The quest for conformal geometric algebra Fourier transformations

Eckhard Hitzer

Department of Material Science, International Christian University, 181-8585 Japan

Abstract. Conformal geometric algebra is preferred in many applications. Clifford Fourier transforms (CFT) allow holistic signal processing of (multi) vector fields, different from marginal (channel wise) processing: Flow fields, color fields, electromagnetic fields, . . . The Clifford algebra sets (manifolds) of \( \sqrt{-1}T \) lead to continuous manifolds of CFTs. A frequently asked question is: What does a Clifford Fourier transform of conformal geometric algebra look like? We try to give a first answer.

Keywords: Clifford geometric algebra, Clifford Fourier transform, conformal geometric algebra, horosphere.

PACS: AMS Subj. Class. 15A66, 42A38

INTRODUCTION

Note: Please respect the Creative Peace License [13] when applying this work. Conformal geometric algebra is widely used in applications [8, 21]. Reasons are the elegant representation of geometric objects by products of points. The products of these objects form in turn new objects (via intersection, union, projection, . . .). Conformal transformations are products of reflections at hyperplanes (versors). This even leads to the linearization of translations. Moreover, it is very interesting for the approximation abilities of conformal geometric algebra neural networks.

Clifford’s geometric algebra \( Cl(2,0) \) is a special case of Clifford algebras \( Cl(p,q) \) of \( \mathbb{R}^{p,q} \) over \( \mathbb{R} \). \( Cl(2,0) = Cl(\mathbb{R}^2) \) has a geometric product of vectors \( a, b \in \mathbb{R}^2 \)

\[
ab = a \cdot b + a \wedge b.
\]  

of scalar symmetric inner product and bivector antisymmetric outer product. It is the geometric algebra of the Euclidean plane and unifies 2D vector algebra, complex numbers and spinors in one algebra. Given an orthonormal vector basis \( \{ e_1, e_2 \} \) of \( \mathbb{R}^2 \), the 4D \( (2^2 = 4) \) Clifford algebra \( Cl(2,0) \) has a basis of 1 scalar, 2 vectors, and 1 bivector \( \{ 1, e_1, e_2, e_{12} = i \} \), where we define \( i = e_{12} = e_1 e_2 \). The basis bivector \( i \) squares to \(-1\). Rotation by a versor (rotor) \( R \) is a product of two reflections

\[
x' = R x R, \quad R = e^\frac{i}{2} \phi, \quad R = e^{-\frac{i}{2} \phi}.
\]  

CONFORMAL GEOMETRIC ALGEBRA OF THE EUCLIDEAN PLANE

The conformal model (see [8] and its references) of the Euclidean plane in \( Cl(3,1) \) extends the basis of \( \mathbb{R}^2 \) by adding a plane \( \{ e_+, e_- \} \), \( e_+^2 = 1 \), \( e_-^2 = -1 \), \( e_+ \cdot e_- = 0 \) and thus generates \( Cl(2+1,0+1) = Cl(3,1) \). We choose a null-basis, assigning origin and infinity vectors (like in projective geometry):

\[
e_0 = \frac{1}{2}(e_+ - e_-), \quad e_\infty = e_- + e_+, \quad e_0^2 = e_\infty^2 = 0, \quad e_0 \cdot e_\infty = -1.
\]  

The bivector \( E \) of the added origin and infinity plane is

\[
E = e_+ \wedge e_- \wedge e_\infty \wedge e_0, \quad E^2 = 1.
\]  

There are the multiplication properties

\[
e_0 E = -e_0, \quad E e_0 = e_0, \quad e_\infty E = e_\infty, \quad E e_\infty = -e_\infty.
\]  

The full 16D basis of \( Cl(3,1) \) is

\[
\{ 1, \ e_1, e_2, e_0, e_\infty, \ e_{12} = i, e_1 e_0, e_2 e_0, e_1 e_\infty, e_2 e_\infty, E, \ i e_0, i e_\infty, e_1 E, e_2 E, \ i E \}.
\]
with scalars (1), vectors (4), bivectors (6), trivectors (4), pseudoscalars (1), and Cl(2,0) ⊂ Cl(3,1). Note that only
bivectors and trivectors change sign under reversion (reversing the order of all vector factors).

We have a set of geometric objects GO ⊂ Cl(3,1) described in the conformal model [8] as conformal points
P = p + \frac{1}{2}P e_0 + e_0 and point pairs P1 ∧ P2, flat point pairs P ∧ e_0, circles C = P1 ∧ P2 ∧ P3
through three points, and lines P1 ∧ P2 ∧ e_0 (hyperplane), offset from the origin. A point P is on one of these objects
(with blade Obj): P ∈ Obj ⇔ P ∧ Obj = 0. The products of these objects with their reverse give [8]

$$PP = 0, \quad Pp = -\mathbf{r}^2 D^2, \quad (P ∧ e_0)(P ∧ e_0) = -1, \quad \text{Line} \quad \text{Line} = -1, \quad \text{Circle} \quad \text{Circle} = \mathbf{r}^2 D^2 < 0,$$

(7)

where D = p_1 − p_2 = 2rd, d^2 = 1 for the point pair Pp, and D is a scalar multiple of e_12 with negative square
for the circle Circle. We therefore assume as norm for these objects ||Obj|| = \sqrt{-Obj Obj}. Note that for two conformal
points Po Q the number \sqrt{-PQ} = |p − q|/\sqrt{2} is their distance.

Conformal transformations of \mathbb{R}^2 become versors (products of vectors) in Cl(3,1): rotation around origin: R = e^{i\Omega}
(see 2), translation by t ∈ \mathbb{R}^2: T = e^{\frac{t}{2}e_0}, transversion by t ∈ \mathbb{R}^2: Tv = e^{\frac{t}{2}e_0} (a transversion composes inversion at
the unit sphere around the origin, translation, and a second inversion at the unit sphere around the origin), and scaling
by \gamma^2, \gamma ∈ \mathbb{R}: S = e^{\gamma^2\Omega}. These transformation versors V are applied to conformal objects A as

$$A \to A' = \tilde{V}AV.$$  

(8)

The inner product of f, g : \mathbb{R}^{p,q} \to GO, respectively its symmetric scalar part, are

$$(f,g) = -\int_{\mathbb{R}^{p,q}} f(x)g(x) d^nx, \quad \langle f,g \rangle = -\int_{\mathbb{R}^{p,q}} f(x)∗g(x) d^nx.$$  

(9)

The \mathcal{L}^2(\mathbb{R}^{p,q}; GO)-quasi-norm (indicating distance in the case of conformal points) is

$$\|f\|^2 = (\langle f,f \rangle), \quad \mathcal{L}^2(\mathbb{R}^{p,q}; GO) = \{ f : \mathbb{R}^{p,q} \to GO \mid \|f\| < \infty \}. $$  

(10)

Note, that for ensuring finite basis coefficient values of geometric objects in GO ⊂ Cl(3,0) in the basis (6) of Cl(3,1)
the principal reverse operation of Clifford algebra can be used (reverse combined with changing the sign of every basis
vector with negative square).

The Clifford algebra Cl(3,1) is isomorphic to the (square) matrix algebras \mathscr{M}(4,\mathbb{R}), \mathcal{S}c(f) = 0 for every f = \sqrt{-1}
∈ Cl(3,1) [12, 19]. All \sqrt{-1} ∈ Cl(3,1) are computable with the Maple package CLIFFORD [2, 1, 18]. The square
roots f of −1 constitute a unique conjugacy class of dimension 8, with as many connected components as the group
G(\mathscr{M}(4,\mathbb{R})) of invertible elements in \mathscr{M}(4,\mathbb{R}). For \mathscr{M}(4,\mathbb{R}), the centralizer (all elements in Cl(3,1) commuting with
f) and the conjugacy class of a square root f of −1 both have \mathbb{R}-dimension 8 with two connected components.

**CLIFFORD FOURIER TRANSFORMATIONS**

We now consider a generalization of quaternion and Clifford Fourier transforms CFTs [3, 4, 10, 5, 6, 7, 9, 11, 14, 15, 16, 17, 20, 22] to conformal geometric algebra Cl(3,1).

**Definition 1 (CFT with respect to two square roots of −1).** Let f, g ∈ Cl(3,1), f^2 = g^2 = −1, be any two square roots
of −1. The general Clifford Fourier transform (CFT) of h ∈ L^1(\mathbb{R}^{p,q}; GO), with respect to f, g is

$$\mathcal{F}_{f,g} \{ h \} (\omega) = \int_{\mathbb{R}^{p,q}} e^{-f(x,\omega)}h(x)e^{-g(x,\omega)} d^nx,$$

(11)

where n = p + q, d^nx = dx_1…dx_n, x, \omega ∈ \mathbb{R}^{p,q}, and u, v : \mathbb{R}^{p,q} × \mathbb{R}^{p,q} → \mathbb{R}.

The square roots f, g ∈ Cl(3,1) of −1 may be from any component of any conjugacy class. The above CFT is
steerable in the continuous submanifolds of \sqrt{-1} in Cl(3,1). We have the following properties of the general two-
sided CFT: a Plancherel identity, respectively a Parseval identity, for functions h_1, h_2, h ∈ L^2(\mathbb{R}^{p,q}; GO)

$$\langle h_1, h_2 \rangle = \frac{1}{(2\pi)^n} \langle \mathcal{F}_{f,g} \{ h_1 \}, \mathcal{F}_{f,g} \{ h_2 \} \rangle, \quad \| h \| = \frac{1}{(2\pi)^{n/2}} \| \mathcal{F}_{f,g} \{ h \} \|.$$  

(12)
For these identities to hold we need for \( \Theta = -f, \tilde{g} = -g \).

We now seek to find the most suitable \( \sqrt{-1} \in \text{Cl}(3,1) \). We can write every real \( f = \sqrt{-1}, f \in \text{Cl}(3,1) \) as

\[
f = \alpha + b + \beta i + (\alpha_e + b_e + \beta_E i) e_0 + (\alpha_0 + b_0 + \beta_0 i) e_0 + (\alpha_E + b_E + \beta_E i) e_0,
\]

where \( \alpha = 0, \alpha_0, \alpha_e, \beta, \beta_0, \beta_E, b, b_0, b_e, b_E, e_0 \in \mathbb{R} \).

From \( f^2 = -1 \) we obtain the root equation (main condition for \( f \))

\[
f^2 = b^2 - 2\alpha_0 \alpha_e + 2b_0 \cdot b_e + 2\beta_0 \beta_e + \alpha_e^2 + b_e^2 - \beta_e^2 = -1,
\]

plus side conditions for zero non-scalar parts of \( f^2 \). By imposing that \( \tilde{f} = -f \) (necessary for Plancherel and Parseval identities), we abandon scalar, vector and pseudoscalar parts of \( f \):

\[
f = \beta i + b_0 e_0 + \beta_0 i e_0 + \alpha_e E + b_E E.
\]

and retain only bivector and trivector parts.

Regarding the trivector \( \sqrt{-1} \) in \( \text{Cl}(3,1) \) we can calculate the following. For \( f = b_E E, t, t' \in \mathbb{R}^2 \), \( t' = -b_E t \), we obtain

\[
e^{-b_E E e_0} e^{b_E E} = e_0, \quad e^{-b_E E i e_0} e^{b_E E} = \text{ch}^2(|b_E|) t + \text{sh}^2(|b_E|) t' + 2 \text{ch}(|b_E|) \text{sh}(|b_E|) (t \wedge \frac{b_E}{|b_E|}) E.
\]

For \( f = \beta_e i e_0 \) we obtain

\[
e^{-\beta_e i e_0} e^{\beta_e i e_0} = e_0 - 2\beta_e^2 e_0 - 2\beta_e i E.
\]

Like for quaternion Fourier transformations this may help to separate signal symmetry components, but for now we set the trivector parts of \( \sqrt{-1} \) in \( \text{Cl}(3,1) \) aside. We keep only the four bivector parts.

If we only keep the bivector parts of \( f = \sqrt{-1} \) in \( \text{Cl}(3,1) \) we have

\[
f = \beta i + b_0 e_0 + \alpha_e E.
\]

We recognize that \( e^{-\varphi \beta i} h(x) e^{\varphi \beta i} \) means a local rotation by \( 2\varphi \beta \in \mathbb{R} \) of the signal function \( h : \mathbb{R}^p \rightarrow GO \).

\( e^{-b \cdot c} h(x) e^{b \cdot c} \), \( b \in \mathbb{R} \), means a translation by \( 2b \cdot c \in \mathbb{R}^2 \) of the signal function \( h : \mathbb{R}^p \rightarrow GO \).

\( e^{-\beta \varphi b \cdot e_0} h(x) e^{\beta \varphi b \cdot e_0} \), \( c \in \mathbb{R} \), means a transversion over \( 2c b_0 \in \mathbb{R}^2 \) of the signal function \( h : \mathbb{R}^p \rightarrow GO \).

\( e^{-\alpha a e_0} h(x) e^{\alpha a e_0} \), \( a \in \mathbb{R} \), means a scaling by the factor \( e^{2\alpha a e_0} \) of the signal function \( h : \mathbb{R}^p \rightarrow GO \). Thus every term in a bivector \( \sqrt{-1} \) in \( \text{Cl}(3,1) \) has a clear geometric transformation interpretation. This could be of great advantage for the choice and application of conformal CFTs in conformal image and signal processing.

Considering the side conditions for the zero non-scalar components in \( f^2 = -1 \) let the angle \( \Theta = \angle(b_0, b_E) \). For \( \Theta \neq 0, \pi \) we find that

\[
\beta = \frac{1}{\sin \Theta} [\alpha_e \cos \Theta \pm \sqrt{\alpha_e^2 \cos^2 \Theta + \sin^2 \Theta}].
\]

Especially for \( \Theta = \pi/2, 3\pi/2 \) we obtain

\[
\beta = \pm \sqrt{1 + \alpha_e^2}.
\]

For the special case \( \Theta = 0, \alpha_e = 0 \) we have

\[
\beta = \pm \sqrt{1 + 2|b_0||b_E|}.
\]

For \( \Theta = \pi, \alpha_e = 0 \) we have

\[
\beta = \pm \sqrt{1 - 2|b_0||b_E|}.
\]

whereas for \( \Theta = \pi, \beta = 0 \) we have

\[
\alpha_e = \pm \sqrt{2|b_0||b_E| - 1}.
\]
CONCLUSION

We have briefly introduced the conformal geometric algebra $Cl(3,1)$ model of the Euclidean plane. Then we defined Clifford Fourier transforms based on real $f = \sqrt{-1}$ in $Cl(3,1)$, and tried to select a Clifford Fourier transform by studying the manifold of $\sqrt{-1}$ in $Cl(3,1)$. A selection with clear geometric interpretation proved to be the bivector parts of $\sqrt{-1}$ in $Cl(3,1)$ for the construction of a CFT in conformal geometric algebra model of Euclidean plane. Then every term in a bivector $\sqrt{-1}$ in $Cl(3,1)$ has a geometric transformation interpretation. Setting $g = -f$ leads to conformal rotor transformation CFTs. We completed this by a detailed characterization of bivector parts of non-scalar parts of $ff$, to be zero, into account. We therefore hope that this new conformal CFT with real bivector $\sqrt{-1}$ in $Cl(3,1)$ will be tried in applications. It still shares many properties of conventional FT (such as linearity, shift, modulation, partial differential, Plancherel, Parseval, convolution, ...).

ACKNOWLEDGMENTS

Soli deo gloria. I do thank my dear family, S. Sangwine for discussions, the friendly staff of the Wivenhoe public library (Essex, UK), T. Simos, W. Sprössig and K. Gürlebeck.

REFERENCES

2. R. Abłamowicz, Computations with Clifford and Grassmann Algebras, AACA 19, No. 3–4 (2009), 499–545.
10. E. Hitzer, B. Mawardi, Clifford Fourier Transf. on Multivector Fields and Unc. Princ. for Dim. $n = 2$ (mod 4) and $n = 3$ (mod 4), P. Angles (ed.), AACA, 18(S3.4), (2008), 715–736.
12. E. Hitzer, R. Abłamowicz, Geometric Roots of $-1$ in Clifford Algebras $Cl(p,q)$ with $p+q \leq 4$. AACA, 21(1), (2011) 121–144.