value problem for $v = v(x,t)$

$$\begin{aligned} \frac{dv}{dt} - a^2 \Delta v &= \frac{1}{\rho}(-\nabla p + f) \text{ in } \Omega \times (0,\infty) && (6)\\ v(x,0) &= v_0(x) \; x \in \Omega && (7)\\ \frac{\partial v}{\partial n} &= 0 \text{ on } \partial\Omega \times (0,\infty) && (8) \end{aligned}$$

where $p = p(x,t)$ and $f = f(x,t)$, $\Omega \subset \mathbb{R}^n$, n the exterior unit normal at the smooth parts of $\partial\Omega$, a^2 a positive constant and $v_0(x)$ a given function.

So according to [6] equation (4), when x is scaled to $a = 1$, could be rewritten as follow

$$\frac{dv}{dt} = \frac{\partial^2 v}{\partial x^2} + Q(x,t), \;\; x \in \Omega, \;\; t > 0 \tag{9}$$

We expand v and Q in the eigenfunctions $\sin\left(\frac{n\pi x}{L}\right)$ on space $\Omega \in [0, L]$ where $sin(\frac{n\pi x}{L})$ and $sin(\frac{m\pi x}{L})$ functions orthogonality could be applied. So, we obtain

$$Q(x,t) = \sum_{n=1}^{\infty} q_n(t) \sin\left(\frac{n\pi x}{L}\right) \tag{10}$$

with

$$\begin{aligned} q_n(t) &= \frac{1}{I_1} \int_{\Omega} Q(x,t) \sin\left(\frac{n\pi x}{L}\right) dx && (11)\\ I_1 &= \int_{\Omega} \sin^2\left(\frac{n\pi x}{L}\right) dx = \frac{L}{2} && (12) \end{aligned}$$

and

$$v(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi x}{L}\right) \tag{13}$$

Thus we get the inhomogeneous ODE

$$\dot{u}_n(t) + \left(\frac{n\pi}{L}\right)^2 u_n(t) = q_n(t), \tag{14}$$

whose solution is

$$u_n(t) = u_n(0) e^{(-(n\pi/L)^2 t)} + \int_0^t q_n(\tau) e^{(-(n\pi/L)^2 (t-\tau))} d\tau \tag{15}$$

where

$$u_n(0) = \frac{1}{I_1} \int\limits_{\Omega} v_0(x) \sin{(\frac{n\pi x}{L})} dx \tag{16}$$

Again, we substitute all obtained equations into (13) and have

$$v(x,t) = \int\limits_{\Omega} v_0(s) (\sum_{n=1}^{\infty} \frac{1}{I_1} \sin{(\frac{n\pi s}{L})} \sin{(\frac{n\pi x}{L})} e^{(-(n\pi/L)^2 t)}) ds$$
$$+ \int\limits_{\Omega} ds \int_0^t Q(s,\tau) (\sum_{n=1}^{\infty} \frac{1}{I_1} \sin{(\frac{n\pi s}{L})} \sin{(\frac{n\pi x}{L})} e^{(-(n\pi/L)^2 (t-\tau))}) d\tau \tag{17}$$

Now we must apply continuity condition $\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x} v(x,t) = 0$. This is equation of extreme for coordinate x. Solving this equation gives extreme point x_{ex}. Finally, solution of 1D incompressible Navier-Stokes equation is

$$v(t) = \int\limits_{\Omega} v_0(s) (\sum_{n=1}^{\infty} \frac{1}{I_1} \sin{(\frac{n\pi s}{L})} \sin{(\frac{n\pi x_{ex}}{L})} \sin{(-(n\pi/L)^2 t)}) ds$$
$$+ \int\limits_{\Omega} ds \int_0^t Q(s,\tau) (\sum_{n=1}^{\infty} \frac{1}{I_1} \sin{(\frac{n\pi s}{L})} \sin{(\frac{n\pi x_{ex}}{L})} \sin{(-(n\pi/L)^2 (t-\tau))}) d\tau \tag{18}$$

If we investigate each point of fluid in moving coordinate system of this point, Galilean transform must by applied $\mathbf{v}(\mathbf{r}_0 + \mathbf{v}t, t)$ in case of this equality

$$\frac{dv^x(x,t)}{dt} = \frac{dv^x(x_0 + v^x t, t)}{dt} \equiv \frac{\partial v^x}{\partial t} + v^x \frac{\partial v^x}{\partial x}. \tag{19}$$

4 Two dimensional inhomogeneous solution

Consider the initial-boundary value problem for $v = v(x,y,t)$

$$\frac{dv^i}{dt} - a^2 \Delta v^i = \frac{1}{\rho}(-\nabla_i p + f_i) \text{ in } \Omega \times (0,\infty) \tag{20}$$
$$v^i(x,y,0) = v_0^i(x,y) \; x,y \in \Omega \tag{21}$$
$$\frac{\partial v^i}{\partial n} = 0 \text{ on } \partial\Omega \times (0,\infty) \tag{22}$$

where $p = p(x,y,t)$ and $f = f(x,y,t)$, $\Omega \subset \mathbb{R}^{2n}$, n the exterior unit normal at the smooth parts of $\partial\Omega$, a^2 a positive constant and $v_0^x(x,y), v_0^y(x,y)$ a given function.

So, when x and y scale was determined to $a = 1$, equation (4) could be rewritten as follow

$$\frac{dv^i}{dt} = \frac{\partial^2 v^i}{\partial x^2} + \frac{\partial^2 v^i}{\partial y^2} + Q^i(x,y,t), \; x,y \in \Omega, \; t > 0 \tag{23}$$

4.1 Rectangular boundary

We will expand v and Q in base of orthogonal functions. At first, we must find vector potential $\mathbf{Q}_o$ of $\mathbf{Q}$ so that $\mathbf{Q} = \nabla \times \mathbf{Q}_o$. To any potential $\mathbf{Q}_o$, an arbitrary gradient field can be added to get another vector potential with the same curl everywhere. For simplicity, the second component of $\mathbf{Q}_o$ can be taken to be zero, since a gradient field can take care of that if needed. This means that some equations simplify

$$-\frac{\partial Q_o^1}{\partial x_1} = Q^1 \tag{24}$$
$$\frac{\partial Q_o^1}{\partial x_2} = Q^2 \tag{25}$$

These can be solved sequentially, namely Q_o^1 is determined using the first equation up to a function of x_1, while Q_o^1 is determined by the second equation, up to a function of x_2. Than we expand $\mathbf{Q}_o$ as follow

$$Q_o^x = \sum_{m,n=1}^{\infty} q_{mn}^x(t) \sin(\frac{n\pi x}{L_x}) \sin(\frac{m\pi y}{L_y}) \tag{26}$$

Now we could express vector $\mathbf{Q}$ as follow

$$Q^x = -\sum_{m,n=1}^{\infty} q_{mnp}^x(t) \frac{\partial}{\partial x} \sin(\frac{n\pi x}{L_x}) \sin(\frac{m\pi y}{L_y}) \tag{27}$$

$$Q^y = \sum_{m,n=1}^{\infty} q_{mnp}^x(t) \frac{\partial}{\partial y} \sin(\frac{n\pi x}{L_x}) \sin(\frac{m\pi y}{L_y}) \tag{28}$$

where

$$q_{mn}^x(t) = \frac{1}{I_{mn}} \iint_{\Omega} ds_1 ds_2 Q_o^1(s_1, s_2, t) \sin(\frac{n\pi s_1}{L_{s1}}) \sin(\frac{m\pi s_2}{L_{s2}}) \tag{29}$$

and

$$I_{mn} = \iint_{\Omega} d\Omega (\sin(\frac{n\pi s_1}{L_{s1}}) \sin(\frac{m\pi s_2}{L_{s2}}))^2 \tag{30}$$

The same way we will find vector $\mathbf{v}$ for $t = 0$ by using equations (24) and (25). We must find vector potential $\mathbf{v}_o$ of $\mathbf{v}$ so that $\mathbf{v} = \nabla \times \mathbf{v}_o$. Than we expand $\mathbf{v}_o$ as follow

$$v_o^x = \sum_{m,n=1}^{\infty} u_{mn}^x(0) \sin(\frac{n\pi x}{L_x}) \sin(\frac{m\pi y}{L_y}) \tag{31}$$

Now we could express vector $\mathbf{v}$ for any t as follow

$$v^x = -\sum_{m,n=1}^{\infty} u_{mn}^x(t) \frac{\partial}{\partial x} \sin(\frac{n\pi x}{L_x}) \sin(\frac{m\pi y}{L_y}) \tag{32}$$

$$v^y = \sum_{m,n=1}^{\infty} u_{mn}^x(t) \frac{\partial}{\partial y} \sin(\frac{n\pi x}{L_x}) \sin(\frac{m\pi y}{L_y}) \tag{33}$$

Thus we get the inhomogeneous ODE

$$\dot{u}_{mn}^x(t) + k_{mn}^2 u_{mn}^x(t) = q_{mn}^x(t), \tag{34}$$

$$k_{mn}^2 = \left(\frac{n\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2 \tag{35}$$

whose solution is

$$u_{mn}^x(t) = u_{mn}^x(0) e^{(-k_{mn}^2 t)} + \int_0^t q_{mn}^x(\tau) e^{(-k_{mn}^2 (t-\tau))} d\tau \tag{36}$$

where

$$u_{mn}^x(0) = \frac{1}{I_{mn}} \iint_{\Omega} v_{0o}^x(s_1, s_2) \sin(\frac{n\pi s_1}{L_{s1}}) \sin(\frac{m\pi s_2}{L_{s2}}) ds_1 ds_2 \tag{37}$$

Again, we substitute all obtained equations into (51) and have

$$\begin{aligned} v^i(x,y,t) &= (-1)^i \iint_{\Omega} v_{0o}^x(s_1,s_2)(\sum_{m,n=1}^{\infty} \tfrac{1}{I_{mn}} S(ns_1, ms_2) \tfrac{\partial}{\partial x_j} S(nx, my) e^{(-k_{mnp}^2 t)}) ds_1 ds_2 \\ &+ (-1)^i \iint_{\Omega} ds_1 ds_2 \int_0^t Q_o^x(s_1, s_2, \tau)(\sum_{m,n=1}^{\infty} \tfrac{1}{I_{mn}} S(ns_1, ms_2) \tfrac{\partial}{\partial x_j} S(nx, my) e^{(-k_{mnp}^2 (t-\tau))}) d\tau \end{aligned} \tag{38}$$

$\forall [i,j] \in [[x,y],[y,x]]$ where

$$S(nx_1, mx_2) = \sin(\tfrac{n\pi x}{L_x}) \sin(\tfrac{m\pi y}{L_y}) \tag{39}$$

If we investigate each point of fluid in moving coordinate system of this point, Galilean transform must by applied $\mathbf{v}(\mathbf{r}_0 + \mathbf{v}t, t)$ in case of this equalities

$$\frac{dv^x(x,y,t)}{dt} = \frac{dv^x(x_0 + v^x t, y_0 + v^y t, t)}{dt} \equiv \frac{\partial v^x}{\partial t} + v^x \frac{\partial v^x}{\partial x} + v^y \frac{\partial v^x}{\partial y}, \tag{40}$$

$$\frac{dv^y(x,y,t)}{dt} = \frac{dv^y(x_0 + v^x t, y_0 + v^y t, t)}{dt} \equiv \frac{\partial v^y}{\partial t} + v^x \frac{\partial v^y}{\partial x} + v^y \frac{\partial v^y}{\partial y}. \tag{41}$$

4.2 Any shaped boundary

For any shaped boundary $\partial\Omega$ we will use the similar equations (27), (28)

$$Q^i(x_1,x_2,t) = (-1)^i \sum_{m,n=1}^{\infty} q^1_{mn}(t) \frac{\partial}{\partial x_j} H_{\partial\Omega,k}(nx_1, mx_2) \tag{42}$$

where $i \neq j$ and equations (32) and (33) for velocities

$$v^i(x_1,x_2,t) = (-1)^i \sum_{m,n=1}^{\infty} u^1_{mn}(t) \frac{\partial}{\partial x_j} H_{\partial\Omega,k}(nx_1, mx_2) \tag{43}$$

where $H_{\partial\Omega,k}(nx, my)$ are partial solutions of Helmholtz 2D equation for given boundary $\partial\Omega$. and could be taken for example from [8]. The equations of inverse curl operator in any coordinate system could be obtained by solving equations (24) and (25). So, equation (38) transforms to

$$v^i(x_1,x_2,t) = (-1)^i \sum_{m,n=1}^{\infty} \frac{(v^1_{0mnf} + Q^1_{mn})}{I_{mn}} \frac{\partial}{\partial x_j} H_{\partial\Omega,k}(nx_1, mx_2) e^{(-k^2_{mnp} t)} \tag{44}$$

$$v^1_{0mnf} = \iint\limits_{\Omega} v^1_{0o}(s_1,s_2) H_{\partial\Omega,k}(ns_1, ms_2) d\Omega \tag{45}$$

$$Q^1_{mn} = \iint\limits_{\Omega} d\Omega \int_0^t Q^1_o(s_1,s_2,\tau) H_{\partial\Omega,k}(ns_1, ms_2) e^{(k^2_{mn}\tau)} d\tau \tag{46}$$

$$I_{mn} = \iint\limits_{\Omega} d\Omega \left(H_{\partial\Omega,k}(ns_1, ms_2)\right)^2 \tag{47}$$

where $\forall [i,j] \in [[x_1,x_2],[x_2,x_1]]$ and denotes coordinate indexes. If we investigate each point of fluid in moving coordinate system of this point, Galilean transform must by applied $\mathbf{v}(\mathbf{r}_0 + \mathbf{v}t, t)$.

5 Three dimensional inhomogeneous solution

Consider the initial-boundary value problem for $v = v(x,y,z,t)$

$$\frac{dv^i}{dt} - a^2 \Delta v^i = \frac{1}{\rho}(-\nabla_i p + f_i) \text{ in } \Omega \times (0,\infty) \tag{48}$$

$$v^i(x,y,z,0) = v^i_0(x,y,z) \;\; x,y,z \in \Omega \tag{49}$$

$$\frac{\partial v^i}{\partial n} = 0 \text{ on } \partial\Omega \times (0,\infty) \tag{50}$$

where $p = p(x,y,z,t)$ and $f = f(x,y,z,t)$, $\Omega \subset \mathbb{R}^{3n}$, n the exterior unit normal at the smooth parts of $\partial\Omega$, a^2 a positive constant and $v^x_0(x,y,z), v^y_0(x,y,z), v^z_0(x,y,z)$ a given function.

So, when x, y and z scale was determined to $a = 1$, equation (4) could be rewritten as follow

$$\frac{dv^i}{dt} = \frac{\partial^2 v^i}{\partial x^2} + \frac{\partial^2 v^i}{\partial y^2} + \frac{\partial^2 v^i}{\partial z^2} + Q^i(x,y,z,t), \;\; x,y,z \in \Omega, \;\; t > 0. \tag{51}$$

5.1 Rectangular boundary

We will expand v and Q in base of orthogonal functions. At first, we must find vector potential $\mathbf{Q}_o$ of $\mathbf{Q}$ so that $\mathbf{Q} = \nabla \times \mathbf{Q}_o$. To any potential $\mathbf{Q}_o$, an arbitrary gradient field can be added to get another vector potential with the same curl everywhere. For simplicity, the third component of $\mathbf{Q}_o$ can be taken to be zero, since a gradient field can take care of that if needed. This means that some equations simplify

$$-\frac{\partial Q_o^2}{\partial x_3} = Q^1 \tag{52}$$

$$\frac{\partial Q_o^1}{\partial x_3} = Q^2 \tag{53}$$

$$\frac{\partial Q_o^2}{\partial x_1} - \frac{\partial Q_o^1}{\partial x_2} = Q^3 \tag{54}$$

These can be solved sequentially, namely Q_o^2 is determined using the first equation up to a function of x_1 and x_3, while Q_o^1 is determined by the second equation, up to a function of x_2 and x_3. The third equation can then be solved provided our solvability conditions holds. Than we expand $\mathbf{Q}_o$ as follow

$$Q_o^i = \sum_{m,n,p=1}^{\infty} q_{mnp}^i(t) \sin(\frac{n\pi x}{L_x}) \sin(\frac{m\pi y}{L_y}) \sin(\frac{p\pi z}{L_z}),\ \forall i \in [1,2] \tag{55}$$

where $q_{mnp}^z = 0$. Now we could express vector $\mathbf{Q}$ as follow

$$Q^x = \sum_{m,n,p=1}^{\infty} (q_{mnp}^y(t)\frac{\partial}{\partial y} - q_{mnp}^z(t)\frac{\partial}{\partial z}) \sin(\frac{n\pi x}{L_x}) \sin(\frac{m\pi y}{L_y}) \sin(\frac{p\pi z}{L_z}) \tag{56}$$

$$Q^y = \sum_{m,n,p=1}^{\infty} (q_{mnp}^z(t)\frac{\partial}{\partial z} - q_{mnp}^x(t)\frac{\partial}{\partial x}) \sin(\frac{n\pi x}{L_x}) \sin(\frac{m\pi y}{L_y}) \sin(\frac{p\pi z}{L_z}) \tag{57}$$

$$Q^z = \sum_{m,n,p=1}^{\infty} (q_{mnp}^x(t)\frac{\partial}{\partial x} - q_{mnp}^y(t)\frac{\partial}{\partial y}) \sin(\frac{n\pi x}{L_x}) \sin(\frac{m\pi y}{L_y}) \sin(\frac{p\pi z}{L_z}) \tag{58}$$

where

$$q_{mnp}^i(t) = \frac{1}{I_{mnp}} \iiint\limits_{\Omega} ds_1 ds_2 ds_3 Q_o^i(s_1, s_2, s_3, t) \sin(\frac{n\pi s_1}{L_{s1}}) \sin(\frac{m\pi s_2}{L_{s2}}) \sin(\frac{p\pi s_3}{L_{s3}}) \tag{59}$$

and

$$I_{mnp} = \iiint\limits_{\Omega} ds_1 ds_2 ds_3 (\sin(\frac{n\pi s_1}{L_{s1}}) \sin(\frac{m\pi s_2}{L_{s2}}) \sin(\frac{p\pi s_3}{L_{s3}}))^2 \tag{60}$$

The same way we will find vector $\mathbf{v}$ for $t = 0$ by using equation (52), (53) and (54). Than we expand $\mathbf{v}_o$ as follow

$$v_o^i = \sum_{m,n,p=1}^{\infty} u_{mnp}^i(0) \sin(\frac{n\pi x}{L_x}) \sin(\frac{m\pi y}{L_y}) \sin(\frac{p\pi z}{L_z}), \forall i \in [1,2] \tag{61}$$

Now we could express vector $\mathbf{v}$ for any t as follow

$$v^x = \sum_{m,n,p=1}^{\infty} (u_{mnp}^y(t)\frac{\partial}{\partial y} - u_{mnp}^z(t)\frac{\partial}{\partial z}) \sin(\frac{n\pi x}{L_x}) \sin(\frac{m\pi y}{L_y}) \sin(\frac{p\pi z}{L_z}) \tag{62}$$

$$v^y = \sum_{m,n,p=1}^{\infty} (u_{mnp}^z(t)\frac{\partial}{\partial z} - u_{mnp}^x(t)\frac{\partial}{\partial x}) \sin(\frac{n\pi x}{L_x}) \sin(\frac{m\pi y}{L_y}) \sin(\frac{p\pi z}{L_z}) \tag{63}$$

$$v^z = \sum_{m,n,p=1}^{\infty} (u_{mnp}^x(t)\frac{\partial}{\partial x} - u_{mnp}^y(t)\frac{\partial}{\partial y}) \sin(\frac{n\pi x}{L_x}) \sin(\frac{m\pi y}{L_y}) \sin(\frac{p\pi z}{L_z}) \tag{64}$$

where $u_{mnp}^z(t) = 0$. Thus we get the inhomogeneous ODE

$$\dot{u}_{mnp}^i(t) + k_{mnp}^2 u_{mnp}^i(t) = q_{mnp}^i(t), \tag{65}$$

$$k_{mnp}^2 = \left(\frac{n\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{p\pi}{L_y}\right)^2 \tag{66}$$

whose solution is

$$u^i_{mnp}(t) = u^i_{mnp}(0)e^{(-k^2_{mnp}t)} + \int_0^t q^i_{mnp}(\tau)e^{(-k^2_{mnp}(t-\tau))}d\tau \tag{67}$$

where

$$u^i_{mnp}(0) = \frac{1}{I_{mnp}}\iiint\limits_{\Omega} v^i_{0o}(s_1,s_2,s_3)\sin(\frac{n\pi s_1}{L_{s1}})\sin(\frac{m\pi s_2}{L_{s2}})\sin(\frac{p\pi s_3}{L_{s3}})ds_1ds_2ds_3 \tag{68}$$

Again, we substitute all obtained equations into (51) and have

$$\begin{aligned}
v^i(x,y,z,t) = &\iiint\limits_{\Omega} v^j_{0o}(s_1,s_2,s_3)(\sum_{m,n,p=1}^{\infty}\tfrac{1}{I_{mnp}}S(ns_1,ms_2,ps_3)\tfrac{\partial}{\partial x_j}S(nx,my,pz)e^{(-k^2_{mnp}t)})ds_1ds_2ds_3\\
&-\iiint\limits_{\Omega} v^l_{0o}(s_1,s_2,s_3)(\sum_{m,n,p=1}^{\infty}\tfrac{1}{I_{mnp}}S(ns_1,ms_2,s_3)\tfrac{\partial}{\partial x_l}S(nx,my,pz)e^{(-k^2_{mnp}t)})ds_1ds_2ds_3\\
&+\iiint\limits_{\Omega} ds_1ds_2ds_3\int_0^t Q^i_o(s_1,s_2,s_3,\tau)(\sum_{m,n,p=1}^{\infty}\tfrac{1}{I_{mnp}}S(ns_1,ms_2,ps_3)\tfrac{\partial}{\partial x_j}S(nx,my,pz)e^{(-k^2_{mnp}(t-\tau))})d\tau\\
&-\iiint\limits_{\Omega} ds_1ds_2ds_3\int_0^t Q^i_o(s_1,s_2,s_3,\tau)(\sum_{m,n,p=1}^{\infty}\tfrac{1}{I_{mnp}}S_{mnp}(ns_1,ms_2,ps_3)\tfrac{\partial}{\partial x_l}S(nx,my,pz)e^{(-k^2_{mnp}(t-\tau))})d\tau
\end{aligned} \tag{69}$$

$\forall[i,j,k] \in [[x,y,z],[y,z,x],[z,x,y]]$ where

$$S(nx_1,mx_2,px_3) = \sin(\tfrac{n\pi x}{L_x})\sin(\tfrac{m\pi y}{L_y})\sin(\tfrac{p\pi z}{L_z}) \tag{70}$$

If we investigate each point of fluid in moving coordinate system of this point, Galilean transform must by applied $\mathbf{v}(\mathbf{r}_0 + \mathbf{v}t, t)$ in case of this equalities

$$\frac{dv^x(x,y,z,t)}{dt} = \frac{dv^x(x_0+v^xt,y_0+v^yt,z_0+v^zt,t)}{dt} \equiv \frac{\partial v^x}{\partial t} + v^x\frac{\partial v^x}{\partial x} + v^y\frac{\partial v^x}{\partial y} + v^z\frac{\partial v^x}{\partial z}, \tag{71}$$

$$\frac{dv^y(x,y,z,t)}{dt} = \frac{dv^y(x_0+v^xt,y_0+v^yt,z_0+v^zt,t)}{dt} \equiv \frac{\partial v^y}{\partial t} + v^x\frac{\partial v^y}{\partial x} + v^y\frac{\partial v^y}{\partial y} + v^z\frac{\partial v^y}{\partial z}, \tag{72}$$

$$\frac{dv^z(x,y,z,t)}{dt} = \frac{dv^z(x_0+v^xt,y_0+v^yt,z_0+v^zt,t)}{dt} \equiv \frac{\partial v^z}{\partial t} + v^x\frac{\partial v^z}{\partial x} + v^y\frac{\partial v^z}{\partial y} + v^z\frac{\partial v^z}{\partial z}. \tag{73}$$

5.2 Any shaped boundary

For any shaped boundary $\partial\Omega$ we will use the similar equations (56), (57), (58)

$$Q^i(x_1,x_2,x_3,t) = \sum_{m,n,p=1}^{\infty}\left(q^j_{mnp}(t)\frac{\partial}{\partial x_j} - q^l_{mnp}(t)\frac{\partial}{\partial x_l}\right)H_{\partial\Omega,k}(nx_1,mx_2,px_3) \tag{74}$$

where $\forall[i,j,l] \in [[x1,x2,x3],[x2,x3,x1],[x3,x1,x2]]$ and equations (62), (63) and (64) for velocities

$$v^i(x_1,x_2,x_3,t) = \sum_{m,n,p=1}^{\infty}\left(u^j_{mnp}(t)\frac{\partial}{\partial x_j} - u^l_{mnp}(t)\frac{\partial}{\partial x_l}\right)H_{\partial\Omega,k}(nx_1,mx_2,px_3) \tag{75}$$

where $H_{\partial\Omega,k}(nx,my,pz)$ are partial solutions of Helmholtz 3D equation for given boundary $\partial\Omega$. and could be taken for example from [8]. The equations of inverse curl operator in any coordinate system could be obtained from [9]. So, equation (69) transforms to

$$\begin{aligned}
v^i(x_1,x_2,x_3,t) &= \sum_{m,n,p=1}^{\infty}(\frac{(v^j_{0mnpf}+Q^j_{mnp})}{I_{mnp}}\frac{\partial}{\partial x_j} - \frac{(v^l_{0mnpf}+Q^l_{mnp})}{I_{mnp}}\frac{\partial}{\partial x_l})H_{\partial\Omega,k}(nx_1,mx_2,px_3)e^{(-k^2_{mnp}t)} && (76)\\
v^i_{0mnpf} &= \iiint\limits_{\Omega} v^i_{0o}(s_1,s_2,s_3)H_{\partial\Omega,k}(ns_1,ms_2,ps_3)d\Omega && (77)\\
Q^i_{mnp} &= \iiint\limits_{\Omega} d\Omega\int_0^t Q^i_o(s_1,s_2,s_3,\tau)H_{\partial\Omega,k}(ns_1,ms_2,ps_3)e^{(k^2_{mnp}\tau)}d\tau && (78)\\
I_{mnp} &= \iiint\limits_{\Omega} d\Omega\,(H_{\partial\Omega,k}(ns_1,ms_2,ps_3))^2 && (79)
\end{aligned}$$

where $\forall[i,j,k] \in [[x_1,x_2,x_3],[x_2,x_3,x_1],[x_3,x_1,x_2]]$ and denotes coordinate indexes. If we investigate each point of fluid in moving coordinate system of this point, Galilean transform must by applied $\mathbf{v}(\mathbf{r}_0 + \mathbf{v}t, t)$.

6 Conclusions

Due to the form of fluid equations they could be transformed into the full/partial inhomogeneous parabolic differential equations: partial differential equations in respect to space variables and full differential equations in respect to the time variable and inhomogeneous time dependent part. Velocity and outer forces density components were expressed in form of curl for obtaining solution satisfying continuity condition. Orthogonal polynomials as the partial solutions of obtained Helmholtz equations were used for derivation of analytical solution of velocities for incompressible fluid in 1D, 2D and 3D space for rectangular boundary. Solution in 2D and 3D space for any shaped boundary was expressed in term of 2D and 3D global solution of Helmholtz equation accordantly.

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