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Discrete filtration of Multivariate Correlated Nonstationary Processes

Contents

Summary

We consider a vector stochastic process with stationary increments of a predetermined order, whose components are linearly dependent, i.e. in the absence of noise vector process components are constrained by a system of linear equations (constraints). The interdependence of stochastic processes can be determined by a static or a dynamic model. The constraints can be maintained rigidly or with a specified error. We offer a method allowing in these conditions synthesis of an optimum filter structure. This method works in cases where no information about signal and noise static properties is available.

1. Introduction

The works [1, 2, 3, 4] review various problems of synthesis of an optimum discrete filter for random series with stationary increments of a predetermined order (RSSI) for brevity's sake hereinafter referred to as *RSSI filters*. Below, these problems will be generalized, supplemented and classified. For this purpose only end results needed by a developer of a control system containing filters.

For a situation characterized by nonstationarity of measurements and the static model of a control object, the problem was formerly considered on the basis of an empirically introduced filter structure, and its solution was reduced to finding filter parameter values. The method proposed allows to *synthesize a* ***structure*** *of an optimum filter*. The method works even when no information about signal and noise static properties is available. It is specifically found that an exponential smoothing filter is a special case of the proposed filter and is optimal according to the criterion laid down in this work. We do not know of analogues to the algorithm proposed. The method covers *interdependent* RSSI's. The interdependence between them yields some additional information that improves the quality of filtration. If we know the filter structure, we can filter such processes in real-time systems.

For *dynamic systems* Kalman filter is widely used. As compared with it, the filter proposed

- has the capability of finding filter parameters in the absense of information necessary for Kalman filter synthesis (knowledge of auto- and intercorrelation functions of signals and noise),
- requires considerably less amount of on-line calculations owing to the less complex structure of the synthesized filter.

Furthermore, there is no solution to the problem in the mentioned formulation using Kalman filtration theory.

The synthesis problem can be formulated in the following way We consider a vector stochastic process with stationary increments of the p-th order whose components are linearly dependent, , i.e. in the absence of noise vector process components are constrained by a system of linear equations (constraints). This system can be either static or dynamic. The constraints can be maintained rigidly or with a specified error. We need to synthesize a filter in such a way that this dependence would remain for signal estimates. Various versions of this problem have been solved [1, 2, 3, 4] by methods that are used for control design synthesis [5].

The meaning of this formulation is as follows. Let us assume that there is a time function $Z(t)$, whose p-th derivative has the small value of $\Delta(t) \approx 0$. The filter is constructed in such a way that in the process of noise filtration of function $Z(t)$, i.e. during the calculation of function $L(t)$ estimate, the value of $(1-\beta)\cdot\Delta(t)+\beta\cdot[Z(t)-L(t)]$ would get minimized, where β is a weight coefficient and $0 \le \beta \le 1$.

2. Multiply correlated vector stochastic processes

Below we consider a vector RSSI-*p* with stationary increments of the *p*-th order whose components are linearly dependent, i.e. in the absense of noise the system of linear equations comprising a control object model is given by $Model\ (Z)=0$. E.g., it may be of the form $A \cdot Z = C$, where A is a known matrix, and C is a known vector. The constraints can be maintained with different degrees of strictness (including strong constraints). We need to synthesize a filter in such a way that that this dependence would remain for signal estimates. At every instant j for each i-th process an additive mixture $Z_i(j)$ of useful signal and noise is observed. This mixture will be the input of the filter whose output we shall call L_j. As a result of the synthesis a <u>time **independent** matrix *B*</u> is formed. With a known matrix *B*, the filtration, i.e. the calculation of optimum estimate $L_i(j)$, is performed by formula $L(j) = B \cdot W(j)$, where $W(j)$ is a determined function of measurements vector $Z(j)$ and values of vectors $Z(k)$ and $L(k)$ at past instants of time.

So the mathematical model of a control object $Model\ (X)=0$ determines the interdependence between the components of the measurable parameters vector. The components of a vector stochastic process are, evidently, also constrained by this this dependence $Model\ (Z)=0$. This interdependence, on the one hand, complicates the filtration problem and the structure of the synthesized filter, but, on the other hand, yields some additional information that improves the quality of filtration. Below we consider various static and dynamic mathematical models. The filtration is performed in such a way that model correlations survive in filtered values, i.e. $Model\ (L)=0$.

3. Optimum filter structure

The following conventional signs will be used:

i – vector component number $Z_i(j)$, $0 \le i \le I$,

j – vector measurement instant $Z_i(j)$, $0 \le j \le J$,

p – order of increment,

Δ^p – increment of the p-th order,

M – mathematical expectation,

$0 \le \beta_i \le 1$ – weight coefficients,

$L_i(j)$ – optimum estimation of vector $Z_i(j)$ at instant j,

$E_i(j)$ – vector $Z_i(j)$ filtering error at instant j-момент, where

$$E_i(j) = Z_i(j) - L_i(j) . \quad (1)$$

The first increment of the discrete stochastic process

$$\Delta^1 Z(j) = Z(j) - Z(j-1) .$$

Increments of higher orders are found by the recurrent formula

$$\Delta^{p+1} Z(j) = \Delta^p Z(j) - \Delta^p Z(j-1) .$$

The stochastic process $Z(j)$ with p-th stationary increments Δ^p is characterized by the fact that mathematical expectation

$$M\left(\Delta^{p+1} Z(j)\right) = 0 .$$

During the process of filter synthesis there is a good reason to pursue a similar condition for the filter output signal, i.e.

$$M\left(\Delta^{p+1} L(j)\right) = 0 . \quad (2)$$

Besides, it is necessary to try to meet the condition

$$\overline{M}\left(E^2(j)\right) = 0 . \quad (3)$$

In this connection the following filtering quality characteristic has been chosen:

$$R = \overline{M} \sum_{i,j} [(1-\beta_i)(\Delta^{p+1} L_i(j))^2 + \beta_i E_i^{\,2}(j)] , \quad (4)$$

The filter structure is independent on *J*. Therefore, with the given p and β_i it can be pre-synthesized and used in real-time systems. An optimum filter in terms of minimum quality characteristic R, is

$$L(j) = B \cdot W(j) , \quad (5)$$

where

B – filter matrix,

$W(j)$ – detemined vector-function of measurements vector $Z(j)$ and values of vectors $Z(k)$ and $L(k)$ at past instants $k = j\text{-}1, j\text{-}2, \ldots, j\text{-}p$; this vector is of the following form:

$$W(j) = \begin{vmatrix} \Delta^p Z(j) \\ \Delta^{p-1} Z(j) \\ \ldots \\ \Delta^2 Z(j) \\ \Delta^1 Z(j) \\ Z(j) \\ L(j-1) \\ \Delta^1 L(j-1) \\ \Delta^2 L(j-1) \\ \ldots \\ \Delta^{p-1} L(j-1) \\ \Delta^p L(j-1) \end{vmatrix} \qquad (6)$$

The lentgth of this vector is

$$G = 2 * p + 2. \qquad (7)$$

4. Classification of RSSI filters

Table 1 contains a classification of stochastic processes and related filters, Table 2 contains major properties of these filters. They will be described in more detail in one of the subsequent sections of this work.

Table 1.

process			
stationary	nonstationary		
1. scalar	2. scalar	vector	
		3. simple (no model)	multiply correlated (see below)

Multiply Correlated Vector Process					
Static Model				Dynamic Model	
Type 1 – with strong constraints		6. Type 1 – with weak constraints	7. Type 2	8. Type 1	9. Type 2
4. General case	5. Synchronous processes				

Table 2.

№	Process	b	Filter	S
1	Scalar stationary process, exponential smoothing filter, *p=0, I=1*	2	(5), (7a)	β
2	Scalar nonstationary process	G	(5)	p,β
3	Vector process without a model	[I]*[G*I]	(5)	p,I,β
4	Vector multiply correlated process with the static model of type 1 (8)	[I]*[G*I+U]	(5)	$A,C,p,$ U,I,β
5	Synchronous processes	[I]*[G*I+U]	(5)	p,I,β
6	Vector process with a static nonstrict model of type 1 (8), (12)	[I]*[G*I+U]	(5)	$A,D,p,$ U,I,β
7	Vector process with the static model of type 2 (13)	[I]*[G*(I+U)]	(14)	$A,p,U,$ I,β',β''
8	Vector process with a dynamic model of type 1 (16)	[I]*[G*(I+U)+U]	(17)	$A,p,U,$ I,β',β''
9	Vector process with a dynamic model of type 2 (19)	[I]*[G*I+I]	(5)	$A,p,$ I,β

In Table 2:

b – dimension of matrix *B,*

I – dimension of measurements vector*,*

G – dimension of vector *W* – s. (7),

U – number of object model equations*,*

S – data for synthesis

4.1. Scalar stationary processes

In this case we consider a single stochastic process at *p=0, I=1*. An optimum filter in terms of minimum quality characteristic (4), is of the form (5), where

$$W(j) = \begin{vmatrix} Z(j) \\ L(j-1) \end{vmatrix}, \; B = \begin{vmatrix} \beta & (1-\beta) \end{vmatrix}.$$

Thus,

$$L(j) = \beta \cdot Z(j) + (1-\beta)L(j-1) \tag{7a}$$

i.e. filter RSSI-0 is the same as the wide-spread exponential smoothing filter.

4.2. Scalar nonstationary processes

In this case we consider a single stochastic process at *p>0, I=1*. An optimum filter in terms of minimum quality characteristic (4), is of the form (5), where *G=2p+2*.

4.3. Uncorrelated vector processes

Let us consider a vector stochastic process with independent components at *I>1*. It is obvious that each component's filter is synthesized independently. If components have the same orders, then matrix B can be constructed for a vector process as a whole.

4.4. Vector process with the static model of type 1

In this case the static model takes the form

$$A \cdot Z(j) = C(j) \tag{8}$$

where

A – known matrix of dimension *U*I,*

C(j) – known vector that can be time variant

U – number of model equations and the dimension of vector *C*.

In this case filter (5) takes the form

$$L(j) = B \cdot W'(j), \tag{9}$$

where

$$W'(j) = \begin{vmatrix} W(j) \\ C(j) \end{vmatrix}, \qquad (10)$$

i.e. the dimension of vector $W'(j)$ equals *I*G+U,* where *G=2p+2.* The dimension of matrix *B* of filter (9) equals *[I]*[I *G+U].*

For filtered vector values a condition similar to condition (8) is fulfilled:

$$A \cdot L(j) = C(j) \qquad (11)$$

Let us remark here that matrix *B* is independent on vector $C(j)$.
Therefore,

‖ vector $C(j)$ can be time variant.

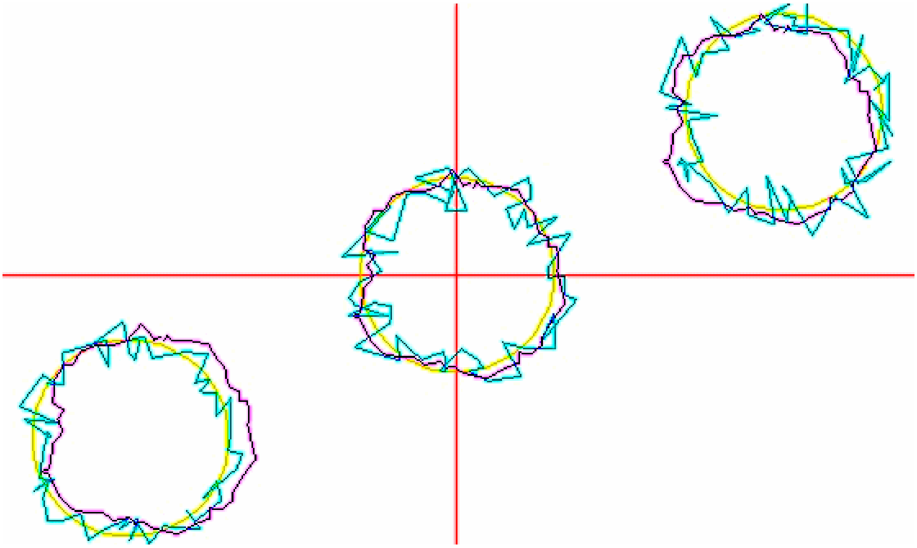
4.6. Synchronous processes

Synchronous processes are characterized by the fact that between each pair of processes there remains the dependence $Z_a(j) - Z_b(j) = Y_k,\ k = func(a,b)$. A model of such processes is a special case of the static model of type 1, and matrix *A* and vector *C* are formed automatically at a given *I.* E.g., at *I* =4 we obtain: *U* = 10. Let us remark that in this case matrix *B* is independent on vector $\{Y_k\}$. Therefore,

‖ values $\{Y_k\}$ can be time variant.

A	C
$A = \begin{vmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{vmatrix}$	C[0]=Z[0]-Z[1]; C[1]=Z[0]-Z[2]; C[2]=Z[0]-Z[3]; C[3]=Z[0]-Z[4]; C[4]=Z[1]-Z[2]; C[5]=Z[1]-Z[3]; C[6]=Z[1]-Z[4]; C[7]=Z[2]-Z[3]; C[8]=Z[2]-Z[4]; C[9]=Z[3]-Z[4];

In Figure 1 graphically shown is the result of the filtration of a synchronous 6-dimensional vector process with stationary increments of the second order, synchronous circular movement of three points (yellow curve). Here each stochastic process is changed coordinates of a point, on which noise is superimposed (blue curve). The filtered process (red curve) comes close to the ideal circle.



4.6. Static model of type 1 at weak constraints

For the static model of type 1 the quality coefficient can be supplemented and presented as

$$T = R + \overline{M}\sum_{j}[[A \cdot L(j) - C(j)]^T D \cdot [A \cdot L(j) - C(j)]], \qquad (12)$$

where D is a diagonal matrix of the dimension $U*U$ of known weight coefficients d_k. At $d_k >> 1$ the model equations are strongly fulfilled. At $d_k \approx 1$ the model equations can be violated to the same degree as filtering errors, i.e. conditions (2) and (3). Finally, at $d_k \approx 1$ and different k the model equations can be fulfilled with different degrees of accuracy.

4.7. Vector process with the static model of type 2

The static model of type 2 takes the following form:

$$Z''(j) = A \cdot Z'(j), \qquad (13)$$

where

$Z''(j)$, $Z'(j)$ – two different vectors (here primes identify two different RSSI-*p*'s and all values associated with them),

U – number of model equations and the dimension of measurements vector $Z''(j)$,

I – dimension of measurements vector $Z'(j)$,

A – known model matrix of the dimension *U*I.*

An optimum filter in terms of quality characteristic (4) is

$$\begin{aligned} L'(j) &= B \cdot W, \\ L''(j) &= A \cdot L'(j), \end{aligned} \tag{14}$$

where

B – matrix of the dimension *[I]*[G*(I+U)],*

W – vector of the dimension *G*(I +U),* where *G=2p+2,* and

$$W(j) = \begin{vmatrix} W'(j) \\ W''(j) \end{vmatrix} \tag{15}$$

4.8. Vector process with the dynamic model of type 1

The dynamic model of type 1 takes the form

$$Z''(j+1) = A \cdot Z'(j), \tag{16}$$

i.e., it connects nearest objects in terms of measurement time that belong to two different processes.

Here

$Z''(j),\ Z'(j)$ – two different vectors (here primes identify two different RSSI-*p*'s and all values associated with them),

U – number of model equations and the dimension of measurements vector $Z''(j)$,

I – dimension of measurements vector $Z'(j)$,

A – known model matrix of the dimension *U*I.*

An optimum filter in terms of quality characteristic (4) is

$$\begin{aligned} L'(j) &= B \cdot W, \\ L''(j+1) &= A \cdot L'(j), \end{aligned} \tag{17}$$

where

B – matrix of the dimension *[I]*[G*(I+U)+U],*

W – vector of the dimension *[G*(I+U)+U],* where *G=2p+2,* and

$$W(j) = \begin{vmatrix} W'(j) \\ W''(j) \\ L''(j) \end{vmatrix} \tag{18}$$

4.9. Vector process with the dynamic model of type 2

The dynamic model of type 2 takes the form

$$Z(j+1) = A \cdot Z(j), \quad (19)$$

i.e., it connects nearest objects in terms of measurement time that belong to one process. Such an object can be one with a known movement model.

Here A is a known square matrix of the dimension *I*I.* An optimum filter in terms of quality characteristic (4) takes the form (5), where

B – matrix of the dimension *[I]*[G*I+I],*

W – vector of the dimension *[G*I+I],* where *G=2p+2,* and

$$W(j) = \begin{vmatrix} W(j) \\ L''(j-1) \end{vmatrix} \quad (20)$$

5. Filtration procedure

On the completion of filter synthesis, i.e. calculation of matrix B, filtration of measurements is performed on-line by the following procedure:

- ✓ receipt of vectors Z(j), C(j);
- ✓ formation of new vector W(j) based on known vectors W(j-1), L(j-1), C(j), Z(j);
- ✓ calculation of a vector of smoothed L(j) based on known vectors W(j), B;
- ✓ repetition of the above calculation group for the next value of *j*, etc.

6. Programming technology

Programming of a RSSI-filter using the method proposed consists of the following stages:

- given process analysis and selection from *mathematical models of filters* of such a model that would be adequate to the given process;
- *filter synthesis* for the selected mathematical model; this stage is fulfilled by means of the proposed program;
- programming of a real-time filter; for this purpose a *function library* is proposed.

7. Conclusion

An indication can be made of a number of practically useful properties of the proposed method:

- optimality of a multivariate filter for nonstationary stochastic processes.
- feasibility at any order of stationary increments.
- applicability to real-time systems.
- applicability to cases where no information about signal and noise static properties is available.
- preservation for filtered values of a known dependence between RSSI components; this dependence can be static, i.e. time-independent, or dynamic, i.e. account for the association between component values at consequent instants of time;
- improvement in filtration quality owing to the additional information provided by a known interdependence between RSSI's.

The proposed method can be used in real-time systems for controlling objects with determined mathematical models, e.g.:

- in power grid supervisory control systems, oil and gas pipelines (it is known that appropriate models are adequately described by a linear equation system);
- in industrial continuous process control (mathematical models for these processes can often be linarized);
- in recognition of objects whose shape changes affinely;
- in particular, in radar systems tracking multiple object groups (it can be shown that in such systems the process is described by equations resulting from the limited maneuvring of objects within a group; the criterion of filter optimality in this case is a rephrased requirement for filtering error minimization at a group acceleration limited due to physical restrictions).

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