A COROLLARY OF RIEMANN HYPOTHESIS

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Abstract. This paper use the results of the value distribution theory, got a significant conclusion by Riemann hypothesis

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First, we give some signs, definition and theorem in the value distribution theory, its contents see the references [1] and [2].

Definition.

\[ \log^+ x = \begin{cases} 
  \log x & 1 \leq x \\
  0 & 0 \leq x < 1 
\end{cases} \]

It is easy to see that \( \log x \leq \log^+ x \).

Set \( f(z) \) is a meromorphic function in the region \( |z| < R, 0 < R \leq \infty \), and not identical to zero.

\( n(r, f) \) represents the poles number of \( f(z) \) on the circle \( |z| \leq r \) (\( 0 < r < R \)), multiple poles being repeated. \( n(0, f) \) represents the order of pole of \( f(z) \) in the origin. For arbitrary complex number \( a \neq \infty \), \( n(r, \frac{1}{f-a}) \) represents the zeros number of \( f(z) - a \) in the circle \( |z| \leq r \) (\( 0 < r < R \)), multiple zeros being repeated. \( n(0, \frac{1}{f-a}) \) represents the order of zero of \( f(z) - a \) in the origin.

Definition.

\[ m(r, f) = \frac{1}{2\pi} \int_{0}^{2\pi} \log^+ |f(re^{i\varphi})| \, d\varphi \]
\[ N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} \, dt + n(0, f) \log r \]

Definition. \( T(r, f) = m(r, f) + N(r, f) \).
\( T(r, f) \) is called the characteristic function of \( f(z) \).

**Lemma 1.** If \( f(z) \) is an analytical function in the region \(|z| < R \ (0 < R \leq \infty)\), then
\[
T(r, f) \leq \log^+ M(r, f) \leq \frac{\rho + r}{\rho - r} T(\rho, f)(0 < r < \rho < R)
\]
where \( M(r, f) = \max_{|z|=r} |f(z)| \).
The proof of the lemma see the page 57 of the references [1].

**Lemma 2.** Set \( f(z) \) is a meromorphic function in the region \(|z| < R \ (0 < R \leq \infty)\), not identical to zero. Set \(|z| < \rho \ (0 < \rho < R)\) is a circle, \( a_\lambda \ (\lambda = 1, 2, ..., h) \) and \( b_\mu \ (\mu = 1, 2, ..., k) \) respectively is the zeros and the poles of \( f(z) \) in the circle, appeared number of every zero or every pole and its order the same, and that \( z = 0 \) is not the zero or the pole of function \( f(z) \), then in the circle \(|z| < \rho\), we have the following formula
\[
\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(\rho e^{i\varphi})| \, d\varphi - \sum_{\lambda=1}^{h} \log \frac{\rho}{|a_\lambda|} - \sum_{\mu=1}^{k} \log \frac{\rho}{|b_\mu|}
\]
this formula is called Jensen formula.
The proof of the lemma see the page 48 of the references [1].

**Lemma 3.** Set function \( f(z) \) is the meromorphic function in \(|z| \leq R\), and
\( f(0) \neq 0, \infty, 1, \ f'(0) \neq 0 \)
then when \( 0 < r < R \), have
\[
T(r, f) < 2 \left\{ N(R, \frac{1}{f}) + N(R, f) + N\left(R, \frac{1}{f-1}\right) \right\}
\]
+ 4 \log^+ |f(0)| + 2 \log^+ \frac{1}{R|f''(0)|} + 24 \log \frac{R}{R - r} + 2328

This is a form of Nevanlinna second basic theorems.

The proof of the lemma see the theorem 3.1 of the page 75 of the references [1].

The need for behind, We will make some preparations.

**Lemma 4.** If when \( x \geq a \), \( f(x) \) is a nonnegative degressive function, then below limits exist

\[
\lim_{N \to \infty} \left( \sum_{n=a}^{N} f(n) - \int_{a}^{N} f(x) \, dx \right) = \alpha
\]

where \( 0 \leq \alpha \leq f(a) \). in addition, if when \( x \to \infty \), have \( f(x) \to 0 \), then

\[
\left| \sum_{a \leq n \leq \xi} f(n) - \int_{a}^{\xi} f(\nu) \, d\nu - \alpha \right| \leq f(\xi - 1), \quad (\xi \geq a + 1)
\]

The proof of the lemma see the theorem 2 of page 91 of the references [3].

Set \( s = \sigma + it \) is the complex number, when \( \sigma > 1 \), the definition of Riemann Zeta function is

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}
\]

When \( \sigma > 1 \), from the page 90 of the references [4], have

\[
\log \zeta(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s \log n}
\]

where \( \Lambda(n) \) is Mangoldt function.
LEMMA 5. For any real number $t$, have

(1) \[ 0.0426 \leq | \log \zeta(4 + it) | \leq 0.0824 \]

(2) \[ | \zeta(4 + it) - 1 | \geq 0.0426 \]

(3) \[ 0.917 \leq | \zeta(4 + it) | \leq 1.0824 \]

(4) \[ | \zeta'(4 + it) | \geq 0.012 \]

PROOF.

(1)
\[
| \log \zeta(4 + it) | \leq \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^4 \log n} \leq \sum_{n=2}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} - 1 \leq 0.0824
\]

\[
| \log \zeta(4 + it) | \geq \frac{1}{2^4} - \sum_{n=3}^{\infty} \frac{1}{n^4} = 1 + \frac{2}{2^4} - \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{9}{8} - \frac{\pi^4}{90} \geq 0.0426
\]

(2)
\[
| \zeta(4 + it) - 1 | = \left| \sum_{n=2}^{\infty} \frac{1}{n^4 + it} \right| \geq \frac{1}{2^4} - \sum_{n=3}^{\infty} \frac{1}{n^4}
\]
\[
= 1 + \frac{2}{2^4} - \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{9}{8} - \frac{\pi^4}{90} \geq 0.0426
\]

(3)
\[
| \zeta(4 + it) | = \left| \sum_{n=1}^{\infty} \frac{1}{n^4 + it} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \leq 1.0824
\]
\[
| \zeta(4 + it) | = \left| \sum_{n=1}^{\infty} \frac{1}{n^4 + it} \right| \geq 1 - \sum_{n=2}^{\infty} \frac{1}{n^4} = 2 - \sum_{n=1}^{\infty} \frac{1}{n^4} = 2 - \frac{\pi^4}{90} \geq 0.917
\]
\begin{align*}
|\zeta'(4 + it)| &= \left| \sum_{n=2}^{\infty} \frac{\log n}{n^{4+it}} \right| \\
&\geq \frac{\log 2}{2^4} - \sum_{n=3}^{\infty} \frac{\log n}{n^4}
\end{align*}

from lemma 4, have

\[
\sum_{n=3}^{\infty} \frac{\log n}{n^4} = \int_{3}^{\infty} \frac{\log x}{x^4} \, dx + \alpha
\]

where \(0 \leq \alpha \leq \frac{\log 3}{3^4}\)

\[
\int_{3}^{\infty} \frac{\log x}{x^4} \, dx = -\frac{1}{3} \int_{3}^{\infty} \log x \, dx = \frac{\log 3}{3^4} + \frac{1}{3} \int_{3}^{\infty} x^{-4} \, dx
\]

\[
= \frac{\log 3}{3^4} - \frac{1}{3^2} \int_{3}^{\infty} dx = \frac{\log 3}{3^4} + \frac{1}{3^5}
\]

therefore

\[
\sum_{n=3}^{\infty} \frac{\log n}{n^4} \leq \frac{\log 3}{3^4} + \frac{1}{3^5} + \frac{\log 3}{3^4}
\]

therefore

\[
|\zeta'(4 + it)| \geq \frac{\log 2}{2^4} - \frac{2\log 3}{3^4} - \frac{1}{3^5} \geq 0.012
\]

The proof is complete.

Set \(0 < \delta \leq \frac{1}{100}\), \(c_1, c_2, \ldots\), represents positive constant with only \(\delta\) relevant in the article below.

**Lemma 6.** When \(\sigma \geq \frac{1}{2}\), \(|t| \geq 2\), have

\[
|\zeta(\sigma + it)| \leq c_1 |t|^\frac{1}{2}
\]

The proof of the lemma see the theorem 2 of page 140 and the theorem 4 of page 142, of the references [4].
**LEMMA 7.** Set \( f(z) \) is the analytic function in the circle \( |z - z_0| \leq R \), then for any \( 0 < r < R \), in the circle \( |z - z_0| \leq r \), have

\[
|f(z) - f(z_0)| \leq \frac{2r}{R - r} \left( A(R) - Re f(z_0) \right)
\]

where \( A(R) = \max_{|z - z_0| \leq R} Re f(z) \).

The proof of the lemma see the theorem 2 of page 61 of the references [4].

Now assume Riemann hypothesis is correct, abbreviation for RH. In other words, when \( \sigma > \frac{1}{2} \), the function \( \zeta(\sigma + it) \) has no zeros. Set the union set of the region \( \sigma \geq \frac{1}{2} + \delta, |t| > 1 \) and the region \( \sigma > 2, |t| \leq 1 \) is the region \( D \).

Therefore, the function \( \zeta(\sigma + it) \) have neither zero nor poles in the region \( D \), so, function \( \log \zeta(\sigma + it) \) is a defined multi-valued analytic function in the region \( D \). Every single value analytic branch differ \( 2\pi i \) integer times.

Assuming there are the points \( s_0 \) in the region \( D \), satisfy \( \zeta(s_0) = 1 \) (If there is not such point \( s_0 \), then the result of lemma 9 turns into \( N(\rho, \frac{1}{1}) = 0 \), the results of the theorem of this article can be obtained directly). For different single value analytic branch, the value of \( \log \zeta(s_0) = \log 1 \) are different, it can value 0, \( 2\pi ki, (k = \pm 1, \pm 2, \ldots) \). We select the single valued analytic branch of \( \log \zeta(s_0) = \log 1 = 0 \).

Because the region \( D \) is simple connected region, so the according to the single value theorem of analytic continuation (the theorem see the theorem 2 of page 276 of the references [5] and theorem 1 of page 155 of the references [6]), \( \log \zeta(\sigma + it) \) is the single valued analytic function in the region \( D \). In addition, when \( \zeta(\sigma + it) = 1 \), have \( \log \zeta(\sigma + it) = 0 \). In other words, 1 value point of \( \zeta(\sigma + it) \) is the zero of \( \log \zeta(\sigma + it) \).

Below, \( \log \zeta(\sigma + it) \) always express a single valued analytic branch for we selected.

**LEMMA 8.** If RH is correct, then when \( 0 < \delta \leq \frac{1}{100}, \sigma \geq \frac{1}{2} + 2\delta, |t| \geq 16 \), we have

\[
|\log \zeta(\sigma + it)| \leq c_2 \log |t| + c_3
\]

**proof.** In the lemma 7, we choose \( z_0 = 0 \), \( f(z) = \log \zeta(z + 4 + it) \), \( |t| \geq 16 \), \( R = \frac{7}{2} - \delta \), \( r = \frac{7}{2} - 2\delta \). Because \( \log \zeta(z + 4 + it) \) is the analytic function
in the circle $|z - z_0| \leq R$, so, from the lemma 7, in the circle $|z - z_0| \leq r$, we have

$$| \log \zeta(z + 4 + it) - \log \zeta(4 + it)| \leq \frac{7}{\delta} \left( A(R) - Re \log \zeta(4 + it) \right)$$

hence

$$| \log \zeta(z + 4 + it)| \leq \frac{7}{\delta} \left( A(R) + | \log \zeta(4 + it)| \right) + | \log \zeta(4 + it)|$$

from the lemma 6, have

$$A(R) = \max_{|z - z_0| \leq R} \log | \zeta(z + 4 + it)| \leq \frac{1}{2} \log |t| + \log c_1$$

from the lemma 5, have

$$| \log \zeta(z + 4 + it)| \leq c_2 \log |t| + c_3$$

because $|t| \geq 16$ is real number arbitrarily, so when $\sigma \geq \frac{1}{2} + 2\delta$, we have

$$| \log \zeta(\sigma + it)| \leq c_2 \log |t| + c_3$$

The proof is complete.

**Lemma 9.** If RH is correct, then when $0 < \delta \leq \frac{1}{100}$, $|t| \geq 16$, $\rho = \frac{7}{2} - 2\delta$, in the circle $|z| \leq \rho$, we have

$$N \left( \rho, \frac{1}{\zeta(z + 4 + it) - 1} \right) \leq \log \log |t| + c_4$$

**Proof.** In the lemma 2, we choose $f(z) = \log \zeta(z + 4 + it)$, $R = \frac{7}{2} - \delta$, $\rho = \frac{7}{2} - 2\delta$, $a_\lambda$ ($\lambda = 1, 2, ..., h$) is the zeros of function $\log \zeta(z + 4 + it)$ in the circle $|z| < \rho$, multiple zeros being repeated. The function $\log \zeta(z + 4 + it)$ has no poles in the the circle $|z| < \rho$, and $\log \zeta(4 + it)$ not equal to zero, therefore we have

$$\log | \log \zeta(4 + it)| = \frac{1}{2\pi} \int_0^{2\pi} \log | \log \zeta(4 + it + \rho e^{i\varphi})| d\varphi - \sum_{\lambda=1}^{h} \log \frac{\rho}{|a_\lambda|}$$
from the lemma 5 and the lemma 8, have

\[
\sum_{\lambda=1}^{h} \log \frac{\rho_{\lambda}}{|a_{\lambda}|} \leq \log \log |t| + c_4
\]

because \( z = 0 \) is neither the zero, nor pole of the function \( \log \zeta(z + 4 + it) \), so if \( r_0 \) is a sufficiently small positive number, then

\[
\sum_{\lambda=1}^{h} \log \frac{\rho_{\lambda}}{|a_{\lambda}|} = \int_{r_0}^{\rho} \left( \log \frac{\rho_{\lambda}}{t} \right) \, dn(t, \frac{1}{f}) = \left[ \left( \log \frac{\rho_{\lambda}}{t} \right) n(t, \frac{1}{f}) \right]_{r_0}^{\rho} + \int_{r_0}^{\rho} \frac{n(t, \frac{1}{f})}{t} \, dt = \int_{0}^{\rho} \frac{n(t, \frac{1}{f})}{t} \, dt = N\left( \rho, \frac{1}{f} \right) = N\left( \frac{1}{\log \zeta(z + 4 + it)} \right) \geq N\left( \rho, \frac{1}{\zeta(z + 4 + it) - 1} \right)
\]

The proof is complete.

**THEOREM.** If RH is correct, then when \( \sigma \geq \frac{1}{2} + 4\delta, \quad 0 < \delta \leq \frac{1}{100}, \quad |t| \geq 16 \), we have

\[
|\zeta(\sigma + it)| \leq c_8 (\log |t|)^{c_6}
\]

**proof.** In the lemma 3, we choose \( f(z) = \zeta(z + 4 + it), \quad |t| \geq 16 \), from the lemma 5, have \( f(0) = \zeta(4 + it) \neq 0, \quad \infty, \quad 1, \quad f'(0) = \zeta'(4 + it) \neq 0 \), and \( f'(0) = \zeta'(4 + it) \geq 0.012, \quad |f(0)| = |\zeta(4 + it)| \leq 1.0824 \). We choose \( R = \frac{7}{2} - 2\delta, \quad r = \frac{7}{2} - 3\delta \). Because \( \zeta(z + 4 + it) \) is the analytic function, and have neither zero nor the poles in the circle \( |z| \leq R \), therefore

\[
N\left( R, \frac{1}{f} \right) = 0, \quad N\left( R, f \right) = 0
\]

from the lemma 9, have

\[
T\left( r, \zeta(z + 4 + it) \right) \leq 2 \log \log |t| + c_5
\]
In the lemma 1, we choose \( R = \frac{7}{2} - 2\delta, \rho = \frac{7}{2} - 3\delta, r = \frac{7}{2} - 4\delta \), from the maximal principle, in the circle \(|z| \leq r\), we have

\[
\log^+ |\zeta(z + 4 + it)| \leq c_6 \log \log |t| + c_7
\]

Since \(|t| \geq 16\) is arbitrary real number, so when \(\sigma \geq \frac{1}{2} + 4\delta\), have

\[
\log^+ |\zeta(\sigma + it)| \leq c_6 \log \log |t| + c_7
\]

therefore

\[
\log |\zeta(\sigma + it)| \leq c_6 \log \log |t| + c_7
\]

therefore

\[
|\zeta(\sigma + it)| \leq c_8 (\log |t|)^{c_6}
\]

The proof is complete.

The result of this theorem is better than known results.

REFERENCES


