

# On Global Solution of Incompressible Navier-Stokes equations

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## Abstract

The fluid equations, named after Claude-Louis Navier and George Gabriel Stokes, describe the motion of fluid substances. These equations arise from applying Newton's second law to fluid motion, together with the assumption that the stress in the fluid is the sum of a diffusing viscous term (proportional to the gradient of velocity) and a pressure term - hence describing viscous flow. Due to specific of NS equations they could be transformed to full/partial inhomogeneous parabolic differential equations: differential equations in respect of space variables and the full differential equation in respect of time variable and time dependent inhomogeneous part. Finally, orthogonal polynomials as the partial solutions of obtained Helmholtz equations were used for derivation of analytical solution of incompressible fluid equations in 1D, 2D and 3D space for rectangular boundary. Solution in 3D space for any shaped boundary is expressed in term of 3D global solution of 3D Helmholtz equation accordantly.

## 1 Introduction

In physics, the fluid equations, named after Claude-Louis Navier and George Gabriel Stokes, describe fluid substances motion. These equations arise from applying Newton's second law to fluid motion, together with the assumption that the stress in the fluid is the sum of a diffusing viscous term (proportional to the gradient of velocity) and a pressure term - hence describing viscous flow. Equations were introduced in 1822 by the French engineer Claude Louis Marie Henri Navier [1] and successively re-obtained, by different arguments, by a several authors including Augustin-Louis Cauchy in 1823 [2], Simeon Denis Poisson in 1829, Adhemar Jean Claude Barre de Saint-Venant in 1837, and, finally, George Gabriel Stokes in 1845 [3]. Detailed and thorough analysis of the history of the fluid equations could be found in by Olivier Darrigol [4]. The invention of the digital computer led to many changes. John von Neumann, one of the CFD founding fathers, predicted already in 1946 that automatic computing machines' would replace the analytic solution of simplified flow equations by a numerical' solution of the full nonlinear flow equations for arbitrary geometries. Von Neumann suggested that this numerical approach would even make experimental fluid dynamics obsolete. Von Neumann's prediction did not fully come true, in the sense that both analytic theoretical and experimental research still coexist with CFD. Crucial properties of CFD methods such as consistency, stability and convergence need mathematical study [5].

Aims of this article are to propose new approach for solution of incompressible fluid equations. The article has three basic parts: first part explains how to solve NS in one dimension, second part extend solution to two-dimensional space and, finally, third part summarize with three-dimensional space.

## 2 Parabolic formulation of equations

Incompressible fluid equations are expressed as follow

$$\rho \left( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) - \mu \Delta \mathbf{v} + \nabla p = f \quad (1)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (2)$$

where equation (2) for incompressible flow reduces to  $\frac{d\rho}{dt} = 0$  or  $\rho = \text{const}$  due to  $\nabla \cdot \mathbf{v} = 0$ . Equations of fluid motion (1) could be expressed in full time derivative replacing covariant time derivative by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla) \quad (3)$$

So, we obtain

$$\frac{d\mathbf{v}}{dt} - a^2 \Delta \mathbf{v} = \frac{1}{\rho} (-\nabla p + f) \quad (4)$$

3 inhomogeneous parabolic like equation for full time derivative, where  $a = \sqrt{\mu/\rho}$ .

### 3 One dimensional inhomogeneous solution

Consider the initial-boundary value problem for  $v = v(x, t)$

$$\frac{dv}{dt} - a^2 \Delta v = \frac{1}{\rho} (-\nabla p + f) \text{ in } \Omega \times (0, \infty) \quad (5)$$

$$v(x, 0) = v_0(x) \quad x \in \Omega \quad (6)$$

$$\frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega \times (0, \infty) \quad (7)$$

where  $p = p(x, t)$  and  $f = f(x, t)$ ,  $\Omega \subset \mathbb{R}^n$ ,  $\mathbf{n}$  the exterior unit normal at the smooth parts of  $\partial\Omega$ ,  $a^2$  a positive constant and  $v_0(x)$  a given function.

So according to [6] equation (4), when  $x$  is scaled to  $a = 1$ , could be rewritten as follow

$$\frac{dv}{dt} = v_{xx} + Q(x, t), \quad x \in \Omega, \quad t > 0 \quad (8)$$

We expand  $v$  and  $Q$  in the eigenfunctions  $\sin(\frac{n\pi x}{L})$  on space  $\Omega \in [0, L]$  where  $\sin(\frac{n\pi x}{L})$  and  $\sin(\frac{m\pi x}{L})$  functions orthogonality could be applied. So, we obtain

$$Q(x, t) = \sum_{n=1}^{\infty} q_n(t) \sin\left(\frac{n\pi x}{L}\right) \quad (9)$$

with

$$q_n(t) = \frac{1}{I_1} \int_{\Omega} Q(x, t) \sin\left(\frac{n\pi x}{L}\right) dx \quad (10)$$

$$I_1 = \int_{\Omega} \sin^2\left(\frac{n\pi x}{L}\right) dx = \frac{L}{2} \quad (11)$$

and

$$v(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi x}{L}\right) \quad (12)$$

Thus we get the inhomogeneous ODE

$$\dot{u}_n(t) + \left(\frac{n\pi}{L}\right)^2 u_n(t) = q_n(t), \quad (13)$$

whose solution is

$$u_n(t) = u_n(0) \exp(-(n\pi/L)^2 t) + \int_0^t q_n(\tau) \exp(-(n\pi/L)^2 (t - \tau)) d\tau \quad (14)$$

where

$$u_n(0) = \frac{1}{I_1} \int_{\Omega} v_0(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad (15)$$

Again, we substitute all obtained equations into (12) and have

$$\begin{aligned} v(x, t) &= \int_{\Omega} v_0(s) \left( \sum_{n=1}^{\infty} \frac{1}{I_1} \sin\left(\frac{n\pi s}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \exp(-(n\pi/L)^2 t) \right) ds \\ &+ \int_{\Omega} ds \int_0^t Q(s, \tau) \left( \sum_{n=1}^{\infty} \frac{1}{I_1} \sin\left(\frac{n\pi s}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \exp(-(n\pi/L)^2 (t - \tau)) \right) d\tau \end{aligned} \quad (16)$$

Now we must apply continuity condition  $\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x} v(x, t) = 0$ . This is equation of extreme for coordinate  $x$ . Solving this equation gives extreme point  $x_{ex}$ . Finally, solution of 1D incompressible Navier-Stokes equation is

$$v(t) = \int_{\Omega} v_0(s) \left( \sum_{n=1}^{\infty} \frac{1}{I_1} \sin\left(\frac{n\pi s}{L}\right) \sin\left(\frac{n\pi x_{ex}}{L}\right) \exp(-(n\pi/L)^2 t) \right) ds \\ + \int_{\Omega} ds \int_0^t Q(s, \tau) \left( \sum_{n=1}^{\infty} \frac{1}{I_1} \sin\left(\frac{n\pi s}{L}\right) \sin\left(\frac{n\pi x_{ex}}{L}\right) \exp(-(n\pi/L)^2 (t - \tau)) \right) d\tau \quad (17)$$

If we investigate each point of fluid in moving coordinate system of this point, Galilean transform must be applied  $\mathbf{v}(\mathbf{r}_0 - \mathbf{v}t, t)$ .

## 4 Two dimensional inhomogeneous solution

Consider the initial-boundary value problem for  $v = v(x, y, t)$

$$\frac{dv^i}{dt} - a^2 \Delta v^i = \frac{1}{\rho} (-\nabla_i p + f_i) \text{ in } \Omega \times (0, \infty) \quad (18)$$

$$v^i(x, y, 0) = v_0^i(x, y) \quad x, y \in \Omega \quad (19)$$

$$\frac{\partial v^i}{\partial n} = 0 \text{ on } \partial\Omega \times (0, \infty) \quad (20)$$

where  $p = p(x, y, t)$  and  $f = f(x, y, t)$ ,  $\Omega \subset \mathbb{R}^{2n}$ ,  $\mathbf{n}$  the exterior unit normal at the smooth parts of  $\partial\Omega$ ,  $a^2$  a positive constant and  $v_0^x(x, y), v_0^y(x, y)$  a given function.

So, when  $x$  and  $y$  scale was determined to  $a = 1$ , equation (4) could be rewritten as follow

$$\frac{dv^i}{dt} = v_{xx}^i + v_{yy}^i + Q^i(x, y, t), \quad x, y \in \Omega, \quad t > 0 \quad (21)$$

### 4.1 Rectangular boundary

We expand  $v$  and  $Q$  in the eigenfunctions  $\exp(\frac{jn\pi x}{L_x}) \exp(\frac{jm\pi y}{L_y})$  on space  $\Omega \in [0, L_x] \times [0, L_y]$  where  $\exp(\frac{jn\pi x}{L_x}) \exp(\frac{jm\pi y}{L_y})$  and  $\exp(\frac{jn'\pi x}{L_x}) \exp(\frac{jm'\pi y}{L_y})$  functions orthogonality could be applied, where  $j = \sqrt{-1}$ . So, we obtain

$$Q^i(x, y, t) = \sum_{m,n=1}^{\infty} q_{mn}^i(t) \exp\left(\frac{jn\pi x}{L_x}\right) \exp\left(\frac{jm\pi y}{L_y}\right) \quad (22)$$

with

$$q_{mn}^i(t) = \frac{1}{I_2} \iint_{\Omega} Q^i(x, y, t) \exp\left(\frac{jn\pi x}{L_x}\right) \exp\left(\frac{jm\pi y}{L_y}\right) dx dy \quad (23)$$

$$I_2 = \iint_{\Omega} \left( \exp\left(\frac{jn\pi x}{L_x}\right) \exp\left(\frac{jm\pi y}{L_y}\right) \right)^2 dx dy \quad (24)$$

and

$$v^i(x, y, t) = \sum_{m,n=1}^{\infty} u_{mn}^i(t) \exp\left(\frac{jn\pi x}{L_x}\right) \exp\left(\frac{jm\pi y}{L_y}\right) \quad (25)$$

Thus we get the inhomogeneous ODE

$$\dot{u}_{mn}^i(t) + k_{mn}^2 u_{mn}^i(t) = q_{mn}^i(t), \quad (26)$$

$$k_{mn}^2 = \left(\frac{n\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2 \quad (27)$$

whose solution is

$$u_{mn}^i(t) = u_{mn}^i(0) \exp(-k_{mn}^2 t) + \int_0^t q_{mn}^i(\tau) \exp(-k_{mn}^2 (t - \tau)) d\tau \quad (28)$$

where

$$u_{mn}^i(0) = \frac{1}{I_2} \iint_{\Omega} v_0^i(x, y) \exp\left(\frac{jn\pi x}{L_x}\right) \exp\left(\frac{jm\pi y}{L_y}\right) dx dy \quad (29)$$

Again, we substitute all obtained equations into (25) and have

$$\begin{aligned} v^i(x, y, t) &= \iint_{\Omega} v_0^i(s', s) \left( \sum_{m,n=1}^{\infty} \frac{1}{I_2} \exp\left(\frac{jn\pi s}{L_x} + \frac{jm\pi s'}{L_y} + \frac{jn\pi x}{L_x} + \frac{jm\pi y}{L_y} - k_{mn}^2 t\right) \right) ds' ds \\ &+ \iint_{\Omega} ds' ds \int_0^t Q^i(s', s, \tau) \left( \sum_{m,n=1}^{\infty} \frac{1}{I_2} \exp\left(\frac{jn\pi s}{L_x} + \frac{jm\pi s'}{L_y} + \frac{jn\pi x}{L_x} + \frac{jm\pi y}{L_y} - k_{mn}^2 (t - \tau)\right) \right) d\tau \end{aligned} \quad (30)$$

Now we must apply continuity condition  $\nabla \cdot \mathbf{v} = \frac{\partial v^x(x,y,t)}{\partial x} + \frac{\partial v^y(x,y,t)}{\partial y} = 0$ . So we obtain relation conditions between  $v_{0mnf}^x$  and  $v_{0mnf}^y$ ,  $Q_{mnf}^x$  and  $Q_{mnf}^y$

$$(v_{0nmf}^x + Q_{nmf}^x) \frac{n\pi}{L_x} + (v_{0nmf}^y + Q_{nmf}^y) \frac{m\pi}{L_y} = 0 \quad (31)$$

where  $v_{0f}^i, Q_f^i$  are

$$v_{0nmf}^i = \frac{1}{I_2} \iint_{\Omega} v_0^i(s', s) \exp\left(\frac{jn\pi s}{L_x} + \frac{jm\pi s'}{L_y}\right) ds' ds \quad (32)$$

$$Q_{nmf}^i = \frac{1}{I_2} \iint_{\Omega} ds' ds \int_0^t Q^i(s', s, \tau) \exp\left(\frac{jn\pi s}{L_x} + \frac{jm\pi s'}{L_y} + k_{mn}^2 \tau\right) d\tau \quad (33)$$

Finally, solutions of 2D incompressible Navier-Stokes equations are

$$v^x(x, y, t) = \sum_{m,n=1}^{\infty} (v_{0mnf}^x + Q_{mnf}^x) \exp\left(-\frac{j\pi m x (v_{0mnf}^y + Q_{mnf}^y)}{L_y (v_{0mnf}^x + Q_{mnf}^x)} - \frac{j\pi n y (v_{0mnf}^x + Q_{mnf}^x)}{L_x (v_{0mnf}^y + Q_{mnf}^y)} - k_{mn}^2 t\right), \quad (34)$$

$$v^y(x, y, t) = \sum_{m,n=1}^{\infty} (v_{0mnf}^y + Q_{mnf}^y) \exp\left(-\frac{j\pi m x (v_{0mnf}^y + Q_{mnf}^y)}{L_y (v_{0mnf}^x + Q_{mnf}^x)} - \frac{j\pi n y (v_{0mnf}^x + Q_{mnf}^x)}{L_x (v_{0mnf}^y + Q_{mnf}^y)} - k_{mn}^2 t\right). \quad (35)$$

If we investigate each point of fluid in moving coordinate system of this point, Galilean transform must be applied  $\mathbf{v}(\mathbf{r}_0 - \mathbf{v}t, t)$ .

## 4.2 Any shaped boundary

For any shaped boundary  $\partial\Omega$ , equation (22) could be replaced by

$$Q^i(x, y, t) = \sum_{m,n=1}^{\infty} q_{mn}^i(t) H_{\partial\Omega}^{mn}(x) H_{\partial\Omega}^{mn}(y) \quad (36)$$

and equation (25) by

$$v^i(x, y, t) = \sum_{m,n=1}^{\infty} u_{mn}^i(t) H_{\partial\Omega}^{mn}(x) H_{\partial\Omega}^{mn}(y). \quad (37)$$

where  $H_{\partial\Omega}^{mn}(x) H_{\partial\Omega}^{mn}(y)$  are partial solutions of Helmholtz 2D equation for given boundary  $\partial\Omega$  and could be taken for example from [7]. So equation (30) transforms to

$$v^i(x, y, t) = \sum_{m,n=1}^{\infty} (v_{0mnf}^i + Q_{mnf}^i) H_{\partial\Omega}^{mn}(x) H_{\partial\Omega}^{mn}(y) \exp(-k_{mn}^2 t) \quad (38)$$

$$v_{mnf}^i = \frac{1}{I_{2mn}} \iint_{\Omega} v_0^i(s', s) H_{\partial\Omega}^{mn}(s) H_{\partial\Omega}^{mn}(s') ds' ds \quad (39)$$

$$Q_{mnf}^i = \frac{1}{I_{2mn}} \iint_{\Omega} ds' ds \int_0^t Q^i(s', s, \tau) H_{\partial\Omega}^{mn}(s) H_{\partial\Omega}^{mn}(s') \exp(k_{mn}^2 \tau) d\tau \quad (40)$$

where  $I_{2mn}$  is expressed as follow

$$I_{2mn} = \iint_{\partial\Omega} (H_{\partial\Omega}^{mn}(x)H_{\partial\Omega}^{mn}(y))^2 dx dy. \quad (41)$$

Applying of continuity condition  $\nabla \cdot \mathbf{v} = 0$  gives similar to equations (31)(34)(35) relations between  $n$  and  $m$ . Finally, we obtain

$$v^i(x, y, t) = \sum_{n=1}^{\infty} (v_{0m(n)nf}^i + Q_{m(n)nf}^i) H_{\partial\Omega}^{m(n)n}(x) H_{\partial\Omega}^{m(n)n}(y) \exp(-k_{m(n)n}^2 t) \quad (42)$$

where  $m(n)$  notes dependence of  $m$  and  $n$ . If we investigate each point of fluid in moving coordinate system of this point, Galilean transform must be applied  $\mathbf{v}(\mathbf{r}_0 - \mathbf{v}t, t)$ .

## 5 Three dimensional inhomogeneous solution

Consider the initial-boundary value problem for  $v = v(x, y, z, t)$

$$\frac{dv^i}{dt} - a^2 \Delta v^i = \frac{1}{\rho} (-\nabla_i p + f_i) \text{ in } \Omega \times (0, \infty) \quad (43)$$

$$v^i(x, y, z, 0) = v_0^i(x, y, z) \quad x, y, z \in \Omega \quad (44)$$

$$\frac{\partial v^i}{\partial n} = 0 \text{ on } \partial\Omega \times (0, \infty) \quad (45)$$

where  $p = p(x, y, z, t)$  and  $f = f(x, y, z, t)$ ,  $\Omega \subset \mathbb{R}^{3n}$ ,  $\mathbf{n}$  the exterior unit normal at the smooth parts of  $\partial\Omega$ ,  $a^2$  a positive constant and  $v_0^x(x, y, z)$ ,  $v_0^y(x, y, z)$ ,  $v_0^z(x, y, z)$  a given function.

So, when  $x$ ,  $y$  and  $z$  scale was determined to  $a = 1$ , equation (4) could be rewritten as follow

$$\frac{dv^i}{dt} = v_{xx}^i + v_{yy}^i + v_{zz}^i + Q^i(x, y, z, t), \quad x, y, z \in \Omega, \quad t > 0 \quad (46)$$

If we investigate each point of fluid in moving coordinate system of this point, Galilean transform must be applied  $\mathbf{v}(\mathbf{r}_0 - \mathbf{v}t, t)$ .

### 5.1 Rectangular boundary

We expand  $v$  and  $Q$  in the eigenfunctions  $\exp(\frac{jn\pi x}{L_x}) \exp(\frac{jm\pi y}{L_y}) \exp(\frac{jp\pi z}{L_z})$  on space  $\Omega \in [0, L_x] \times [0, L_y] \times [0, L_z]$  where  $\exp(\frac{jn\pi x}{L_x}) \exp(\frac{jm\pi y}{L_y}) \exp(\frac{jp\pi z}{L_z})$  and  $\exp(\frac{jn'\pi x}{L_x}) \exp(\frac{jm'\pi y}{L_y}) \exp(\frac{jp'\pi z}{L_z})$  functions orthogonality could be applied. So, we obtain

$$Q^i(x, y, z, t) = \sum_{m,n,p=1}^{\infty} q_{mnp}^i(t) \exp(\frac{jn\pi x}{L_x}) \exp(\frac{jm\pi y}{L_y}) \exp(\frac{jp\pi z}{L_z}) \quad (47)$$

with

$$q_{mnp}^i(t) = \frac{1}{I_3} \iiint_{\Omega} Q^i(x, y, z, t) \exp(\frac{jn\pi x}{L_x}) \exp(\frac{jm\pi y}{L_y}) \exp(\frac{jp\pi z}{L_z}) dx dy dz \quad (48)$$

$$I_3 = \iiint_{\Omega} (\exp(\frac{jn\pi x}{L_x}) \exp(\frac{jm\pi y}{L_y}) \exp(\frac{jp\pi z}{L_z}))^2 dx dy dz \quad (49)$$

and

$$v^i(x, y, z, t) = \sum_{m,n,p=1}^{\infty} u_{mnp}^i(t) \exp(\frac{jn\pi x}{L_x}) \exp(\frac{jm\pi y}{L_y}) \exp(\frac{jp\pi z}{L_z}) \quad (50)$$

Thus we get the inhomogeneous ODE

$$\dot{u}_{mnp}^i(t) + k_{mnp}^2 u_{mnp}^i(t) = q_{mnp}^i(t), \quad (51)$$

$$k_{mnp}^2 = \left(\frac{n\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{p\pi}{L_z}\right)^2 \quad (52)$$

whose solution is

$$u_{mnp}^i(t) = u_{mnp}^i(0) \exp(-k_{mnp}^2 t) + \int_0^t q_{mnp}^i(\tau) \exp(-k_{mnp}^2(t-\tau)) d\tau \quad (53)$$

where

$$u_{mnp}^i(0) = \frac{1}{I_3} \iiint_{\Omega} v_0^i(x, y, z) \exp\left(\frac{jn\pi x}{L_x}\right) \exp\left(\frac{jm\pi y}{L_y}\right) \exp\left(\frac{jp\pi z}{L_z}\right) dx dy dz \quad (54)$$

Again, we substitute all obtained equations into (50) and have

$$\begin{aligned} v^i(x, y, z, t) &= \iiint_{\Omega} v_0^i(s'', s', s) \left( \sum_{m,n,p=1}^{\infty} \frac{1}{I_3} S_{mnp}(s, s', s'') S_{mnp}(x, y, z) \exp(-k_{mnp}^2 t) \right) ds'' ds' ds \\ &+ \iiint_{\Omega} ds'' ds' ds \int_0^t Q^i(s'', s', s, \tau) \left( \sum_{m,n,p=1}^{\infty} \frac{1}{I_3} S_{mnp}(s, s', s'') S_{mnp}(x, y, z) \exp(-k_{mnp}^2(t-\tau)) \right) d\tau \end{aligned} \quad (55)$$

$$S_{mnp}(x, y, z) = \exp\left(\frac{jn\pi x}{L_x}\right) \exp\left(\frac{jm\pi y}{L_y}\right) \exp\left(\frac{jp\pi z}{L_z}\right) \quad (56)$$

Now we must apply continuity condition  $\nabla \cdot \mathbf{v} = \frac{\partial v^x(x,y,z,t)}{\partial x} + \frac{\partial v^y(x,y,z,t)}{\partial y} + \frac{\partial v^z(x,y,z,t)}{\partial z} = 0$ . So we obtain relation conditions between  $n$ ,  $m$  and  $p$

$$(v_{0mnpf}^x + Q_{mnpf}^x) \frac{n\pi}{L_x} + (v_{0mnpf}^y + Q_{mnpf}^y) \frac{m\pi}{L_y} + (v_{0mnpf}^z + Q_{mnpf}^z) \frac{p\pi}{L_z} = 0 \quad (57)$$

where  $v_{0f}^i, Q_f^i$  are

$$v_{0mnpf}^i = \frac{1}{I_3} \iiint_{\Omega} v_0^i(s'', s', s) S_{mnp}(s, s', s'') ds'' ds' ds \quad (58)$$

$$Q_{mnpf}^i = \frac{1}{I_3} \iiint_{\Omega} ds'' ds' ds \int_0^t Q^i(s'', s', s, \tau) S_{mnp}(s, s', s'') d\tau \quad (59)$$

Finally, solutions of 3D incompressible Navier-Stokes are

$$v^i(x, y, z, t) = \sum_{m,n=1}^{\infty} (v_{0mnpf}^i + Q_{mnpf}^i) S_{mnp(m,n)}(x, y, z) \exp(-k_{mnp(m,n)}^2 t) \quad (60)$$

$$S_{mnp(m,n)}(x, y, z) = \exp\left(\frac{jn\pi x}{L_x}\right) \exp\left(\frac{jm\pi y}{L_y}\right) \exp\left(\frac{jp(m,n)\pi z}{L_z}\right) \quad (61)$$

where integers  $p(m, n)$  must satisfy (57) relations for each  $m$  and  $n$  and could be expressed by substituting of  $p/L_z$  corresponding expression as the sum of two others  $n/L_x, m/L_y$ . If we investigate each point of fluid in moving coordinate system of this point, Galilean transform must be applied  $\mathbf{v}(\mathbf{r}_0 - \mathbf{v}t, t)$ .

## 5.2 Any shaped boundary

For any shaped boundary  $\partial\Omega$ , equation (47) could be replaced by

$$Q^i(x, y, z, t) = \sum_{m,n,p=1}^{\infty} q^i(t) H_{\partial\Omega,k}^{mnp}(x, y, z) \quad (62)$$

and

$$v^i(x, y, z, t) = \sum_{m,n,p=1}^{\infty} u^i(t) H_{\partial\Omega,k}^{mnp}(x, y, z) \quad (63)$$

where  $H_{\partial\Omega,k}^{mnp}(x, y, z)$  are partial solutions of Helmholtz 3D equation for given boundary  $\partial\Omega$ . and could be taken for example from [8]. So, equation (55) transforms to

$$v^i(x, y, z, t) = \sum_{m,n=1}^{\infty} (v_{0mnpf}^i + Q_{mnpf}^i) H_{\partial\Omega,k}^{mnp}(x, y, z) \exp(-k_{mnp}^2 t) \quad (64)$$

$$v_{0mnpf}^i = \iiint_{\Omega} v_0^i(s'', s', s) H_{\partial\Omega,k}^{mnp}(s, s', s'') ds'' ds' ds \quad (65)$$

$$Q_{mnpf}^i = \iiint_{\Omega} ds'' ds' ds \int_0^t Q^i(s'', s', s, \tau) H_{\partial\Omega,k}^{mnp}(s, s', s'') \exp(k_{mnp}^2 \tau) d\tau \quad (66)$$

Applying of continuity condition  $\nabla \cdot \mathbf{v} = 0$  gives relations between  $n, m$  and  $p$  or  $p = p(m, n)$ . Finally, we obtain

$$v^i(x, y, z, t) = \sum_{m,n=1}^{\infty} (v_{0mnf}^i + Q_{mn}^i) H_{\partial\Omega, k}^{mnp(m,n)}(x, y, z) \exp(-k_{mnp(mn)}^2 t) \quad (67)$$

If we investigate each point of fluid in moving coordinate system of this point, Galilean transform must be applied  $\mathbf{v}(\mathbf{r}_0 - \mathbf{v}t, t)$ .

## 6 Conclusions

Due to the form of fluid equations they could be transformed into the full/partial inhomogeneous parabolic differential equations: differential equations in respect to space variables and full differential equations in respect to the time variable and inhomogeneous time dependent part. Finally, orthogonal polynomials as the partial solutions of obtained Helmholtz equations were used for derivation of analytical solution of velocities for incompressible fluid in 1D, 2D and 3D space for rectangular boundary. Solution in 3D space for any shaped boundary is expressed in term of 3D global solution of 3D Helmholtz equation accordantly.

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