On Global Solution of Incompressible Navier-Stokes equations

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Abstract

The fluid equations, named after Claude-Louis Navier and George Gabriel Stokes, describe the motion of fluid substances. These equations arise from applying Newton's second law to fluid motion, together with the assumption that the stress in the fluid is the sum of a diffusing viscous term (proportional to the gradient of velocity) and a pressure term - hence describing viscous flow. Due to specific of NS equations they could be transformed to full/partial inhomogeneous parabolic differential equations: differential equations in respect of space variables and the full differential equation in respect of time variable and time dependent inhomogeneous part. Finally, orthogonal polynomials as the partial solutions of obtained Helmholtz equations were used for derivation of analytical solution of incompressible fluid equations in 1D, 2D and 3D space for rectangular boundary. Solution in 3D space for any shaped boundary is expressed in term of 3D global solution of 3D Helmholtz equation accordantly.

1 Introduction

In physics, the fluid equations, named after Claude-Louis Navier and George Gabriel Stokes, describe fluid substances motion. These equations arise from applying Newton's second law to fluid motion, together with the assumption that the stress in the fluid is the sum of a diffusing viscous term (proportional to the gradient of velocity) and a pressure term - hence describing viscous flow. Equations were introduced in 1822 by the French engineer Claude Louis Marie Henri Navier [1] and successively re-obtained, by different arguments, by a several authors including Augustin-Louis Cauchy in 1823 [2], Simeon Denis Poisson in 1829, Adhemar Jean Claude Barre de Saint-Venant in 1837, and, finally, George Gabriel Stokes in 1845 [3]. Detailed and thorough analysis of the history of the fluid equations could be found in by Olivier Darrigol [4]. The invention of the digital computer led to many changes. John von Neumann, one of the CFD founding fathers, predicted already in 1946 that automatic computing machines' would replace the analytic solution of simplified flow equations by a numerical' solution of the full nonlinear flow equations for arbitrary geometries. Von Neumann suggested that this numerical approach would even make experimental fluid dynamics obsolete. Von Neumann's prediction did not fully come true, in the sense that both analytic theoretical and experimental research still coexist with CFD. Crucial properties of CFD methods such as consistency, stability and convergence need mathematical study [5].

Aims of this article are to propose new approach for solution of incompressible fluid equations. The article has three basic parts: first part explains how to solve NS in one dimension, second part extend solution to two-dimensional space and, finally, third part summarize with three-dimensional space.

2 Parabolic formulation of equations

Incompressible fluid equations are expressed as follow

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) - \mu\Delta\mathbf{v} + \nabla p = f \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \tag{2}$$

where equation (2) for incompressible flow reduces to $\frac{d\rho}{dt} = 0$ or $\rho = const$ due to $\nabla \mathbf{v} = 0$. Equations of fluid motion (1) could be expressed in full time derivative replacing covariant time derivative by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \quad (\mathbf{v} \cdot \nabla) \tag{3}$$

So, we obtain

$$\frac{d\mathbf{v}}{dt} - a^2 \Delta \mathbf{v} = \frac{1}{\rho}(-\nabla p + f) \tag{4}$$

3 inhomogeneous parabolic like equation for full time derivative, where $a = \sqrt{\mu/\rho}$.

3 One dimensional inhomogeneous solution

Consider the initial-boundary value problem for $v = v(x,t)$

$$\frac{dv}{dt} - a^2 \Delta v = \frac{1}{\rho}(-\nabla p + f) \text{ in } \Omega \times (0,\infty) \tag{5}$$

$$v(x,0) = v_0(x) \; x \in \Omega \tag{6}$$

$$\frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega \times (0,\infty) \tag{7}$$

where $p = p(x,t)$ and $f = f(x,t)$, $\Omega \subset \mathbb{R}^n$, n the exterior unit normal at the smooth parts of $\partial\Omega$, a^2 a positive constant and $v_0(x)$ a given function.

So according to [6] equation (4), when x is normed to $a = 1$, could be rewritten as follow

$$\frac{dv}{dt} = v_{xx} + Q(x,t), \; x \in \Omega, \; t > 0 \tag{8}$$

We expand v and Q in the eigenfunctions $\sin(n\pi x)$ on space $\Omega \in [0, L]$ where $sin(\frac{n\pi x}{L})$ and $sin(\frac{m\pi x}{L})$ functions orthogonality could be applied. So, we obtain

$$Q(x,t) = \sum_{n=1}^{\infty} q_n(t) \sin(\frac{n\pi x}{L}) \tag{9}$$

with

$$q_n(t) = \frac{1}{I_1} \int_\Omega Q(x,t) \sin(\frac{n\pi x}{L}) dx \tag{10}$$

$$I_1 = \int_\Omega \sin^2(\frac{n\pi x}{L}) dx = \frac{L}{2} \tag{11}$$

and

$$v(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin(\frac{n\pi x}{L}) \tag{12}$$

Thus we get the inhomogeneous ODE

$$\dot{u}_n(t) + \left(\frac{n\pi}{L}\right)^2 u_n(t) = q_n(t), \tag{13}$$

whose solution is

$$u_n(t) = u_n(0) e^{-(n\pi/L)^2 t} + \int_0^t q_n(\tau) e^{-(n\pi/L)^2 (t-\tau)} d\tau \tag{14}$$

where

$$u_n(0) = \frac{1}{I_1} \int_\Omega v_0(x) \sin(\frac{n\pi x}{L}) dx \tag{15}$$

Again, we substitute all obtained equations into (12) and have

$$v(x,t) = \int_\Omega v_0(s) (\sum_{n=1}^{\infty} \frac{1}{I_1} \sin(\frac{n\pi s}{L}) \sin(\frac{n\pi x}{L}) e^{-(n\pi/L)^2 t}) ds$$

$$+ \int_\Omega ds \int_0^t Q(s,\tau) (\sum_{n=1}^{\infty} \frac{1}{I_1} \sin(\frac{n\pi s}{L}) \sin(\frac{n\pi x}{L}) e^{-(n\pi/L)^2 (t-\tau)}) d\tau \tag{16}$$

4 Two dimensional inhomogeneous solution

Consider the initial-boundary value problem for $v = v(x, y, t)$

$$\frac{dv^i}{dt} - a^2\Delta v^i = \frac{1}{\rho}(-\nabla_i p + f_i) \text{ in } \Omega \times (0, \infty) \tag{17}$$

$$v^i(x, y, 0) = v_0^i(x, y) \; x, y \in \Omega \tag{18}$$

$$\frac{\partial v^i}{\partial n} = 0 \text{ on } \partial\Omega \times (0, \infty) \tag{19}$$

where $p = p(x, y, t)$ and $f = f(x, y, t)$, $\Omega \subset \mathbb{R}^{2n}$, n the exterior unit normal at the smooth parts of $\partial\Omega$, a^2 a positive constant and $v_0^x(x, y), v_0^y(x, y)$ a given function.

So, when x and y are normed to $a = 1$, equation (4) could be rewritten as follow

$$\frac{dv^i}{dt} = v^i_{xx} + v^i_{yy} + Q^i(x, y, t), \;\; x, y \in \Omega, \;\; t > 0 \tag{20}$$

4.1 Rectangular boundary

We expand v and Q in the eigenfunctions $\sin(\frac{n\pi x}{L_x})\sin(\frac{m\pi y}{L_y})$ on space $\Omega \in [0, L_x] \times [0, L_y]$ where $\sin(\frac{n\pi x}{L_x})\sin(\frac{m\pi y}{L_y})$ and $\sin(\frac{n'\pi x}{L_x})\sin(\frac{m'\pi y}{L_y})$ functions orthogonality could be applied. So, we obtain

$$Q^i(x, y, t) = \sum_{m,n=1}^{\infty} q^i_{mn}(t)\sin(\frac{n\pi x}{L_x})\sin(\frac{m\pi y}{L_y}) \tag{21}$$

with

$$q^i_{mn}(t) = \frac{1}{I_2}\iint_\Omega Q^i(x, y, t)\sin(\frac{n\pi x}{L_x})\sin(\frac{m\pi y}{L_y})dxdy \tag{22}$$

$$I_2 = \iint_\Omega (\sin(\frac{n\pi x}{L_x})\sin(\frac{m\pi y}{L_y}))^2 dxdy = \frac{L_x L_y}{4} \tag{23}$$

and

$$v^i(x, y, t) = \sum_{m,n=1}^{\infty} (u^i_{mn}(t)\sin(\frac{n\pi x}{L_x})\sin(\frac{m\pi y}{L_y}) \tag{24}$$

Thus we get the inhomogeneous ODE

$$\dot{u}^i_{mn}(t) + k^2_{m,n}u^i_{mn}(t) = q^i_{mn}(t), \tag{25}$$

$$k^2_{m,n} = \left(\frac{n\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2 \tag{26}$$

whose solution is

$$u^i_{mn}(t) = u^i_{mn}(0)e^{-k^2_{m,n}t} + \int_0^t q^i_{jmn}(\tau)e^{-k^2_{m,n}(t-\tau)}d\tau \tag{27}$$

where

$$u^i_{mn}(0) = \frac{1}{I_2}\iint_\Omega v_0^i(x, y)\sin(\frac{n\pi x}{L_x})\sin(\frac{m\pi y}{L_y})dxdy \tag{28}$$

Again, we substitute all obtained equations into (24) and have

$$v^i(x, y, t) = \iint_\Omega v_0^i(s', s)(\sum_{m,n=1}^{\infty}\frac{1}{I_2}\sin(\frac{n\pi s}{L_x})\sin(\frac{m\pi s'}{L_y})\sin(\frac{n\pi x}{L_x})\sin(\frac{m\pi y}{L_y})e^{-k^2_{m,n}t})ds'ds$$
$$+ \iint_\Omega ds'ds\int_0^t Q^i(s', s, \tau)(\sum_{m,n=1}^{\infty}\frac{1}{I_2}\sin(\frac{n\pi s}{L_x})\sin(\frac{m\pi s'}{L_y})\sin(\frac{n\pi x}{L_x})\sin(\frac{m\pi y}{L_y})e^{-k^2_{m,n}(t-\tau)})d\tau \tag{29}$$

4.2 Any shaped boundary

For any shaped boundary $\partial\Omega$, equation (21) could be replaced by

$$Q^i(x,y,t) = \sum_{m,n=1}^{\infty} q^i_{mn}(t) H^{mn}_{\partial\Omega}(x) H^{mn}_{\partial\Omega}(y) \tag{30}$$

and equation (24) by

$$v^i(x,y,t) = \sum_{m,n=1}^{\infty} u^i_{mn}(t) H^{mn}_{\partial\Omega}(x) H^{mn}_{\partial\Omega}(y). \tag{31}$$

where $H^{mn}_{\partial\Omega}(x)H^{mn}_{\partial\Omega}(y)$ are partial solutions of Helmholtz 2D equation for given boundary $\partial\Omega$ and could be taken for example from [7]. So equation (29) transforms to

$$\begin{aligned} v^i(x,y,t) &= \iint_\Omega v^i_0(s',s)\Big(\sum_{m,n=1}^{\infty} \frac{1}{I_{2mn}} H^{mn}_{\partial\Omega}(s) H^{mn}_{\partial\Omega}(s') H^{mn}_{\partial\Omega}(x) H^{mn}_{\partial\Omega}(y) e^{-k^2_{m,n}t}\Big) ds' ds \\ &+ \iint_\Omega ds' ds \int_0^t Q^i(s',s,\tau)\Big(\sum_{m,n=1}^{\infty} \frac{1}{I_{2mn}} H^{mn}_{\partial\Omega}(s) H^{mn}_{\partial\Omega}(s') H^{mn}_{\partial\Omega}(x) H^{mn}_{\partial\Omega}(y) e^{-k^2_{m,n}(t-\tau)}\Big) d\tau \end{aligned} \tag{32}$$

where I_{2mn} is expressed as follow

$$I_{2mn} = \iint_{\partial\Omega} (H^{mn}_{\partial\Omega}(x) H^{mn}_{\partial\Omega}(y))^2 dxdy. \tag{33}$$

5 Three dimensional inhomogeneous solution

Consider the initial-boundary value problem for $v = v(x,y,z,t)$

$$\frac{dv^i}{dt} - a^2\Delta v^i = \frac{1}{\rho}(-\nabla_i p + f_i) \text{ in } \Omega \times (0,\infty) \tag{34}$$

$$v^i(x,y,z,0) = v^i_0(x,y,z) \; x,y,z \in \Omega \tag{35}$$

$$\frac{\partial v^i}{\partial n} = 0 \text{ on } \partial\Omega \times (0,\infty) \tag{36}$$

where $p = p(x,y,z,t)$ and $f = f(x,y,z,t)$, $\Omega \subset \mathbb{R}^{3n}$, n the exterior unit normal at the smooth parts of $\partial\Omega$, a^2 a positive constant and $v^x_0(x,y,z), v^y_0(x,y,z), v^z_0(x,y,z)$ a given function.

So, when x, y and z are normed to $a = 1$, equation (4) could be rewritten as follow

$$\frac{dv^i}{dt} = v^i_{xx} + v^i_{yy} + v^i_{zz} + Q^i(x,y,z,t), \;\; x,y,z \in \Omega, \;\; t > 0 \tag{37}$$

5.1 Rectangular boundary

We expand v and Q in the eigenfunctions $\sin(\frac{n\pi x}{L_x})\sin(\frac{m\pi y}{L_y})\sin(\frac{p\pi z}{L_z})$ on space $\Omega \in [0,L_x]\times[0,L_y]\times[0,L_z]$ where $\sin(\frac{n\pi x}{L_x})\sin(\frac{m\pi y}{L_y})\sin(\frac{p\pi z}{L_z})$ and $\sin(\frac{n'\pi x}{L_x})\sin(\frac{m'\pi y}{L_y})\sin(\frac{p'\pi z}{L_z})$ functions orthogonality could be applied. So, we obtain

$$Q^i(x,y,z,t) = \sum_{m,n,p=1}^{\infty} q^i_{mn}(t) \sin(\frac{n\pi x}{L_x}) \sin(\frac{m\pi y}{L_y}) \sin(\frac{p\pi z}{L_z}) \tag{38}$$

with

$$q^i_{mnp}(t) = \frac{1}{I_3} \iiint_\Omega Q^i(x,y,z,t) \sin(\frac{n\pi x}{L_x}) \sin(\frac{m\pi y}{L_y}) \sin(\frac{p\pi z}{L_z}) dxdydz \tag{39}$$

$$I_3 = \iiint_\Omega (\sin(\frac{n\pi x}{L_x}) \sin(\frac{m\pi y}{L_y}) \sin(\frac{p\pi z}{L_z}))^2 dxdydz = \frac{L_x L_y L_z}{8} \tag{40}$$

and

$$v^i(x,y,z,t) = \sum_{m,n,p=1}^{\infty} u^i_{mn}(t) \sin(\frac{n\pi x}{L_x}) \sin(\frac{m\pi y}{L_y}) \sin(\frac{p\pi z}{L_z}) \tag{41}$$

Thus we get the inhomogeneous ODE

$$\begin{aligned} \dot{u}^i_{mnp}(t) + k^2_{mnp} u^i_{mnp}(t) &= q^i_{mnp}(t), \qquad (42) \\ k^2_{mnp} &= \left(\frac{n\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{p\pi}{L_y}\right)^2 \qquad (43) \end{aligned}$$

whose solution is

$$u^i_{mnp}(t) = u^i_{mnp}(0) e^{-k^2_{mnp} t} + \int_0^t q^i_{mnp}(\tau) e^{-k^2_{mnp}(t-\tau)} d\tau \qquad (44)$$

where

$$u^i_{mnp}(0) = \frac{1}{I_3} \iiint_\Omega v^i_0(x,y,z) \sin(\frac{n\pi x}{L_x}) \sin(\frac{m\pi y}{L_y}) \sin(\frac{p\pi z}{L_z}) dxdydz \qquad (45)$$

Again, we substitute all obtained equations into (41) and have

$$\begin{aligned} v^i(x,y,z,t) &= \iiint_\Omega v^i_0(s'',s',s)(\sum_{m,n,p=1}^{\infty} \frac{1}{I_3} S_{mnp}(s,s',s'') S_{mnp}(x,y,z) e^{-k^2_{mnp} t}) ds'' ds' ds \\ &+ \iiint_\Omega ds'' ds' ds \int_0^t Q^i(s'',s',s,\tau)(\sum_{m,n,p=1}^{\infty} \frac{1}{I_3} S_{mnp}(s,s',s'') S_{mnp}(x,y,z) e^{-k^2_{mnp}(t-\tau)}) d\tau \qquad (46) \\ S_{mnp}(x,y,z) &= \sin(\frac{n\pi x}{L_x}) \sin(\frac{m\pi y}{L_y}) \sin(\frac{p\pi z}{L_z}) \qquad (47) \end{aligned}$$

5.2 Any shaped boundary

For any shaped boundary $\partial\Omega$, equation (38) could be replaced by

$$Q^i(x,y,z,t) = q^i(t) H_{\partial\Omega,k}(x,y,z) \qquad (48)$$

and

$$v^i(x,y,z,t) = u^i(t) H_{\partial\Omega,k}(x,y,z) \qquad (49)$$

where $H_{\partial\Omega,k}(x,y,z)$ is global solution of Helmholtz 3D equation for given boundary $\partial\Omega$ and could be taken for example from [8] or [9]. So equation (46) transforms to

$$\begin{aligned} v^i(x,y,z,t) &= \iiint_\Omega v^i_0(s'',s',s) H_{\partial\Omega,k}(s,s',s'') H_{\partial\Omega,k}(x,y,z) e^{-k^2 t} ds'' ds' ds \\ &+ \iiint_\Omega ds'' ds' ds \int_0^t Q^i(s'',s',s,\tau) H_{\partial\Omega,k}(s,s',s'') H_{\partial\Omega,k}(x,y,z) e^{-k^2(t-\tau)} d\tau \qquad (50) \end{aligned}$$

6 Conclusions

Due to the form of fluid equations they could be transformed into the full/partial inhomogeneous parabolic differential equations: differential equations in respect to space variables and full differential equations in respect to the time variable and inhomogeneous time dependent part. Finally, orthogonal polynomials as the partial solutions of obtained Helmholtz equations were used for derivation of analytical solution of velocities for incompressible fluid in 1D, 2D and 3D space for rectangular boundary. Solution in 3D space for any shaped boundary is expressed in term of 3D global solution of 3D Helmholtz equation accordantly.

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References

[1] Navier, C. L. M. H. Mem acad. R. sci. paris, Vol. 6 pp 389-416, 1823.

[2] Cauchy, A.L. Exercises de mathematique, p.183, Paris, 1828.

[3] Stokes. G. G. trans. Camb. Phil. Soc., vol 8, pp 287–305, 1845.

[4] Darrigol O., Between Hydrodynamics and Elasticity Theory: The First Five Births of the Navier-Stokes Equation. Arch. Hist. Exact Sci. 56, pp 95–150, 2002.

[5] `http://www.cwi.nl/fluiddynamicshistory`, Centrum Wiskunde and Informatica, History of theoretical fluid dynamics, retrieved 11/29/2013

[6] Beny Neta, Partial Differential Equations, MA 3132 Lecture Notes, Department of Mathematics Naval Postgraduate School, Monterey, California, p. 353, 2012

[7] Read W.W. Analytical solutions for a helmholtz equation with dirichlet boundary conditions and arbitrary boundaries, Mathematical and Computer Modelling, V. 24, Nr 2, pp. 23–34, July 1996.

[8] Hoppe Ronald H.W. Computational Electromagnetics, Handout of the Course Held at the AIMS Muizenberg, South Africa December 17–21, p.110, 2007.

[9] Zapletal J. The Boundary Element Method for the Helmholtz Equation in 3D, echnical University of Ostrava Faculty of Electrical Engineering and Computer Science Department of Applied Mathematics, p. 102, 2011.