Infinity at stake
Selected arguments on the actual infinity hypothesis

Interciencia
Infinity at stake
Selected arguments on the actual infinity

Against dogmatism and intolerance
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1.-Introduction

Some of the most relevant problems of contemporary philosophy were already posed by the pre-Socratic philosophers as early as the VII century BC (partially suggested or directly taken from the cultural precedents developed in the Neolithic Fluvial Cultures.\(^1\) Three among those problems deserve special consideration: the problem of change, infinity, and self-reference. The first of them is surely the most difficult, and at the same time the most relevant, problem ever posed by man. For this reason, it is surprising what little attention we currently pay to that fascinating problem, specially when compared with the attention we pay to the other two.

After more than twenty seven centuries, the problem of change remains unsolved. In spite of its apparent simplicity, no one has been capable of explaining, for instance, how a simple change of position takes place. Physics, the science of change (the science of the regular succession of events, as Maxwell called it [127, page 98]) seems to have forgotten its most fundamental problem. In their turn, some philosophers as Hegel\(^2\) defended that change is an inconsistent notion, while others, as McTaggart, came to the same conclusion as Parmenides [147] on the impossibility of change [132]. Perhaps the (apparent) insolvibility of the problem of change has to do with the continuum spacetime framework where all solutions have been tried. As Appendix A of this book shows, the problem of change could find a solution within a discrete spacetime framework.

While change is an evident characteristic of our continuously evolving universe, both infinity and self-reference are theoretical notions without apparent relation to the natural world. Cantor and Gödel (the princes of infinity and self-reference respectively) were two enthusiastic platonists of scarce devotion to natural sciences and of enormous influence in contemporary mathematics.\(^3\) To illustrate the profound Cantor’s teoplatonic convictions, let us remember some of his words:

\[\text{...in my opinion the absolute reality and legality of natural numbers is much higher than that of the sensory world. This is so because of a unique}\]

\(^{1\text{[22],[169],[144],[183]}}\)
\(^{2\text{[96],[98],[133],[146],[158],[196]}}\)
\(^{3\text{For the case of Cantor see [57],[134],[43, pag. 141]; for that of Gödel [81, pags. 235-236],[83, pag. 359],[73],[59][140],[100],[85]}}\)
and very simple reason, namely, that natural numbers exist in the highest
degree of reality, both separately and collectively in their actual infinitude,
in the form of eternal ideas in Intellectus Divinus. ([134]; reference and
(Spanish) text in [76])

...I am only an instrument of a higher power, which will continue to work
after me in the same way as it manifested itself thousands of years ago in
Euclides and Archimedes . . . ([42, pp 104-105])

...I cannot regards them [the atoms] as existent either in concept or in
reality no matter how many useful things have up to a certain limit been
accomplished by means of this fiction. ([41, p 78], English translation of
[34])

Twenty seven centuries of discussions were not sufficient to prove the consist-
tency (or inconsistency) of the actual infinity hypothesis, which had finally to
be legitimated by the expeditious way of axioms. Set contemporary mathematics
are founded on the believing that infinite sets do exist as complete totalities.

Set theory is a strictly infinitist theory, a theory founded on, and inspired
by, the actual infinity hypothesis. For Georg Cantor, one of its most relevant
founders, the actual infinity was not a simple hypothesis but a firm teo-
platic conviction. Set theory contains, however, the appropriate instruments
to put into question the formal consistency of the actual infinity hypothesis.
Although, until now, they have never been used with that critical intention.
As we will see here, that is the case of \( \omega \), the smallest infinite ordinal, and of
all \( \omega \)-ordered sets and sequences. We will make an extensive use of them in
this book.

Self-reference, in both formal and colloquial languages, is also a theoretical
notion on which there is not a general agreement. Self-reference paradoxes
have been, and continue to be, the source of interminable discussions. One of
those paradoxes, the Liar paradox, led (via Richard paradox, as Gödel himself
recognized [82, p. 56]), to the celebrated Gödel’s first incompleteness theorem.
Many logicians consider it as the most important theorem of all times. From
our perspective of natural sciences, that sounds a little exaggerated.

To put it in a few words, we inherited from Presocratics, among other things,
a promising challenge (the problem of change) and two questionable concepts
(self-reference and the actual infinity). The problem is that while we have
forgotten the challenge, self-reference and the actual infinity form part of the
foundational basis of contemporary logic and mathematics. This book is mainly
devoted to put into question the most annoying of those concepts: the actual

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4 Axiom of Infinity in modern set theories, which, in short, states the existence of an infinite
(denumerable) set.
5 For example the ordered list of natural numbers would exist as a complete totality despite
the fact that no last number completes it.
6 As firm as a rock in Cantor’s own words (Cantor’s letter to Heman, June 21, 1888)
7 In addition to language and meta-language (language on language) we would also have self-
language, language autonomously speaking about itself.
8 In informal terms: This sentence is false.
We must recall at this point that science seems to be excessively self-reverent and scarcely self-critique. To put the personal convictions and interests in front of the objective knowledge of reality is not as unusual as it could be expected. In those conditions, to question a long-established foundational assumption becomes an impossible task, even if that assumption is suspicious of being inconsistent. In my opinion the Axiom of Infinity is one of those inconsistent foundational assumptions.

The consequences of infinitist mathematics are disastrous because it promotes an analogue, and then continuous, model for the physical world which is clearly in conflict with the digital nature revealed until now by all physical observations and measurements: ordinary matter, elementary particles, energy, electric and non-electric charges, seem to be, all of them, discrete entities with indivisible minima. The war of physicists against the infinities is really shocking. They pay a high price in the form of interminable and tedious calculations for getting rid of infinities. While, on the other hand, they don’t spend a single minute to call into question the formal consistency of the hypothesis of the actual infinity.

Thanks to the assumption of the actual infinity hypothesis, experimental sciences are forced to explain a reality that seems to be essentially discrete by means of indiscrete mathematics. A task that could be impossible at certain basic levels were discreteness becomes essential, as is the case of the quantum level. Thus, the tragedy of infinity is that we have not developed an appropriate discrete mathematics to explain a world that seems to be essentially discrete. Even our current discrete mathematics have been built in terms of indiscrete mathematics. Apart from certain particular applications, discrete mathematics are usually interpreted as a mere approximation to the true continuous world of infinitist mathematics. The problem is that no continuous world seems to exist.

In any case, the hypothesis of the actual infinity is just a hypothesis, and one has the right and the duty to bring it into question. That is the main objective of this book. A collection of critical arguments on the hypothesis of the actual infinity developed for the last twenty years. Each chapter consists of a complete and independent argument, so that they can be read in any order. It also includes three appendixes, the first addresses the problem of change to illustrate the consequences of assuming the infinitist spacetime continuum. The second introduces a non platonic alternative to current set theories. The third is a brief criticism of platonic essentialism (the home of the actual infinity) from the perspective of contemporary biology.

Although the discussions on the mathematical infinity may seem intimidating to the general reader, this book is anything but intimidating. It is a book on basic science. That science we learn at High School and first university courses.

\[^9\] Obviously, this independence has a narrative cost in terms of an excessive number of text repetitions, for which I apologize.
The problem is we learn and teach it as a sort of catechism free of criticism. Basic science is rarely put into question because scientists work some steps beyond it. But basic science must also be, at least periodically, questioned. As just noted, here we question one of its basic hypothesis, the hypothesis of the actual infinity, one of the fundamental pillars of contemporary mathematics.

In almost all chapters the infinite in question will be the numerable infinity (the smallest of the infinities\textsuperscript{10}) subsumed within the Axiom of Infinity. But also the infinite that legitimizes the sequences of increasing infinities.\textsuperscript{11} Thus, to prove the inconsistency of the first infinity implies to invalidate all the others.

There is a general agreement in that a contradiction suffices to prove the inconsistency of the hypothesis from which the contradictory results have been deduced. Except in the case of the hypothesis of the actual infinity. And this is not a joke: in Cantor’s words, certain infinities are inconsistent because of their excessive infinitude [34]. An additional reason to deal exclusively with the smallest of them.

\textsuperscript{10}The infinite of the set of natural numbers.

\textsuperscript{11}The sequence of 'alephs': $\aleph_0$, $\aleph_1$, $\aleph_2$ . . .; and the sequence of powers $\aleph_0$, $2^{\aleph_0}$, $2^{2^{\aleph_0}}$ . . .
2.-Conventions

1. To facilitate discussions, all paragraphs in this book will be consecutively numbered (as this one). They will be referred to by their corresponding numbers without parenthesis, such as they appear at the beginning of each paragraph. For the same reason, all equations will also be consecutively numbered within each chapter, although in this case the numbers will be put in brackets on the right side of each equation. Equations will be referred to by their corresponding number in brackets.

2. Theorems, definitions, conclusions, etc. will be numbered with the same number of the paragraph where they are stated. For instance, if a theorem is stated in paragraph 153 it will be referred to as Theorem 153.

3. Most of the sets and sequences we will make use of, will be $\omega$-ordered (as the sequence 1, 2, 3, ... of natural numbers in their natural order of precedence), in a few cases they will be $\omega^*$-ordered (as in the case of the increasing sequence of negative integers ... -3, -3, -1). In many arguments we will also make use of sequences of instants within finite intervals of time, those sequences will always be strictly increasing and convergent, being always the limit of the sequence the corresponding right endpoint of the interval.

4. In most cases we will use the word denumerable to refer to the infinity of the set $\mathbb{N}$ of natural numbers and to the infinity of any other set or sequence that can be put into a one to one correspondence with $\mathbb{N}$. The words enumerable or numerable can also be used with the same meaning. Although the word countable is also used to refer to finite or denumerable infinite sets we will not use it here in order to avoid confusions. Finally, the terms non-countable or non-denumerable will be used to refer to the infinities greater than the denumerable infinity.

5. Needless to say, all arguments in this book are of a conceptual nature, even when they make use of material artifacts as machines, boxes, balls and the like, all of which have to be understood as theoretical devices to facilitate discussions.
Part I: The Paradise

3.- The actual infinity

INTRODUCTION

6 This book deals exclusively with the actual infinity, although some references to the potential infinity will be inevitable. This is why we begin by introducing the distinction between the potential and the actual infinity. Once introduced, we will define the actual infinity in set-theoretical terms, and then the distinction between transfinite cardinals and ordinals. This is all we will need in order to follow the arguments on the actual infinity hypothesis that will be developed in the remainder of the book. Most of those arguments will be related to $\omega$, the smallest infinite ordinal; the ordinal of the set $\mathbb{N}$ of natural numbers when ordered in their natural order of precedence: $\mathbb{N} = \{1, 2, 3, \ldots \}$ (see below).

7 'Infinite' is a common word we use to refer to the quality of being huge, immense, unbounded etc. In this way, and according to Gauss\(^1\) the infinite is a manner of speaking. But the word 'infinite' ('infinity', 'the infinite') has also a precise mathematical meaning: a set is said infinite if it can be put into a one to one correspondence with one of its proper subsets. This is the well known Dedekind’s definition that, together with Cantor’s works on transfinite numbers, inaugurated modern transfinite mathematics at the end of the XIX century. Although the history of the mathematical infinity began twenty seven centuries before.

8 Fortunately there is an excellent literature on the history of infinity.\(^2\) I will not even give a summary of that history, though we could arbitrarily choose three of its most significant protagonists as historical references:

1. Zeno of Elea (490-430 BC), a presocratic philosopher that made use for the first time of the mathematical infinity when defending Parmenides’ thesis on the impossibility of change. We know Zeno’s work (near forty arguments, including his famous paradoxes against the possibility of change \([2], [52]\)) through his doxographers (Plato, Aristotle, Diogenes Laertius or Simplicius). The infinite in Zeno’s arguments seems to be the countable

\(^1\)C.F. Gauss, Letter to astronomer H.C. Shumacher, 12 July 1831
\(^2\)For instance: [208], [124], [171], [23], [163], [51], [115], [135], [138], [110], [111], [1], [136], [50], [197], [15].
actual infinity, although obviously Zeno is not doing infinitist mathematics but logical argumentations in which appear infinite collections of points and instants. Zeno’s arguments work properly only if those collections are considered as complete infinite totalities (see Chapter 20 on Zeno’s Dichotomies).

2. Aristotle (384-322 BC), one of the most influential thinkers in western culture. Philosopher and naturalist, he introduced the notion of one to one correspondence just when trying to solve some of the Zeno’s paradoxes. He also introduced the basic distinction between the potential and the actual infinity. We will analyze that distinction in the next section.

3. Georg Cantor (1845-1918), German mathematician cofounder, together with R. Dedekind and G. Frege, of set theory at the end of the XIX century. His work on transfinite numbers (cardinals and ordinals) lays the foundations of modern transfinite mathematics. He inaugurated the so called paradise of the actual infinity, where, according to D. Hilbert, infinitists will inhabit forever.

9 From Zeno to Aristotle the only infinity was the actual infinity, although that notion was far from being clearly established. From Aristotle to Cantor we find defenders of both types of infinities (actual and potential) although with a certain hegemony of the potential infinity, particularly from the XIII century, once Aristotle was ‘christianized’ by the medieval scholastics. In those preinfinitist times, the same arguments could be used in support of one or of the other hypothesis (for instance the arguments based on the correspondence between the points of a circumference and the points of a straight line). But there is not still a theory of the mathematical infinity. The first mathematical theory of infinity appears at the end of the XIX century, being Bolzano, Dedekind and, specially, Cantor its most relevant founders. From Cantor to nowadays the hegemony of the actual infinity has been almost absolute and, in addition, free of serious criticism.

**Actual and potential infinity**

10 The distinction between the actual and the potential infinity is due to Aristotle [11], [10]. We will now explain it in modern set-theoretical terms. It goes without saying that the only infinite in modern transfinite mathematics, including Dedekind’s foundational definition of the infinite sets, is the actual infinity.

11 Consider the list of natural numbers in their natural order of precedence: 1, 2, 3, . . . According to the hypothesis of the actual infinity that list exists as a complete totality, i.e as a totality that contains, all at once, all natural numbers. The ellipsis in:

\[ N = \{1, 2, 3, \ldots \} \tag{1} \]

stands for all natural numbers. Notice the list of natural numbers is considered as a complete totality despite the fact that no last number completes the list.
12 To emphasize this sense of completeness let us consider the task of counting the successive natural numbers 1, 2, 3, ... In agreement with the hypothesis of the actual infinity we could count all natural numbers in a finite time by performing the following supertask:\(^3\)

Count each of the successive numbers 1, 2, 3, ... at each of the successive instants \(t_1, t_2, t_3, \ldots\) of a strictly increasing sequence of instants within the finite real interval \((t_a, t_b)\), being \(t_b\) the limit of the sequence. For instance the classical sequence defined by:

\[
t_n = t_a + (t_b - t_a) \frac{2^n - 1}{2^n}.
\]

In these conditions, at \(t_b\) all natural numbers would have been counted. All(!)

13 The above task of counting all natural numbers is an example of supertask. They will be discussed later in this book. Meanwhile note that the fact of pairing the elements of two infinite sequences does not prove both sequences exist as complete totalities. They could also be potentially infinite.

14 The alternative to the actual infinity hypothesis is the hypothesis of the potential infinity, which rejects the existence of complete infinite totalities and then the possibility of counting all natural numbers. From this perspective, natural numbers result from the endless process of counting: it is always possible to count numbers greater than any given number. But it is impossible to complete the process of counting all of them, so that the complete list of all natural numbers makes no sense.

15 In short, the actual infinite hypothesis states that the infinite totalities are complete totalities, even if no last element completes them, as in the case of the ordered list of natural numbers. From this perspective it is possible to complete a sequence of steps in which no last step completes the sequence; or even without a first step to start the sequence, as in the case of \(\omega\) -ordered sequences (see below), for instance, the increasing sequence of negative integers \(\ldots, -3, -2, -1\). From the perspective of the potential infinite both possibilities are impossible. From this perspective the only complete totalities are finite totalities, as large as we wish but always finite.

16 The potential infinity (the 'improper' or 'non-genuine' infinity as Cantor called it [41, p. 70]) has never deserved the attention of contemporary mathematicians. The infinity in Dedekind’s definition of the infinite sets is the actual infinity. The infinitely many elements of an infinite set exist all at once, as a complete totality. Dedekind’s definition is, therefore, based on the violation of the old Euclidian Axiom of the Whole and the Part [71]. Set theory has been built on that violation.

17 The hegemony of the actual infinity in contemporary mathematics is al-

\(^3\)An infinite sequence of actions carried out in a finite interval of time. See, for instance, [154]. See also the chapter on Thomson’s lamp in this book.
most absolute. As absolute as the submission of physics to infinitist mathematics. One has the impression that a significant number of physicists believe the existence of complete infinite totalities has been formally proven. Obviously, if that were the case we would not need the Axiom of Infinity to legitimate them (see below). The actual infinity hypothesis is just a hypothesis.

18 The three most influential ‘proofs’ of the existence of actual infinite totalities (by Bolzano, Dedekind and Cantor) are illustrative of what could be called naive infinitism. They also explain why infinitist mathematics had finally to establish the existence of actual infinite sets in axiomatic terms.

19 Bolzano’s proof goes as follow (taken from [136, p 112]):

One truth is the proposition that Plato was Greek. Call this \( p_1 \). But then there is another truth \( p_2 \), namely the proposition that \( p_1 \) is true. But then there is another truth \( p_3 \), namely the proposition that \( p_2 \) is true. And so ad infinitum. Thus the set of truths is infinite.

The problem here is that the existence of an endless process (\( p_1 \) is true, then \( p_2 \) is true, then \( p_3 \) is true, then . . . ) does by no means prove the existence of its finished result as a complete totality.

20 Dedekind’s proof is similar (taken from [136, p 113]):

Given some arbitrary thought \( s_1 \), there is a separate thought \( s_2 \), namely that \( s_1 \) can be object of thought [there is a separate thought \( s_3 \), namely that \( s_2 \) can be object of thought]. And so ad infinitum. Thus the set of thoughts is infinite.

The above comment on Bolzano proof also applies here. Dedekind gave another proof something more detailed, albeit with the same formal defect, based on his definition of infinite set [60, p. 112].

21 And finally, Cantor’s proof: ([95, p 25], [136, p. 117]):

Each potential infinite presupposes an actual infinity.

or ([39, p. 404] English translation [164, p. 3]):

... in truth the potential infinity has only a borrowed reality, insofar as a potentially infinite concept always points towards a logically prior actually infinite concept whose existence it depends on.

It is now clear why the existence of an actually infinite set had to be finally established by means of an Axiom.

THE AXIOM OF INFINITY

22 Nothing in nature seems to be actually infinite. Until now, all things we have been capable of observing and measuring are finite. Twenty seven centuries of discussions, on the other hand, were not sufficient to prove the existence of actual infinities. So that, finally, infinitists had no other choice but to declare its existence in axiomatic terms by means of the so called Axiom of Infinity, one of the foundational axioms in all modern axiomatic set theories (see
below). Set theory is then the gateway of the actual infinity in contemporary mathematics.

23 Since sets will be present in almost all our arguments, it seems appropriate to make the following consideration on the different ways an element can belong to a set. We usually assume that a particular element belongs or does not belong to a given set, although we could also consider the so-called fuzzy sets \[\text{[205], [65]},\] whose elements can have different degrees of membership. In this book, however, we will exclusively deal with complete membership, i.e., with sets whose elements belong completely to the sets.

24 That said, let us recall that the Axiom of Infinity states:

\[\exists N(\emptyset \in N \land \forall x \in N (x \cup \{x\} \in N))\]  

that reads: there exist a set \(N\) such that \(\emptyset\) belongs to \(N\) and for all elements \(x\) in \(N\), the element \(x \cup \{x\}\) also belongs to \(N\). In a less abstract way it could also be written as:

\[\exists N (0 \in N \land \forall x \in N (s(x) \in N))\]  

where \(s(x)\) is the successor of \(x\). In arithmetical terms we could write:

\[s(0) = 1; \quad s(1) = 2; \quad s(2) = 3; \ldots\]  

Thus, to put it in informal terms, the Axiom of Infinity states the existence of a denumerable infinite set, where denumerable (or enumerable) means that it can be put into a one to one correspondence with the set \(\mathbb{N} = \{1, 2, 3 \ldots\}\) of natural numbers,\(^4\) and infinite stands for the actual infinity: the elements of that set exist all at once, as a complete totality.

25 Unnecessary as it may seems, let us recall that an axiom is just an axiom. That is to say, an statement you can either accept or reject. Although the election will have important consequences on the resulting theory. In the case of the actual infinity hypothesis some relevant authors as Kronecker, Poincaré, Brouwer, Wittgenstein, Kleene, among others, rejected it. Other thing is the criticism against the actual infinity once set theory was axiomatically established and formally developed. This criticism has been basically non-existent during the last sixty years, and the few attempts carried out were always naive and frequently based on misconceptions of transfinite numbers.

**Cardinals and ordinals**

26 For the same reason we need axioms and fundamental laws in science,\(^5\) we also need primitive concepts in language, i.e., concepts that cannot be defined

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\(^4\)Two sets that can be put into a one to one correspondence are said to be equipotent or equinumerous.

\(^5\)The aristotelian infinite regress [9].
in terms of other concepts without falling into circular definitions (dictionaries are finite). Most basic mathematical concepts belong to this category: number, point, line, plane, set, and some others. So, to say the cardinal of a set is the number of its element is to say nothing. Notwithstanding, everyone knows what we mean when we say the set \{a, b, c\} has three elements, or that its cardinal is three. Even what we mean when we say the cardinal of a denumerable set, as the set \(\mathbb{N}\) of natural numbers, is \(\aleph_0\) (aleph null).

27 Although in informal terms, we will say the cardinal \(C\) of a set \(X\) is the number of its elements; in symbols \(C = |X|\). For obvious reasons, the cardinals of finite sets are said finite, and the cardinals of actually infinite sets are said infinite. Although we will not do it here, it can easily be proved the number of subsets of a set whose cardinal is \(C\), is just \(2^C\) (including the set itself and the empty set).

28 Cantor took it for granted the existence of the totality of finite cardinals (natural numbers) [40, pp 103-104]:

The first example of a transfinite aggregate is given by the totality of finite cardinal numbers \(v\); we call its cardinal number 'Aleph-zero' and denote it by \(\aleph_0\); thus we define

\[\aleph_0 = \{\overline{v}\}\]

where \(\{\overline{v}\}\) is Cantor’s notation for the cardinal of the set \(\{v\}\) of all finite cardinals (\(|\mathbb{N}|\) in modern notation). Obviously \(\aleph_0\) is an infinite cardinal. Cantor proved it is the less cardinal greater than all finite cardinals [40, § 6].

29 The successive natural numbers 1, 2, 3, . . . can be defined as the cardinals of the successive finite sets of the sequence of sets \(S = \{0\}, \{0, 1\}, \{0, 1, 2\}, . . .\), or as the cardinals of any succession of finite sets whose successive terms are equipotent with the successive terms of \(S\) (see Von Neumann operational definition of natural numbers in Appendix B). Natural numbers can still be used in informal terms as the counting numbers 1, 2, 3, . . . At the end, we say the cardinal of a finite set is \(n\) after counting its finitely many elements, or after pairing them with the elements of a set that have been previously counted or in any form successively considered or even arithmetically calculated or processed.

30 All denumerable sets, on the other hand, have the same cardinal \(\aleph_0\). Thus, as noted above, the cardinal of the set \(\mathbb{N}\) of natural numbers is \(\aleph_0\). The cardinal of the power set \(P(\mathbb{N})\), the set of all subsets of \(\mathbb{N}\) (including \(\mathbb{N}\) and the empty set), is not \(\aleph_0\) but \(2^{\aleph_0}\), which is also the cardinal of the set \(\mathbb{R}\) of real numbers. The cardinal of the set \(P(P(\mathbb{N}))\) of all subsets of \(P(\mathbb{N})\) is not \(2^{\aleph_0}\) but \(2^{2^{\aleph_0}}\). The same applies to the set \(P(P(P(\mathbb{N})))\) of all subsets of \(P(P(\mathbb{N}))\) and son on. We have then an increasing sequence of infinite cardinals (the power sequence):

\[\aleph_0 < 2^{\aleph_0} < 2^{2^{\aleph_0}} < 2^{2^{2^{\aleph_0}}} < \ldots\]

(6)

In this book we will dealt exclusively with \(\aleph_0\), except in a few number of arguments in which \(2^{\aleph_0}\), called the power of the continuum, will also be involved.
Ordinal numbers are a little more subtle. An ordinal is the order type of a well-ordered set. All finite sets with the same number of elements have the same ordinal, for instance the ordinal of the set \{a, b, c\} is the same as the ordinal of the set \{2, 3, 1\} because their elements can only be well-ordered as first, second and third (independently of which element is the first, second and third). And the same applies to any finite set of \(n\) elements. The successive cardinals and ordinals of finite sets are represented by the following numerals (symbols):

\[
\begin{align*}
\emptyset &: \text{Cardinal 0 Ordinal 0} \\
\{0\} &: \text{Cardinal 1 Ordinal 1} \\
\{0, 1\} &: \text{Cardinal 2 Ordinal 2} \\
\{0, 1, 2\} &: \text{Cardinal 3 Ordinal 3} \\
&\vdots & & \vdots
\end{align*}
\]

This is an important characteristic of finite sets: they only have one cardinal and only one ordinal and we use the same symbol (numeral) for both of them. According to Cantor’s terminology, finite ordinals are called ordinals of the first class.

Things are quite different with the infinite sets. For example, all denumerable sets have the same cardinal \(\aleph_0\), but they can be well-ordered in infinitely many different ways:

\[
\begin{align*}
\{1, 2, 3, \ldots\} &: \text{Ordinal } \omega \\
\{2, 3, 4, \ldots 1\} &: \text{Ordinal } \omega + 1 \\
\{3, 4, 5, \ldots 1, 2\} &: \text{Ordinal } \omega + 2 \\
\{1, 3, 5, \ldots 2, 4, 6, \ldots\} &: \text{Ordinal } \omega_2 \\
\{1, 4, 7, \ldots, 2, 5, 8, \ldots 3, 6, 9 \ldots\} &: \text{Ordinal } \omega_3 \\
&\vdots & & \vdots
\end{align*}
\]

being \(\omega < \omega + 1 < \omega + 2 < \ldots < \omega_2 < \omega_2 + 1 < \ldots < \omega_3 < \ldots\)

The ordinal numbers of denumerable sets are called ordinals of the second class. There are two types of ordinals of the second class:

1. Ordinals of the first kind: ordinals \(\alpha\) that have an immediate predecessor \(\alpha'\) such that \(\alpha = \alpha' + 1\), where ’1’ is the first finite ordinal. All ordinals of the first kind can then be written in the form \(\alpha + n\), being \(\alpha\) infinite and \(n\) finite.

2. Ordinals of the second kind: these ordinals are limits of infinite increasing sequences either of finite ordinals or of infinite ordinals of the first kind. For example:

\[
\omega = \lim_{n} (n); \quad n = 1, 2, 3, \ldots
\]  

\(^6\)A set with a total order relation between its elements and such that all its subsets have a first element.
Almost all arguments in this book will be arguments on $\omega$, the first ordinal of the second class, second kind; the smallest of the infinite ordinals.

For the sake of clarity and simplicity, in the remainder of the book we will say that a set, or a sequence, is $\alpha$-ordered to express it is a well ordered set (or sequence) whose ordinal is $\alpha$, being $\alpha$ any transfinite ordinal, that almost always will be $\omega$.

The ordinals of the second class define a new set: the set of all ordinals of the second class (the set of all ordinals whose sets have the same cardinal $\aleph_0$), whose cardinal is $\aleph_1$ [40, Theorem 16-F]. In its turn, the set of all ordinals whose sets have the same cardinal $\aleph_1$ is another set whose cardinal is $\aleph_2$. The set of all ordinals whose sets have the same cardinal $\aleph_2$ is another set whose cardinal is $\aleph_3$. And so on. Thus, according to Cantor there are two increasing sequences of infinite cardinals:

\[ \aleph_0 < 2^{\aleph_0} < 2^{2^{\aleph_0}} < \ldots \quad \text{(Power sequence)} \quad (10) \]

\[ \aleph_0 < \aleph_1 < \aleph_2 < \aleph_3 < \ldots \quad \text{(Aleph sequence)} \quad (11) \]

The famous (and still unproved) continuum hypothesis asserts: $\aleph_1 = 2^{\aleph_0}$. The generalized version asserts that, for all $i$, the $i$th term of the first sequence is equal to the $i$th term of the second one. Fortunately we will not have to address that question in this book.

Obviously this is only a short and schematic introduction to Cantor’s theory of transfinite numbers [40]. But this is all we need to know in order to follow the arguments in this book. As noted above, we will focus our attention on $\omega$-ordered objects (sets and sequences), i.e on objects whose elements are ordered in the same way as the natural numbers in their natural order of precedence. Objects as, for instance, the sequence $a_1, a_2, a_3, \ldots$ This type of ordering ($\omega$-order from now on) is characterized by:

1. There is a first element $a_1$.
2. Each element $a_n$ has an immediate predecessor $a_{n-1}$, except the first one $a_1$.
3. Each element $a_n$ has an immediate successor $a_{n+1}$.
4. Between any two successive elements $a_n, a_{n+1}$, no other element exists (immediate successiveness).
5. There is not last element, in spite of which $\omega$-ordered objects are considered as complete totalities.

In a few occasions, we will also deal with $\omega^*-$ordered objects, i.e. objects whose elements are ordered in the same way as the increasing sequence of
negative integers . . . , -3, -2, -1. In this type of ordering we will use the notation $a_{n^*}$ to refer to the last but $n - 1$ element. $\omega^*$-Order is characterized by:

1. There is a last element $a_{1^*}$.
2. Each element $a_{n^*}$ has an immediate successor $a_{(n-1)^*}$, except the last one $a_{1^*}$.
3. Each element $a_{n^*}$ has an immediate predecessor $a_{(n+1)^*}$.
4. Between any two successive elements $a_{(n+1)^*}$, $a_{n^*}$ no other element exists (immediate successiveness).
5. There is not first element, in spite of which $\omega^*$-ordered objects as considered as complete totalities.

As noted above, all transfinite numbers (cardinals and ordinals) are built on the assumption that a denumerable $\omega$-ordered set does exist. This is why almost all the following arguments will deal with $\omega$-ordered objects. If that infinitist assumption were proved to be inconsistent then the whole edifice of transfinite mathematics would fall down like a house of cards.
4.-Reinterpreting the paradoxes of reflexivity

INTRODUCTION

39 If after pairing each and every element of a set \( A \) with a different element of another set \( B \) all elements of \( B \) result paired, we say both sets have the same cardinality (the same number of elements). But if one or more elements of \( B \) result unpaired and \( B \) is infinite, we are not allowed to say both sets have different cardinality. In this chapter we discuss why we are not allowed to do it. As we will see, the existence of both exhaustive and non-exhaustive injections\(^1\) between two infinite sets could be indicating they have and not have the same cardinality. Thus, the arbitrary distinction of the exhaustive injections to the detriment of the non-exhaustive ones could be concealing a fundamental contradiction in set theory.

40 Most of the paradoxes related to the actual infinity result from the violation of the Euclidian Axiom of the Whole and the Part,\(^2\) among them the so called paradoxes of reflexivity in which the elements of a whole are paired off with the elements of one of its proper parts.\(^3\) Galileo’s paradox\(^4\) is a well known example of reflexive paradox. Authors as Proclus, J. Filopón, Thabit ibn Qurra al-Harani, R. Grosseteste, G. of Rimini, W. of Ockham etc. found many other examples \([171]\).

41 The strategy of pairing off the elements of two sets is not just a modern invention. Aristotle used it when trying to solve Zeno’s Dichotomies.\(^5\) And since then, it has been extensively used by numerous authors with different discursive purposes, although, before Dedekind and Cantor, they were never used (including the case of Bolzano \([26]\)) as an instrument to consummate the violation of the old Euclidian axiom. Of course, the existence of a one to one correspondence between two infinite sets does not prove both set are actually infinite because they could also be potentially infinite.

\(^1\) An injection is a correspondence between the elements of two sets \( A \) and \( B \) such that each and every element of \( A \) is paired off with a different element of \( B \).

\(^2\) The assumption that the whole is greater than the part is one of the common notions that appears in the first book of Euclid’s Elements \([71, p\ 19]\).

\(^3\)\([171]\), \([63]\).

\(^4\) The elements of the set of natural numbers can be paired with the elements of one of its proper subsets: the subset of their squares: \( 1 \leftrightarrow 1^2, 2 \leftrightarrow 2^2, 3 \leftrightarrow 3^2, 4 \leftrightarrow 4^2, 5 \leftrightarrow 5^2 \ldots \) \([78]\).

\(^5\) Aristotle finally rejected his pairing method and proposed the distinction between the actual and the potential infinity \([11]\), \([10]\).
Things began to change with Dedekind, who stated the definition of the infinite sets just on the basis of that violation: a set is infinite if its elements can be put into a one to one correspondence with the elements of one of its proper subsets [60]. Dedekind and Cantor inaugurated the so called paradise of the actual infinity, where exhaustive injections (bijections or one to one correspondences) play a capital role.

**Paradoxes or contradictions?**

An exhaustive injection between two sets $A$ and $B$ is a correspondence between the elements of both sets in which each element of $A$ is paired off with a different element of $B$, and all elements of $A$ and $B$ result paired. When at least one element of the set $B$ results unpaired the injection is said non-exhaustive. Exhaustive and non-exhaustive injections can be used to compare the cardinality of finite sets. But if the compared sets are infinite, then only exhaustive injections are permitted. No reason has ever been given as to justify that arbitrary distinction (see below 45-48) except that by definition infinite sets violate the Euclidian axiom.

But, since definitions can also be inconsistent, the infinite sets could have been defined inconsistently on the basis of one of the terms of a contradiction: there is an exhaustive injection between the set and one of its proper subsets. The other part of the contradiction would be: there is a non-exhaustive injection between the set and the same proper subset. No one has ever explained why to have an exhaustive injection with a proper subset and at the same time to have a non-exhaustive injection with the same proper subset is not contradictory. The problem has simply been ignored and set theory has been raised on the basis of that ignorance.

It seems reasonable to assume that if after pairing every element of a set $A$ with a different element of a set $B$, all elements of $B$ result paired, then $A$ and $B$ do have the same number of elements. But it seems also reasonable to assume, and for the same elementary reasons, that if after pairing every element of a set $A$ with a different element of a set $B$ one or more elements of the set $B$ remain unpaired, then $A$ and $B$ don’t have the same number of elements. It is worth noting that both exhaustive and non-exhaustive injections make use of the same basic method of pairing elements, without carrying out any finite or transfinite arithmetic operation. We are not counting but pairing, we are discussing at the most basic foundational level of set theory.

It should be recalled at this point that the arithmetic peculiarities of transfinite cardinals, as $\mathfrak{N}_\omega = \mathfrak{N}_\omega + \mathfrak{N}_\omega$ and the like, are of all them derived from the hypothetical existence (Axiom of Infinity) of the infinite sets, i.e. of sets whose elements can, by definition, be paired with the elements of some of their proper subsets. So, under penalty of circular reasoning, we cannot infer

---

6Specially when the definition is based on the violation of a basic axiom, as is the case of the Dedekind’s definition of the infinite sets.
from the deduced existence of those arithmetical ‘peculiarities’ (that could be used to justify the existence of exhaustive and non-exhaustive injections between an infinite set and some of its proper subsets), the existence of just the sets from which those arithmetic peculiarities of infinite cardinals have been deduced. Here, we are simply discussing if the method of pairing the elements of two sets is appropriate to compare their respective cardinalities; and if it is, why non-exhaustive injections are rejected, because that rejection could be concealing a fundamental contradiction.

47 Exhaustive and non-exhaustive injections would have to have the same validity as instruments to compare the cardinality of infinite sets just because they use exactly the same comparison method. However, only exhaustive injections can be used with that purpose. The problem here is that the existence of both exhaustive and non-exhaustive injections between two infinite sets could be indicating the existence of an elementary contradiction (that both infinite sets have and have not the same cardinality), in which case the distinction of exhaustive injections would be the distinction of a term of a contradiction to the detriment of the other.

48 At the very least, the alternative of considering a set inconsistent because of the existence of both exhaustive and non-exhaustive injections with the elements of one of its proper subsets is as legitime as the alternative of considering it consistent. Thus, at the very least, the arbitrary selection of the second alternative should be explicitly declared at the foundational level of the theory, which is not the case in current set theories. Current set theories systematically ignore the first alternative. It could be argued that Dedekind’s definition implies to assume the existence of sets for which there exist both exhaustive and non-exhaustive injections with at least one of its proper subsets, but a simple definition does not guarantee the defined object is consistent, and then the alternative of the inconsistency has also to be considered. The proposal of this consideration is just the main objective of this discussion. A consideration that, for all I know, has never been seriously proposed.

Figure 4.1: The suspicious power of the ellipsis: the sets S and N have (left) and not have (right) the same number of elements.

49 Assume, only for a moment, that exhaustive and non-exhaustive injections
were valid instruments to compare the cardinality of any two sets. In those conditions, let $B$ be an infinite set. By definition, there exists a proper subset $A$ of $B$ and an exhaustive injection $f$ from $A$ to $B$ proving both sets have the same number of elements. Consider now the injection $g$ from $A$ to $B$ defined by:

$$g(x) = x, \forall x \in A$$

(1)

which evidently is non-exhaustive (the elements of the nonempty set $B-A$ remain unpaired). The injections $f$ and $g$ would be proving that $A$ and $B$ have $(f)$ and not have $(g)$ the same number of elements, i.e. that infinite sets are inconsistent.

50 We must therefore decide if exhaustive and non-exhaustive injections do have the same validity as instruments to compare the number of elements of any two sets. If they do, then the actually infinite sets are inconsistent. If they don’t, at least one (non-circular) reason should be given to explain why they don’t. And, if no reason can be given, then the arbitrary distinction in favor of the exhaustive injections should be declared in an appropriate ad hoc axiom. Until then, the foundation of set theory rests on the basis of one of the terms of a possible contradiction.7

51 As might be expected from a theory with such foundations, inconsistencies appeared from its very beginning: the set of all ordinals and the set of all cardinals were proved to be inconsistent by Burali-Forti [29] and Cantor respectively. According to Cantor those sets are inconsistent because of their excessive infinitude.8 One can be infinite but only within certain limits. By the appropriate axiomatic restrictions, it was finally stated that some infinite totalities, as the totality of cardinals or the totality of ordinals, don’t exist because they lead to contradictions. It can easily be proved, as we will see in the next chapter, that in a naive (not limited by axiomatic restrictions) infinitist set theory, as Cantor’s set theory, each set of cardinal $C$ originates nothing less than $2^C$ inconsistent totalities.

52 In Chapter 18 we will see that Riemann’s series theorem can also be reinterpreted as a proof of the inconsistency of the actual infinity hypothesis. In the remainder of the book, more than twenty arguments will be developed all of them suggesting the same conclusion.

7Unbelievable as it may seem, the axiomatic foundation of set theory has always ignored this problem.

8Letter to Dedekind quoted in [57, pag. 245], [79], [75].
5.-Extending Cantor paradox

INTRODUCTION

Cantor’s paradox is not a paradox but a true inconsistency related to the set of all cardinals. This is why that set is explicitly rejected in modern axiomatic set theories. The following discussion proves, however, that not only the set of all cardinals is inconsistent, it proves that in Cantor’s naive set theory each set of cardinality $C$ originates at least $2^C$ inconsistent sets.

Although Burali-Forti was the first to publish an inconsistency related to transfinite sets [28], [79], Cantor was the first to discover a paradox in the nascent set theory: the maximum cardinal paradox [79], [57]. There is no agreement regarding the date Cantor discovered his paradox [79] (the proposed dates range from 1883 [156] to 1896 [87]). Burali-Forti paradox on the set of all ordinals and Cantor paradox on the set of all cardinals are both related to the size of the considered totalities, perhaps too big as to be consistent according to Cantor. It seems somewhat ironic that an infinite set may be inconsistent just because of its excessive size. By the way, note we use the euphemism ‘paradox’ to denote what really is an inconsistency, i.e. a pair of contradictory results that surely derive from a common precedent assumption. From which assumption? could we also ask. Perhaps from the hypothesis that infinite sets do exist as complete totalities?

Indeed, the simplest explanation for both paradoxes is that they really are inconsistencies derived from the hypothesis of the actual infinity, i.e. from assuming the existence of infinite sets as complete totalities. But no one has dared to analyze this alternative. It was finally accepted that some infinite totalities (as the totality of real numbers) do exist while others (as the totality of cardinals, or the totality of ordinals) do not because they lead to contradictions.

CANTOR PARADOX

The easiest and shortest version of Cantor paradox¹ (for a detailed analysis see [79, pp. 66-74]) goes as follows: Let $U$ be the set of all sets, the so called

¹Note that, usual as it may be, the expression ‘Cantor’s paradox’ is at least confusing since it is not a paradox but a true contradiction.
universal set\(^2\) and \(P(U)\) its power set, the set of all its subsets. Let us denote by \(|U|\) and \(|P(U)|\) their respective cardinals. Being \(U\) the set of all sets it must contain all sets and then we can write:

\[
|U| \geq |P(U)| \tag{1}
\]

On the other hand, and according to Cantor theorem on the power set \([36]\), it holds:

\[
|U| < |P(U)| \tag{2}
\]

which contradicts (1). This is Cantor’s inconsistency or paradox.

57 As is well known, Cantor gave no importance to that inconsistency \([75]\) and clinched the argument by assuming the existence of two types of infinite totalities, the consistent and the inconsistent ones \([34]\). As noted above, in Cantor’s opinion the inconsistency of those infinite totalities would surely due to their excessive size. In fact, we would be in the face of the mother of all infinities, the absolute infinity which, according to Cantor, leads directly to God, being just the divine nature of this absolute infinitude what makes it inconsistent for our poor human minds \([34]\).

58 As we will immediately see, it is possible to extend Cantor paradox to other sets much more modest than the set of all sets. But neither Cantor nor his successors considered such a possibility. We will do it here. This is just the objective of the discussion that follows. A discussion that will take place within the framework of Cantor naive, and then non axiomatized, set theory.

**An extension of Cantor’s paradox**

59 Since the elements of a set in naive set theory can be sets, sets of sets, sets of sets of sets and so on, we will begin by defining the following binary relation \(\mathcal{R}\) between two sets: we will say that a set \(A\) is \(\mathcal{R}\)-related to a set \(B\), written \(A \mathcal{R} B\), if \(B\) contains at least one element which forms part of the definition of at least one element of \(A\). For instance, if:

\[
A = \{ \{\{a\}, \{b\}\}\}, \{c\}, d, \{\{\{e\}\}\}, f \} \tag{3}
\]
\[
B = \{1, 2, b\} \tag{4}
\]
\[
C = \{1, 2, 3\} \tag{5}
\]

then \(A\) is \(\mathcal{R}\)-related to \(B\) because the element \(b\) of \(B\) forms part of the definition of the element \(\{\{a\}, \{b\}\}\) of \(A\), while \(A\) is not \(\mathcal{R}\)-related to \(C\) because no element of \(C\) is involved in the definition of \(A\)’s elements.

60 In these conditions, let \(X\) be any non empty set and \(Y\) any of its subsets.

\(^2\)Cantor’s naive set theory admits sets as the universal set \(U\) that are forbidden in modern axiomatic theories.
From $Y$ we define the set $T_Y$ according to:

$$T_Y = \{ Z \mid \neg \exists V (V \cap Y \neq \emptyset \land Z \mathcal{R} V) \}$$  \hspace{1cm} (6)$$

$T_Y$ is, therefore, the set of all sets $Z$ that are not $\mathcal{R}$-related to any set $V$ containing one or more elements of the set $Y$. Notice that if $Y = \emptyset$ then $T_Y$ is the inconsistent universal set.

61 Let us now consider the set $P(T_Y)$, the power set of $T_Y$. The elements of $P(T_Y)$ are all of them subsets of $T_Y$ and therefore sets of sets that are not $\mathcal{R}$-related to sets that contain elements of the set $Y$:

$$\forall D \in P(T_Y) : \neg \exists V (V \cap Y \neq \emptyset \land D \mathcal{R} V)$$  \hspace{1cm} (7)$$

Consequently, it holds:

$$\forall D \in P(T_Y) : D \in T_Y$$  \hspace{1cm} (8)$$

And then:

$$P(T_Y) \subseteq T_Y$$  \hspace{1cm} (9)$$

Accordingly, we can write:

$$|P(T_Y)| \leq |T_Y|$$  \hspace{1cm} (10)$$

62 On the other hand, and in accordance with Cantor’s theorem it holds:

$$|P(T_Y)| > |T_Y|$$  \hspace{1cm} (11)$$

Again a contradiction. But now $X$ is any non empty set and $Y$ any of its subsets. We have therefore proved the following:

**Theorem 62 (of Cantor Paradox).**—In Cantor’s set theory, every set of cardinal $C$ gives rise to at least $2^C$ inconsistent totalities.

where $2^C$ is the number of subsets of a set with $C$ elements.

63 The above argument not only proves the number of inconsistent infinite totalities is much greater than the number of consistent ones, it also suggests the excessive size of the sets could not be the cause of the inconsistency. Consider, for example, the set $X$ of all sets whose elements are exclusively defined by means of the natural number 1:

$$X = \{1, \{1\}, \{1, \{1\}\}, \{\{1\}\}, \{\{1, \{1\}\}\} \ldots \}$$  \hspace{1cm} (12)$$

An argument similar to 60/62 would immediately prove it is an inconsistent totality, although compared with the universal set it is an insignificant totality.\(^3\)

64 Notice that sets as the set $X$ defined by (12) are inconsistent only when

\(^3\)Recall, for instance, that between any two real numbers an uncountable infinitude ($2^{\aleph_0}$) of other different reals numbers do exist. What, as Wittgenstein would surely say, makes one feel dizzy [202]
considered from the perspective of the actual infinity, i.e. when considered as complete totalities. And recall that from the potential infinite point of view these sets make no sense because from this perspective the only complete totalities are the finite totalities, as large as we wish but always finite.

65 Had we know the existence of so many inconsistent infinite totalities, and not necessarily so great as the absolute infinity, and perhaps Cantor transfinite set theory would have been received in a different way. Perhaps the very notion of the actual infinity would have been put into question in set theoretical terms; and perhaps we would have discovered the way to prove it is an inconsistent notion. But, as we know, this was not the case.

66 The history of the set theory reception and the way of dealing with its inconsistencies (all of them promoted by the actual infinity hypothesis and by self-reference) is well known. From the beginnings of the XX century a great deal of effort has been carried out to found set theory on a consistent background free of inconsistencies. Although the objective could only be reached with the aid of the appropriate axiomatic patching. At least half a dozen of axiomatic set theories have been developed ever since. Some hundreds of pages are needed to explain in detail all axiomatic restrictions of contemporary axiomatic set theories. Just the contrary one could expect from the axiomatic foundation of a formal science.

67 As noted above, the simplest explanation of Cantor and Burali-Forti paradoxes is that they are true contradictions derived from the inconsistency of the hypothesis of the actual infinity. The same applies to the set of all sets, and to the set of all sets that are not member of themselves (Russell paradox), although in this case there is an additional cause of inconsistency related to self-reference. All sets involved in the paradoxes of naive set theory were finally removed from the theory by the opportune axiomatic restrictions. No one dared to suggest the possibility that those paradoxes were in fact contradictions derived from the hypothesis of the actual infinity; i.e. from assuming the existence of infinite sets as complete totalities.

68 What is really true is that Cantor set of all cardinals, Burali-Forti set of all ordinals, the set of all sets, and Russell set of all sets that are not members of themselves, are all of them inconsistent totalities when considered from the perspective of the actual infinity hypothesis. Even Turing’s famous halting problem is related to the hypothesis of the actual infinity because it also assumes the existence of all pairs (programs, inputs) as a complete infinite totality [192]. Under the hypothesis of the potential infinity, on the other hand, none of those totalities makes sense because from this perspective only finite totalities can be considered, indefinitely extensible, but always finite.

4There are also some contemporary attempts to recover naive set theory [104]
INTRODUCTION

69 The set \( \mathbb{Q} \) of rational numbers, in its natural ordering, is densely ordered: between any two rationals infinitely many different rationals do exist. But, being denumerable \([32]\), it can also be \( \omega \)-ordered: between any two successive rationals no other rational exists. The argument that follows takes advantage of this sort of numerical schizophrenia.

DISCUSSION

70 For the sake of simplicity, we will deal with the set \( \mathbb{Q}^+ \) of positive rationals greater than zero, which is also denumerable and densely ordered. Let then \( f \) be a one to one correspondence between the set \( \mathbb{N} \) of natural numbers and the set \( \mathbb{Q}^+ \). Evidently, \( f \) induces an \( \omega \)-order in \( \mathbb{Q}^+ \) so that the set of all positive rational numbers can be written as \( \{q_1, q_2, q_3, \ldots\} \), being \( q_i = f(i) \), \( \forall i \in \mathbb{N} \).

71 Let \( x \) and \( y \) be two rational variables whose initial values are \( x = 1 \), \( y = 1 \). Then consider the following sequence of (re)definitions \( \langle D_i \rangle \) of \( x \) and \( y \):

\[
i = 1, 2, 3, \ldots \begin{cases}
D_i(y) = 1 \\
|q_{i+1} - q_1| < x \Rightarrow D_i(x) = |q_{i+1} - q_1|
\end{cases}
\]  \hspace{1cm} (13)

where we denote by \( D_i(y) \) and \( D_i(x) \) the \( i \)th definition of \( y \) and \( x \) respectively, and where \( |q_{i+1} - q_1| \) is the absolute value of \( q_{i+1} - q_1 \), and ‘<’ stands for the usual dense ordering of \( \mathbb{Q} \); i.e \( |q_{i+1} - q_1| < x \) means \( |q_{i+1} - q_1| - x < 0 \).

72 The successive definitions \( D_i \) of \( \langle D_i \rangle \) define the variable \( y \) always with the same value 1, and the variable \( x \) as \( |q_{i+1} - q_1| \) if \( |q_{i+1} - q_1| \) is less than the
current value of $x$. Definitions and procedures consisting of infinitely many successive steps, as Definition 13, are usual in infinitist mathematics (see, for instance, Cantor 1874 argument, or Cantor ternary set, later in this book). We will impose to the successive definitions $\langle D_i \rangle$ the following:

Restriction 75.-Each successive definition $D_i$ will be carried out if, and only if, $x$ results defined as a positive rational number.

73 By induction, it is immediate to prove that for each natural number $v$, the first $v$ successive definitions $D_{1...v}$ can be carried out. Evidently the first definition $D_1$ can be carried out since $x$ is defined as $|q_2 - q_1|$ if $|q_2 - q_1| < 1$ or as 1 if it is not, being the result of each alternative a positive rational number. Assume that, being $n$ any natural number, the first $n$ successive definitions $D_{1...n}$ can be carried out, which means they define $x$ as a positive rational number. Since $|q_{n+2} - q_1|$ is a well defined positive rational number it will be, or not, less than the current rational value of $x$; consequently $x$ could be defined as $|q_{n+2} - q_1|$ if this number is less than its current value or retain its current value if it is less than $|q_{n+2} - q_1|$. Consequently, the first $n+1$ successive definitions $D_{1...n+1}$ can also be carried out. We have just proved that the first definition $D_1$ can be carried out, and that if, for any natural number $n$, the first $n$ successive definitions $D_{1...n}$ can be carried out, then the first $(n + 1)$ successive definitions $D_{1...n+1}$ can also be carried out. In consequence, for any natural number $v$, the first $v$ successive definitions $D_{1...v}$ can be carried out.

74 We will begin by proving that once performed all possible successive definitions $D_i$ the rational $q_1 + x$ is not the less rational greater than $q_1$. In fact, whatsoever be the value of $x$ once performed all possible successive definitions $D_i$, the rational $q_1 + 0.1x$, for instance, is greater than $q_1$ and less then $q_1 + x$. Notice this argument is a consequence of the natural dense ordering of $\mathbb{Q}^+$. 

75 We will prove now, however, that once performed all possible successive definitions $D_i$ the rational $q_1 + x$ is the less rational greater than $q_1$. In effect, assume that while the successive definitions $D_i$ that observe Restriction 75 can be carried out, they are carried out. Assume once performed all possible successive definitions $D_i$ the rational $q_1 + x$ is not the less rational greater than $q_1$. In such a case there would be a positive rational $q_v$ greater than $q_1$ and less that $q_1 + x$:

$$q_1 < q_v < q_1 + x \quad (14)$$

and then, by subtracting $q_1$:

$$0 < q_v - q_1 < x \quad (15)$$

which is impossible because:

1. The index $v$ of $q_v$ is a natural number.

5Note that if it were impossible to perform all possible successive definitions $D_i$ we would be in the face of the elementary contradiction of an impossible possibility
2. In accordance with 73 the first $v$ successive definitions $D_{1...v}$ can be carried out.

3. All possible successive definitions $D_i$ have been carried out.

4. So, at least the first $v$ successive definitions $D_{1...v}$ have been carried out.

5. As a consequence of the $(v - 1)$th definition $D_{v-1}(x)$ we can assert that $x \leq q_v - q_1$.

6. It is then impossible that $q_v - q_1 < x$.

In consequence our initial hypothesis must be false and $q_1 + x$ is the less rational greater than $q_1$. Notice this argument is a consequence of the $\omega$—order of $\mathbb{Q}^+$ induced by the one to one correspondence $f$ defined in 70.

76 It could be argued that $x$ results undefined because the number of the successively performed definitions $D_i$ is not finite but infinite, without a last definition completing the sequence of performed definitions. In this case we would have to explain why after defining a variable infinitely many successive times it get finally undefined, being the case that each and every performed definition $D_i(x)$ leaves $x$ well defined. Notice this is not a case of indeterminacy but of indefiniteness. Or in other words, this is not the case that we don’t know the current value of $x$ once performed all possible successive definitions $D_i$, the case is that $x$ cannot have any positive rational value (otherwise the above argument 75-74 would be possible).

77 Some infinitists argue that although each performed definition $D_i$ leaves $x$ well defined, the completion of the sequence $\langle D_i \rangle$, as such a completion, leaves $x$ undefined. In this case we would have to admit that the completion of an infinite sequence of successive definitions, as such a completion, does have arbitrary additional effects on the defined object. But if that were the case, the same arbitrary additional effects could be expected from any other definition or procedure involving an infinite sequence of steps without a last completing step, and then anything could be expected from infinitist mathematics.

78 In our case, once performed all possible successive definitions $D_i$, the variable $y$ would also be undefined or defined as $\sqrt{\pi}$ or as anything else, in spite of the fact that it was always defined with the same value 1. Or we would have to explain why we can define infinitely many successive times a variable with the same value but not with different values.

79 We could even timetable the sequence of definitions $\langle D_i \rangle$ by performing each definition $D_i$ at the precise instant $t_i$ of the $\omega$—ordered and strictly increasing sequence of instants $\langle t_n \rangle = t_1, t_2, t_3 \ldots$ within the finite interval $(t_a, t_b)$, whose limit is $t_b$. In these conditions, $x$ can only get undefined at the precise instant $t_b$, the first instant at which $x$ is no longer redefined. In fact, being $t_b$ the limit of $\langle t_n \rangle$ we will have:

$$\forall t \in [t_a, t_b):$$  \hspace{1cm} (16)

$$\exists v: t_v \leq t < t_{v+1}$$  \hspace{1cm} (17)

$$\therefore \text{ at } t, \text{ } x \text{ is well defined by definition } D_v$$  \hspace{1cm} (18)
and then, at any instant $t$ within $[t_a, t_b)$ $x$ is well defined. In consequence it can only get undefined at $t_b$, which is impossible since at $t_b$ the sequence of definitions $\langle D_i \rangle$ has already finished. At $t_b$ $x$ is no longer defined.

80  The problem with the impossible indefiniton of $x$ is rather a consequence of assuming that it is possible to consider as complete a sequence in which no last element completes the sequence, in our case a sequence of definitions.
7.-Thomson’s lamp revisited

INTRODUCTION

81 Although Benacerraf’s criticism of Thomson’s lamp argument is well founded (see below), it is far from being complete. As we will see here, it is possible to consider a new line of argument, which Benacerraf only incidentally considered, based on the formal definition of the lamp. That line of argument leads to a contradictory result that compromises the formal consistency of the \( \omega \)-order involved in all \( \omega \)-supertasks.

82 To perform an \( \omega \)-supertask (supertask hereafter) means to perform an \( \omega \)-ordered sequence of actions (tasks) in a finite interval of time. Supertasks are useful theoretical devices for the philosophy of mathematics, particularly for the discussions on certain problems related to infinity.\(^1\) Although their physical possibilities and implications have also been discussed.\(^2\) Notwithstanding, here we will only deal with conceptual supertasks.

83 Probably Gregory of Rimini was the first to propose how a supertask could be accomplished ([136], p. 53):

If God can endlessly add a cubic foot to a stone -which He can- then He can create an infinitely big stone. For He need only add one cubic foot at some time, another half an hour later, another a quarter of an hour later than that, and so on \textit{ad infinitum}. He would then have before Him an infinite stone at the end of the hour.

But the term ”supertask” was introduced by J. F. Thomson in his seminal paper of 1954 [191]. Thomson’s paper was motivated by Black’s argument [24] on the impossibility to perform infinitely many successive actions and the rejections of Black’s argument by R. Taylor [190] and J. Watling [198]. In his paper Thomson tried to prove the impossibility of supertasks. Thomson argument was, in turn, criticized in another seminal paper, in this case by P. Benacerraf [18]. Benacerraf’s successful criticism finally motivated the foundation of a

\(^1\)[191], [27], [49], [154], [19], [200], [154]
\(^2\)[149], [150], [154], [165], [92], [94], [93], [150], [151], [152], [69], [153], [143], [5], [6], [155] [200], [103], [67], [68], [143], [66], [174]
The possibilities to perform an uncountable infinitude of actions were examined, and ruled out, by P. Clark and S. Read [49]. Supertasks have also been considered from the perspective of nonstandard analysis, although the possibilities to perform an *hypertask* along an hyperreal interval of time have not been discussed, despite the fact that finite hyperreal intervals can be divided into hypercountably many successive infinitesimal intervals (hyperfinite partitions). But most of the supertasks are $\omega$-supertasks, i.e. $\omega$-ordered sequences of actions performed in a finite (or perceived as finite) interval of time.

The basic idea of Benacerraf’s criticism of Thomson’s argument is the impossibility to derive formal conclusions on the final state of the supermachine that performs the supertask from the sequence of states the machine traverses as a consequence of performing the supertask. But, as we will see, Benacerraf’s analysis of Thomson’s lamp argument is incomplete.

In fact, if the world continues to be the same world it was before the execution of a supertask, and one is still allowed to think in rational terms in the same framework of the laws of logic, then Thomson’s argument can be reoriented towards the formal definition of the machine that performs the supertask. A definition that is assumed to be independent of the number of performed tasks with that machine, and then a definition that holds before, during and after performing the supertask, whenever the execution of a supertask does not arbitrarily changes a legitimate definition previously stated.

**Thomson’s Lamp**

As Thomson did in 1954, in the following discussion we will make use of one of those:

... reading-lamps that have a button in the base. If the lamp is *off* and you press the button the lamp goes *on*, and if the lamp is *on* and you press the button the lamp goes *off*. ([191], p. 5).

Let us complete Thomson’s definition by explicitly declaring the following two conditions regarding the (theoretical) functioning of the lamp:

1. The state of the lamp (on/off) changes if, and only if, its button is pressed down.
2. The pressing down (click) of the button and the corresponding lamp change of state (on/off) are both instantaneous and simultaneous events.
Assume now the lamp button is clicked at each of the infinitely many successive instants $t_i$, and only at them, of a strictly increasing $\omega$—ordered sequence of instants $\langle t_n \rangle$ defined within a finite interval of time $(t_a, t_b)$, being $t_b$ the limit of the sequence $\langle t_n \rangle$. In these conditions, at instant $t_b$ the lamp button will have undergone an $\omega$—ordered sequence $\langle c_n \rangle$ of clicks (each click $c_i$ at the precise instant $t_i$) and, consequently, the state of the lamp will have changed an $\omega$—ordered infinitude of times. Or in other words, at $t_b$ Thomson’s supertask will have been completed. Don’t forget this is a purely conceptual argument, so that we are not concerned here with the physical details.

Thomson tried to derive a contradiction from his supertask by speculating on the final state of the lamp at instant $t_b$ in terms of the sequence of switchings completed along the supertask ([191], p. 5):

[The lamp] cannot be on, because I did not ever turn it on without at once turning it off. It cannot be off, because I did in the first place turn it on, and thereafter I never turned off without at once turning it on. But the lamp must be either on or off. This is a contradiction.

It is worth noting that Thomson based his argument on the sequence of actions carried out on the lamp: it was never turned on without turning it off after, and vice versa. What Thomson tried to do is to derive the final state of the lamp, the state of the lamp at $t_b$, from the successive changes of state the lamp underwent during the supertask: The reason why the lamp cannot be on is because it was always turned off after turning it on. And for the same reason it cannot be off either. This way of arguing was severely criticized by Benacerraf.

Benacerraf argued against Thomson’s argument as follows ([18], p. 768):

The only reasons Thomson gives for supposing that his lamp will not be off at $t_b$ are ones which hold only for times before $t_b$. The explanation is quite simply that Thomson’s instructions do not cover the state of the lamp at $t_b$, although they do tell us what will be its state at every instant between $t_a$ and $t_b$ (including $t_a$). Certainly, the lamp must be on or off (provided that it hasn’t gone up in a metaphysical puff of smoke in the interval), but nothing we are told implies which it is to be. The arguments to the effect that it can’t be either just have no bearing on the case. To suppose that they do is to suppose that a description of the physical state of the lamp at $t_b$ (with respect to the property of being on or off) is a logical consequence of a description of its state (with respect to the same property) at times prior to $t_b$. ($t_a$ and $t_b$ appear respectively as $t_0$ and $t_1$ in Benacerraf’s paper).

In short, according to Benacerraf, the problem posed by Thomson is not sufficiently described since no constraint have been placed on what happens at $t_b$ [3]. But the only constraint on what happens at $t_b$ is that Thomson’s lamp continue to be Thomson’s lamp. Or in other words, that the execution of a supertask does not change the formal definitions of the involved theoretical artifacts. As we will see, the state of Thomson lamp at $t_b$ is not ‘a logical
consequence of a description of its state (with respect to the same property) at times prior to \( t_b \), it is a logical consequence of being a Thomson lamp. That will be the key of the next argumentation.

93 Consider the instant \( t_b \), the limit of the sequence \( \langle t_n \rangle \) of instants at which the successive clicks \( \langle c_n \rangle \) have been performed. That instant is, therefore, the first instant after completing the sequence of switchings. The first instant at which the lamp button is no longer clicked. Let now \( S_b \) be the state of the lamp at instant \( t_b \). Being the state of a Thomson lamp, it can only be either on or off. And this conclusion has nothing to do with the number of previously performed switchings. The lamp will be on or off because, being a Thomson’s lamp, it has only two states: on and off.

94 Some infinitist claim, however, that at \( t_b \), after performing Thomson’s supertask, the lamp could be in any unknown state, even in an exotic one. But a lamp that can be in an unknown state is not a Thomson’s lamp: the only possible states of a Thomson’s lamp are on and off. No other alternative is possible without arbitrarily violating the legitimate formal definition of Thomson’s lamp. And we presume no formal theory is authorized to violate arbitrarily a formal definition, nor, obviously to change, in the same arbitrary terms, the nature of the world. It goes without saying that if that were the case anything could be expected from the hypothesis of the actual infinity.

95 Others claim the state \( S_b \) is the consequence of completing the \( \omega \)-ordered sequence of clicks \( \langle c_n \rangle \), since that sequence, and only that sequence, has been carried out. But if to complete the sequence of clicks \( \langle c_n \rangle \) means to perform each and every of the infinitely many clicks \( c_i \), and only them, then we have a problem. The problem that no click \( c_i \) of \( \langle c_n \rangle \) originates \( S_b \). None. Indeed, if \( c_v \) is any element of \( \langle c_n \rangle \) it cannot originates \( S_b \) because in such a case the button would have been clicked only a finite number \( v \) of times. Or in other terms, if we remove from \( \langle c_n \rangle \) all clicks that don’t originates \( S_b \) then all of them would be removed.

96 In those conditions, how can it be claimed that the completion of the sequence of clicks \( \langle c_n \rangle \), none of whose elements originates \( S_b \), originates just \( S_b \)? Is the completion of the sequence an additional click different from all elements of \( \langle c_n \rangle \)? If that were the case the sequence of performed clicks would be \( (\omega + 1) \)-ordered in the place of \( \omega \)-ordered, but \( \omega \)-supertasks are \( \omega \)-ordered not \( (\omega + 1) \)-ordered.

97 At this point some infinitists claim the lamp could be at \( S_b \) by reasons unknown. But, once again, that claim violates the definition of the lamp: the state of a Thomson’s lamp changes exclusively by pressing down its button, by clicking its button. So a lamp that changes its state by reasons unknown is not, by definition, a Thomson’s lamp.

98 In any case, the relevant question on the state \( S_b \) is: at which instant Thomson lamp becomes \( S_b \)? It is immediate to prove that instant can only be
the precise instant $t_b$. We know the state of the lamp is $S_b$ at instant $t_b$, but assume the lamp becomes $S_b$ at any instant $t$ prior to $t_b$. Since $t_b$ is the limit of the sequence $\langle t_n \rangle$, we will have:

$$\exists v : t_v \leq t < t_{v+1}$$

which means that at $t$ only a finite number $v$ of clicks have been carried out, and then that infinitely many clickings still remain to be performed. Therefore, and taking into account $t$ is any instant in the interval $(t_a, t_b)$, Thomson lamp can only becomes $S_b$ at the precise instant $t_b$.

99 But $t_b$ is not the instant at which the sequence of switchings is completed; $t_b$ is the first instant after completing the sequence. There is not, in fact, an instant at which that sequence is completed because that sequence is $\omega$-ordered and $\omega$-ordered sequences have not last element. At $t_b$ the sequence $\langle c_n \rangle$ of clicks and then sequence $\langle S_n \rangle$ of changes of state of the lamp have already been completed. At $t_b$ the lamp button is not clicked. At $t_b$ nothing happens that can produce a lamp change of state.

100 It makes no sense to argue about the last term of an $\omega$-ordered sequence simply because such a last term does not exist. By contrast, we could always argue about the limit of an $\omega$-ordered sequence, whenever that limit exists, because it is a well defined object, though it is not an element of the sequence. Similarly, whilst it makes no sense to argue about the last instant at which Thomson lamp button is clicked, the instant $t_b$ is plenty of meaning: it is limit of the sequence of instants at which the successive switchings are carried out; it is the first instant after completing the sequence of switchings. It is the first instant at which the button of the lamp is no longer clicked.

101 In accordance with 98-100, it cannot be claimed that $S_b$ results from completing the sequence $\langle c_n \rangle$ of clicks: Thomson lamp becomes $S_b$ just at $t_b$ and at $t_b$ the sequence of clicks has already been completed; $t_b$ is posterior to the completion of the sequence $\langle c_n \rangle$ of clicks. Thomson lamp becomes $S_b$ at the precise instant $t_b$, but nothing happens at the precise instant $t_b$ for the lamp to become $S_b$:

1. At $t_b$ the sequence of clicks $\langle c_n \rangle$ has already been completed.
2. At $t_b$ the lamp button is not clicked.

$S_b$ is then an impossible state, a consequence of assuming that it is possible to complete an uncompletable sequence of actions, uncompletable in the sense that no last element completes the sequence.

102 The fact that the elements of two incompletable sequences can be paired off by a one by one correspondence, as in the case of the above sequences of clicks and of instants, does not prove both sequences exist as complete infinite totalities: they could also be potentially infinite. The possibility of pairing off

\[\text{This poses an additional problem: how it is possible to complete a sequence of actions within the interval } (t_a, t_b) \text{ if there is no instant within } (t_a, t_b) \text{ at which the sequence is completed?}\]
the elements of two impossible totalities does not make them possible

103 At this point, all that one can expect from infinitists is to be declared incompetent to understand the meaning of the sentence: 'the state of the lamp at \( t_b \) is the result of completing the \( \omega \)-ordered sequence \( \langle c_n \rangle \) of clicks, a result that manifests for the first time just at \( t_b \)'. But, wait a moment, is not \( S_b \) the result of a button click, of a pressing down the lamp button? Don’t forget that Thomson’s lamp can only change its state if, and only if, you press down its button, if you click it. And that both events, the clicking and the corresponding lamp change of state, are instantaneous and simultaneous by definition.

104 So if \( S_b \) appears for the first time at the precise instant \( t_b \) and at \( t_b \) the lamp button is not clicked, what is the cause of \( S_b \)? where does \( S_b \) come from?

105 In short, \( S_b \) must of necessity be originated just at instant \( t_b \), otherwise only a finite number if clicks would have been performed, according to 98-100. But, on the other hand, it cannot be originated at \( t_b \) because:

1. The state of the lamp changes only by clicking its button.
2. The clicking of the button and the corresponding lamp change of state are instantaneous and simultaneous events.
3. Being the clicking of the button and the corresponding lamp change of state instantaneous and simultaneous events, and being the state \( S_b \) originated at the precise instant \( t_b \), the lamp button must be clicked at that precise instant \( t_b \).
4. But at \( t_b \) the lamp button is not clicked.

106 \( S_b \) could only be, therefore, the impossible last state of an \( \omega \)-ordered sequence of states in which no last state exists. The consequence of assuming the hypothesis of the actual infinity from which derives the existence of \( \omega \)-ordered sequences as complete totalities in spite of the fact that no last element completes them.

107 Thomson’s lamp is a theoretical device intentionally invented to facilitate a formal discussion on the actual infinity hypothesis that legitimizes the existence of \( \omega \)-ordered sequences as complete totalities [38], [40, Theorem 15-A]. Supertasks are an example of such sequences, and contradiction 105 clearly suggests the hypothesis on which they are founded could be inconsistent.

**The Counting Machine**

108 The Counting Machine (CM) we will examine in this section poses a problem similar to the one posed by Thomson lamp we have just examined. As its name suggests, CM count natural numbers, and it does it by counting the successive numbers 1, 2, 3... at each of the successive instants \( t_1, t_2, t_3 \ldots \) of the above sequence \( \langle t_n \rangle \). It counts each number \( n \) at the precise instants \( t_n \). In addition, the machine has a blue LED that turns on when, and only when, the machine counts an even number; and turns off when, and only when, the machine counts an odd number.
The one to one correspondence \( f : \langle t_n \rangle \mapsto \mathbb{N} \) proves that at \( t_b \), our machine will have counted all natural numbers. Thus, if after performing the supertask, our counting machine \( CM \) continues to be the same counting machine it was before beginning the supertask, i.e. if performing a supertask does not change the nature of the world nor implies the arbitrary violation of a legitimate formal definition, as that of our \( CM \), then the blue LED of \( CM \) can only be on or off, simply because an LED can only be on or off, independently of the number of times it has been turned on and off.

Assume then that at \( t_b \) the blue LED is on (a similar argument would apply if it were off). One of the following two exhaustive and mutually exclusive alternatives must be true:

1. The blue LED is on because \( CM \) counted a last even number.
2. The blue LED is on because of any other reason.

The first alternative is impossible if all natural numbers have been in fact counted: each even number has an immediate odd successor and then there is not a last natural number, neither even nor odd. The second alternative would imply the formal definition of \( CM \) has been arbitrarily violated: its blue LED turns on when, and only when, the machine counts an even number, which excludes the possibility of being turned on by any other reason.

If the \( \omega \)-ordered list of natural numbers exists as a complete totality in spite of the fact that no last number complete the list, then our modest blue LED has be and not to be on. Otherwise, a legitimate definition would be arbitrarily violated with the only purpose of justifying that our LED can be turned on by any reason different from counting an even number. In this case anything could be expected from the hypothesis of the actual infinity.
8.-Cantor’s 1874-argument revisited

**INTRODUCTION**

112 This chapter examines the conditions under which Cantor’s 1874 argument on the uncountable nature of real numbers could also be applied to rational numbers. It will necessary, therefore, to prove those conditions can never be fulfilled in order to ensure the impossibility of a contradiction on the cardinality of the set of rational numbers, that was proved to be denumerable by Cantor himself [32]. A short rational variant of Cantor’s argument is also included.

**CANTOR’S 1874-ARGUMENT**

113 This section explains in detail the first Cantor’s proof of the uncountable nature of the set \( \mathbb{R} \) of real numbers, published in the year 1874 in a short paper that also included a proof of the denumerable nature of the set \( \mathbb{A} \) (also denoted by \( \overline{\mathbb{Q}} \)) of algebraic numbers and then of the set of rational numbers \( \mathbb{Q} \), a subset of \( \mathbb{A} \) [32], (French edition [33], Spanish edition [44]).

114 Assume the set \( \mathbb{R} \) were denumerable. In those conditions there would be a one to one correspondence \( f \) between the set \( \mathbb{N} \) of natural numbers and \( \mathbb{R} \). Consequently, the elements of \( \mathbb{R} \) could be \( \omega \)-ordered\(^1\) by \( f \) as:

\[
    r_1, r_2, r_3, \ldots
\]

being \( r_i = f(i), \forall i \in \mathbb{N} \). Obviously, the sequence \( \langle r_n \rangle \) defined by \( f \) would contain all real numbers if \( \mathbb{R} \) were actually denumerable.

115 Consider now any real interval \( (a, b) \). Cantor’s 1874-argument consists in proving the existence of a real number \( \eta \) in \( (a, b) \) which is not in the \( \omega \)-ordered sequence \( \langle r_n \rangle \). The existence of \( \eta \) would prove that \( \langle r_n \rangle \) does not contain all real numbers and, therefore, that the initial assumption on the denumerable

\(^1\)As is well known, in the \( \omega \)-ordered sequences there exists a first element and each element has an immediate predecessor, except the first one, and an immediate successor so that no last element exists.
nature of $\mathbb{R}$ is false. The proof goes as follows.

116 Starting from $r_1$, find the first two elements of $\langle r_n \rangle$ within $(a, b)$. Denote the smallest of them by $a_1$ and the greater by $b_1$. Define the real interval $(a_1, b_1)$ (see Figure 8.1).

117 Starting from $r_1$, find the first two elements of $\langle r_n \rangle$ within $(a_1, b_1)$. Denote the smallest of them by $a_2$ and the greater by $b_2$. Define the real interval $(a_2, b_2)$. Evidently it holds:

\[(a_1, b_1) \supset (a_2, b_2)\]  (2)

118 Starting from $r_1$, find the first two elements of $\langle r_n \rangle$ within $(a_2, b_2)$. Denote the smallest of them by $a_3$ and the greater by $b_3$. Define the real interval $(a_3, b_3)$. Evidently it holds:

\[(a_1, b_1) \supset (a_2, b_2) \supset (a_3, b_3)\]  (3)

119 The continuation of the above procedure (R-procedure from now on) defines a sequence of real nested intervals (R-intervals):

\[(a_1, b_1) \supset (a_2, b_2) \supset (a_3, b_3) \supset \ldots\]  (4)

whose left endpoints $a_1, a_2, a_3, \ldots$ form a strictly increasing sequence of real numbers, and whose right endpoints $b_1, b_2, b_3, \ldots$ form a strictly decreasing sequence also of real numbers, being every element of the first sequence smaller than every element of the second one.

120 From the $\omega$-order of $\langle r_n \rangle$ and the ordered way by which R-procedure defines the successive R-intervals (starting from $r_1$ find the first two elements. . . ), it immediately follows that if $r_n$ defines an endpoint $a_i$ or $b_i$, then it must hold $i \leq n$. In consequence, if $r_n$ is any element of $\langle r_n \rangle$, it will not belong to the successive R-intervals:

\[(a_n, b_n) \supset (a_{n+1}, b_{n+1}) \supset (a_{n+2}, b_{n+2}) \supset \ldots\]  (5)

121 The number of R-intervals will be be finite or infinite, and both possibilities have to be considered. Assume in the first place the number of R-intervals is finite. In this case there would be a last R-interval (a, b) in the sequence
of R-intervals. This last R-interval would contain, at best, one element $r_v$ of $\langle r_n \rangle$, otherwise it would be possible to define at least one new R-interval $(a_{n+1}, b_{n+1})$. Let, therefore, $\eta$ be any element within $(a_n, b_n)$, different from $r_v$ if $r_v$ does exist. Evidently $\eta$ is a real number within $(a, b)$ which does not belong to the sequence $\langle r_n \rangle$. Consequently, the sequence $\langle r_n \rangle$ does not contain all real numbers, and then the initial assumption on the denumerable nature of $\mathbb{R}$ must be false.

122 Consider now the number of R-intervals is infinite. Since the sequence $\langle a_n \rangle$ is strictly increasing and upper bounded by every element of $\langle b_n \rangle$, the limit $L_a$ of $\langle a_n \rangle$ does exist. On its part, the sequence $\langle b_n \rangle$ is strictly decreasing and lower bounded by every element of $\langle a_n \rangle$, in consequence the limit $L_b$ of this sequence also exists. Taking into account that every $a_i$ is less than every $b_i$ it must hold: $L_a \leq L_b$.

123 Assume that $L_a < L_b$. In this case any of the infinitely many elements within the real interval $(L_a, L_b)$ is a real number within $(a, b)$ which does not belong to the sequence $\langle r_n \rangle$, and then a proof of the falseness of the initial hypothesis on the denumerable nature of $\mathbb{R}$.

124 Finally, assume that $L_a = L_b = L$. It is immediate to prove that $L$ is a real number within $(a, b)$ which is not in $\langle r_n \rangle$. In fact, assume that $L$ is an element $r_v$ of $\langle r_n \rangle$. According to 120, $r_v$ does not belong to the successive R-intervals:

$$(a_v, b_v) \supset (a_{v+1}, b_{v+1}) \supset (a_{v+2}, b_{v+2}) \supset \ldots, \quad (6)$$

while $L$ belongs to all of them. Therefore, $L$ cannot be $r_v$. The limit $L$ is a real number in $(a, b)$ which is not in $\langle r_n \rangle$, and then a proof of the falseness of the initial assumption on the denumerable nature of $\mathbb{R}$.

Q-V ERSION OF CANTOR’S 1874-ARGUMENT

125 This section develops an argument that is identical to the previous one, except in that it applies to the set $\mathbb{Q}$ of rational numbers.

126 Assume the set $\mathbb{Q}$ of rational numbers were denumerable. In these conditions there would be a one to one correspondence $f$ between the set $\mathbb{N}$ of natural numbers and $\mathbb{Q}$ so that the elements of $\mathbb{Q}$ could be $\omega$-ordered by $f$ as:

$$q_1, q_2, q_3, \ldots \quad (7)$$

being $q_i = f(i), \forall i \in \mathbb{N}$. Obviously, the sequence $\langle q_n \rangle$ defined by $f$ would contain all rational numbers if $\mathbb{Q}$ were actually denumerable.

4Note this case implies the completion of a procedure of infinitely many successive steps.
127 Consider any rational interval \((a, b)\). Starting from \(q_1\), find the first two elements of \(\langle q_n \rangle\) within \((a, b)\). Denote the smallest of them by \(a_1\) and the greater by \(b_1\). Define the rational interval \((a_1, b_1)\).

128 Starting from \(q_1\), find the first two elements of \(\langle q_n \rangle\) within \((a_1, b_1)\). Denote the smallest of them by \(a_2\) and the greater by \(b_2\). Define the rational interval \((a_2, b_2)\). Evidently it holds:

\[(a_1, b_1) \supset (a_2, b_2) \quad (8)\]

129 Starting from \(q_1\), find the first two elements of \(\langle q_n \rangle\) within \((a_1, b_1)\). Denote the smallest of them by \(a_3\) and the greater by \(b_3\). Define the rational interval \((a_3, b_3)\). Evidently it holds:

\[(a_1, b_1) \supset (a_2, b_2) \supset (a_3, b_3)\]

130 The continuation of the above procedure (Q-procedure from now on) defines a sequence of rational nested intervals (Q-intervals):

\[(a_1, b_1) \supset (a_2, b_2) \supset (a_3, b_3) \supset \ldots \quad (10)\]

whose left endpoints \(a_1, a_2, a_3, \ldots\) form a strictly increasing sequence of rational numbers, and whose right endpoints \(b_1, b_2, b_3, \ldots\) form a strictly decreasing sequence of rational numbers, being every element of the first sequence smaller than every element of the second one.

131 From the \(\omega\)-order of \(\langle q_n \rangle\) and the ordered way by which Q-procedure defines the successive Q-intervals (starting from \(q_1\) find the first two elements \ldots), it immediately follows that if \(q_n\) defines an endpoint \(a_i\) or \(b_i\), then it must hold \(i \leq n\). In consequence, if \(q_n\) is any element in \(\langle q_n \rangle\), it will not belong to the successive Q-intervals:

\[(a_n, b_n) \supset (a_{n+1}, b_{n+1}) \supset (a_{n+2}, b_{n+2}) \supset \ldots \quad (11)\]

132 The number of Q-intervals will be be finite or infinite, and both possibilities have to be considered. Assume in the first place the number of Q-intervals is finite\(^6\). In this case there would be a last rational Q-interval\(^7\) \((a_n, b_n)\) in the sequence of Q-intervals. This last Q-interval would contain, at best, one element \(q_v\) of \(\langle q_n \rangle\), otherwise it would be possible to define at least one new Q-interval \((a_{n+1}, b_{n+1})\). Let, therefore, \(\eta\) be any element within \((a_n, b_n)\), different from \(q_v\) in the case that \(q_v\) does exist. Evidently \(\eta\) is a rational number within \((a, b)\) which does not belong to the sequence \(\langle q_n \rangle\). Consequently, the sequence \(\langle q_n \rangle\) does not contain all rational numbers, and then our initial assumption on the denumerable nature of \(\mathbb{Q}\) must be false.

---

\(^5\)The real or rational nature of the successive intervals \((a_i, b_i)\) is irrelevant.

\(^6\)Including the case that Q-procedure defines no Q-interval.

\(^7\)Or the whole interval \((a, b)\) in the case that Q-procedure defines no Q-interval.
Consider now the number of Q-intervals is infinite. Since the sequence \( \langle a_n \rangle \) is strictly increasing and upper bounded by every element of \( \langle b_n \rangle \), the real limit \( L_a \) of \( \langle a_n \rangle \) does exist. On its part, the sequence \( \langle b_n \rangle \) is strictly decreasing and lower bounded by every element of \( \langle a_n \rangle \), in consequence the real limit \( L_b \) of this sequence also exists. Taking into account that every \( a_i \) is less than every \( b_i \), it must hold: \( L_a \leq L_b \), being \( L_a \) and \( L_b \) two real (rational or irrational) numbers.

Assume that \( L_a < L_b \). In this case, any of the infinitely many rationals within the real interval \( (L_a, L_b) \) is a rational number within \( (a, b) \) which does not belong to the sequence \( \langle q_n \rangle \), and then a proof of the falseness of the initial hypothesis on the denumerable nature of \( \mathbb{Q} \).

Finally, assume that \( L_a = L_b = L \). It is immediate that \( L \) is a real number within the real interval \( (a, b) \) which is not in \( \langle q_n \rangle \). In fact, if \( L \) is irrational then it is clear that it is not in \( \langle q_n \rangle \); assume then \( L \) is rational, and assume also it is an element \( q_v \) of \( \langle q_n \rangle \). According to 131, \( q_v \) does not belong to the successive intervals:

\[
(a_v, b_v) \supset (a_{v+1}, b_{v+1}) \supset (a_{v+2}, b_{v+2}) \supset \ldots
\]

while \( L \) belongs to all of them. Therefore, \( L \) cannot be \( q_v \). The limit \( L \) is a real number (rational or irrational) in the real interval \( (a, b) \) which is not in \( \langle q_n \rangle \). Thus, if \( L \) were rational then our initial assumption on the denumerable nature of \( \mathbb{Q} \) would be false.

We have just proved the alternatives of Cantor 1874-argument on the cardinality of the real numbers also applies to the set \( \mathbb{Q} \) of rational numbers, except the last one, that applies only if the common limit of the sequences of left and right rational endpoints of Q-intervals is a rational number.

Evidently, If Cantor’s 1874-argument could be extended to the rational numbers we would have a contradiction: the set \( \mathbb{Q} \) would and would not be denumerable. In consequence, and in order to ensure the impossibility of that contradiction, we must prove that whatsoever be the rational interval \( (a, b) \) and the reordering of \( \langle q_n \rangle \), the number of Q-intervals can never be finite and the sequences of endpoints \( \langle a_n \rangle \) and \( \langle b_n \rangle \) have always a common irrational limit. Until then, the consistency of transfinite set theory will be at stake.

A variant of Cantor’s 1874 argument

The argument that follows is a variant of the above Cantor’s first proof of the uncountable nature of the set of real numbers.

Since, according to Cantor, the set \( \mathbb{Q} \) of rational numbers is denumerable we can consider a one to one correspondence \( f \) between this set and the set \( \mathbb{N} \)

\(^8\)Note this case implies the completion of a procedure of infinitely many successive steps.
of natural numbers. Let \( \langle q_n \rangle \) be the \( \omega \)-ordered sequence of rational numbers defined by:

\[
q_i = f(i), \ \forall i \in \mathbb{N}
\]

(13)

Obviously \( \langle q_n \rangle \) contains all rational numbers.

140 Let \( x \) be a rational variable whose initial value is \( b \), the right endpoint of any rational interval \((a, b)\); and \( \langle q_n \rangle \) the sequence of rational numbers defined by (13). Now consider the following sequence of successive redefinitions of \( x \):

\[
\begin{cases}
i = 1, 2, 3, \ldots \\
q_i \in (a, b) \land q_i < x \Rightarrow x = q_i
\end{cases}
\]

(14)

that compares \( x \) with the successive elements of \( \langle q_n \rangle \) that belong to \((a, b)\), and redefines \( x \) as the compared element each time the compared element is less than the current value of \( x \).

141 Unnecessary as it may seem, we will impose the following restriction to the successive definitions (14):

Restriction 141.-Each successive definition (14) will be carried out if, and only if, \( x \) results defined as a rational number within \((a, b)\).

We will prove now that for any natural number \( v \) the first \( v \) successive definitions (14) can be carried out. In what follows we will assume that whatsoever be the finite or infinite number of definitions (14) that have been carried out, they have been carried out successively.

142 Since \( q_1 \) is a well defined rational number it will be, or not, in \((a, b)\) and it will be, or not, less than \( x \), whose current value is \( b \). Thus, the first definition (14) can be carried out because it defines \( x \) as \( b \) or as \( q_1 \) if \( q_1 \) is in \((a, b)\) and \( q_1 < b \). Assume that, being \( n \) any natural number, the first \( n \) definitions (14) can be carried out, and let \( x_c \) be the corresponding current value of \( x \) once performed the first \( n \) definitions (14). Since \( q_{n+1} \) is a well defined rational number, it will be, or not, in \((a, b)\) and it will be, or not, less than \( x_c \). Thus the \((n + 1)\)th definition can also be carried out because it defines \( x \) as \( x_c \), or as \( q_{n+1} \) if \( q_{n+1} \) is in \((a, b)\) and \( q_{n+1} < x_c \). We have just proved that the first definition (14) can be carried out, and that if for any natural number \( n \) the first \( n \) definitions (14) can be carried out, then the first \((n + 1)\) definitions (14) can also be carried out. Thus, for any natural number \( v \), the first \( v \) definitions (14) can be carried out.

143 Assume that while the successive definitions (14) that observe Restriction 141 can be carried out, they are carried out. The value of \( x \) once performed all possible\(^9\) definitions (14), whatsoever be the finite or infinite number of times it has been redefined, will be a rational number within the interval \((a, b)\] just

\(^9\)If it were impossible to perform all possible definitions (14) we would be in the face of the elementary contradiction of an impossible possibility.
because it was always defined, including the initial definition, as a rational number within \((a, b]\), and only as a rational number within \((a, b]\). Thus, we can assert:

- Whatever be the current value \(x^*\) of \(x\) once performed all possible definitions (14), it will be a rational number within the rational interval \((a, b]\).

Some infinitists argue that although restriction 141 applies to all the infinitely many successive definitions of \(x\), once completed the infinite sequence of those definitions we cannot ensure \(x\) is defined as a rational number within \((a, b]\), despite the fact that each of those definitions defined \(x\) as a rational number within \((a, b]\). As if the completion of an infinite sequence of definitions, as such a completion, had unknown additional effects on the defined object, including the arbitrary violation of any legitimate restriction, as Restriction 141.

The same unknown additional effects on the defined objects could, then, be expected in any other definition of procedure composed of infinitely many successive steps, in which case infinitist mathematics would have no sense. For instance, in Cantor’s 1874 argument if the number of R-intervals is infinite, and due to those unknown additional effects of the completion, as such a completion, on the defined object, we could not ensure all of them are real intervals within \((a, b]\) in spite of the fact that each of them was successively defined as a real interval within \((a, b]\).

Thus, if to complete the infinite sequence of definitions (14) means to perform each and every definition of the sequence (and only them) each of which observes Restriction 141 and then leaves \(x\) defined as a rational number within \((a, b]\), and if the completion of the sequence, as such a completion, has not unknown arbitrary effects on \(x\), then, once performed all possible definitions, \(x\) can only be a rational number \(x^*\) within \((a, b]\). Notice this conclusion is not deduced from the successively performed definitions but from the observance of Restriction 141 by all of them.

Consider the rational interval \((a, x^*)\) and any element \(\eta\) within \((a, x^*)\). It is quite clear that \(\eta \in (a, b]\) and \(\eta < x^*\). We will prove \(\eta\) cannot belong to \(\langle q_n \rangle\). In fact, assume \(\eta\) belongs to \(\langle q_n \rangle\). In such a case there will be an element \(q_v\) in \(\langle q_n \rangle\) such that \(q_v = \eta\), and being \(\eta\) in \((a, x^*)\), we will have \(q_v \in (a, x^*)\), and therefore \(q_v < x^*\). But this is impossible because:

1. The index \(v\) of \(q_v\) is a natural number.
2. According to 142, for each natural number \(v\), it is possible to carry out the first \(v\) definitions (14)
3. All possible definitions (14) have been carried out.
4. At least the first \(v\) definitions (14) have been carried out.
5. Once performed the first \(v\) definitions (14) we will have \(x \leq q_v\). Therefore \(x^* \leq q_v\)
6. It is then impossible that \( q_v < x^* \).

In consequence \( \eta \) cannot be an element of \( \langle q_n \rangle \).

148 The rational number \( \eta \) proves, therefore, the existence of rational numbers within \((a, b)\) which are not in \( \langle q_n \rangle \), which in turn proves the falseness of the initial assumption on the denumerable nature of \( \mathbb{Q} \). And taking into account Cantor’s proof on the denumerability of the set \( \mathbb{Q} \), the final conclusion is that \( \mathbb{Q} \) is and is not denumerable.

Remark 148-1.- The sequence of definitions (14) leads to some other contradictory results the reader could easily find. Evidently, contradictory results do not invalidate one another, they simply prove the existence of contradictions. If, starting from the same hypothesis, two independent arguments lead to contradictory results they prove the inconsistency of the initial hypothesis. It is quite clear then that an argument cannot be refuted by another argument. An argument can only be refuted by indicating where and why that argument fails.

Remark 148-2.- Infinitist mathematics assumes the possibility to perform procedures of infinitely many successive steps. But mathematics is not usually concerned with the way those procedures could be, in fact, carried out; it is only concerned with the consistency of the involved arguments. When the result of a procedure or definition of infinitely many steps is an infinite set (or sequence), then the set (or sequence) is always considered as a complete infinite totality, which implies the completion of the infinitely many steps of the corresponding procedure or definition, as in the above case of Cantor’s 1874 argument. That said, it seems appropriate to recall we dispose of a formal theory one of whose objectives is just to analyze the ways those infinite procedures and definitions could be carried out in a finite or infinite interval of time (supertask and bifurcated supertask respectively).

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10 This obviousness is often ignored in the discussions on the actual infinity.

11 See for instance [27], [49], [154], [66], [174], [12], [122].
9.-Numerical exchanges

ω-EXCHANGES

As we will see in this chapter, it is possible to make disappear a number from a list of numbers if the list is ω-ordered and the number exchanges successively its current position (row) in the table with the number in the next position in the table, while a number in the next position in the table exists to exchange its position. This absurdity is an inevitable consequence of assuming that ω-ordered lists exist as complete totalities. We will also see the conflict does not appear in potentially infinite lists.

Consider the ω-ordered table $T$ of natural numbers in their natural order of precedence, and let $r_1 = 1; r_2 = 2; r_3 = 3 \ldots$ be its successive rows. Assume now we exchange the number 1 with the number 2, and then the number 1 with the number 3, and then the number 1 with the number 4, and so on (Figure 9.1). In symbols:

$$n = 2, 3, 4, \ldots \left\{ \begin{array}{l}
r_{n-1} = n \\
r_n = 1 \end{array} \right.$$

(1) Figure 9.1: Numerical 1-exchanges through the ordered list of natural numbers.

The purpose of the next discussion is to examine the destination of the number 1 once all possible exchanges (1), 1-exchanges hereafter, have been carried out.

The successive 1-exchanges will be subjected to the following:

Restriction 151.-For each natural number $n$, the $n$th 1-exchange will be carried out if, and only if, it places the number 1 in the $(n+1)$th row of $T$

It is immediate to prove that for each natural number $v$ the first $v$ 1-exchanges can be carried out. In fact, it is clear the first 1-exchange between 1 and 2 can be carried out without violating Restriction 151, because it places the number 1 in the second row $r_2$ of $T$ and the number 2 in $r_1$. Assume that,
being \( n \) any natural number, the first \( n \) 1-exchanges can be performed without violating Restriction 151. Once performed, the number 1 will be placed in the row \( r_{n+1} \). Consequently, a new 1-exchange, the \((n+1)\)th one, can be performed because it places 1 in \( r_{n+2} \) (and the number \((n + 2)\) in the row \( r_{n+1} \)). Thus, the first 1-exchange can be performed, and if for any natural number \( n \) the first \( n \) 1-exchanges can be performed, then the first \((n + 1)\) 1-exchanges can also be performed. This inductive reasoning proves that for each natural number \( n \) the first \( n \) 1-exchanges can be carried out without violating Restriction 151.

We will examine the consequences of this conclusion in the following two sections by means of two independent arguments.

**Supertask Argument**

Supertask theory assumes the possibility of performing infinitely many actions in a finite time (see [154] for background details and Chapters 7 and 25 in this book). The short discussion that follows analyzes this assumption by mean of an elementary conditional supertask\(^1\) whose successive tasks consist just in performing the successive 1-exchanges subjected to Restriction 151. As a consequence of those successive 1-exchanges, the number 1, originally placed in the first row, will be successively placed in the 2nd, 3rd, 4th ... row of \( T \).

Let us denote the \( i \)th 1-exchange by \( e_i \) and let \( \langle t_n \rangle \) be a strictly increasing \( \omega \)-ordered sequence of instants within the real interval \((t_a, t_b)\) whose limit is \( t_b \). Assume each possible 1-exchange \( e_i \) is performed at the precise instants \( t_i \) of \( \langle t_n \rangle \). Obviously, at instant \( t_b \) all possible 1-exchanges will have been carried out. The problem is: in which row will be placed the number 1 at \( t_b \)? If \( r_v \) is any row of \( T \) it is quite clear that 1 is not in \( r_v \) because in such a case the first \( v \) 1-exchanges would not have been carried out,\(^2\) which according to 152 is impossible. Thus, and being, \( r_v \) any row, we must conclude that at instant \( t_b \) the number 1 has disappeared from the table. At \( t_b \), therefore, Restriction 1 has been arbitrarily violated, in spite of the fact that none of the performed 1-exchanges violated it. While all numbers greater than 1 remain in the table, the number 1 has mysteriously disappeared in an 'infinitist puff of smoke'.

It is worth noting the conclusion on the disappearance of the number 1 has not been derived from the successively performed 1-exchanges. We have simply proved that once completed the supertask, the number 1 cannot be in any of the \( T \) rows, otherwise, if it were in a row \( r_v \), only a finite number \( v - 1 \) of successive 1-exchanges would have been carried out. This is the type or result one can expect when assuming the list of natural numbers does exist as a complete totality without a last number completing the list.

\(^1\)In a conditional supertask each successive task if performed if, and only if, a certain condition is met, in our case Restriction 151.

\(^2\)The \( v \)th 1-exchange places \( x \) in the row \( r_{v+1} \).
Modus Tollens Argument

Consider the following two propositions regarding the completion of all possible 1-exchanges:

- **p**: Once performed all possible 1-exchanges the number 1 remains in $T$.
- **q**: Once performed all possible 1-exchanges the number 1 is in a certain row $r_v$ of $T$.

It is quite clear that $p \Rightarrow q$ because if once performed all possible 1-exchanges the number 1 remains in $T$, then it must be in one of its rows $r_v$.

We will prove now $q$ is false. Let $r_v$ be any row of $T$. If once performed all possible 1-exchanges the number 1 is in $r_v$ then the $v$th 1-exchange $e_v$ has not be carried out. But this is false because:

1. The index $v$ of $e_v$ is a natural number.
2. According to 152, for each natural number $v$, it is possible to carry out the first $v$ 1-exchanges.
3. All possible 1-exchanges have been carried out.
4. At least the first $v$ 1-exchanges have been carried out.
5. The $v$th 1-exchange $e_v$ placed the number 1 in the row $r_{v+1}$.

In consequence the number 1 is not in $r_v$. Therefore, and being $r_v$ any row, we must conclude $q$ is false.

We can therefore write:

\[
\begin{align*}
p & \Rightarrow q \\
\neg q & \\
\therefore \neg p
\end{align*}
\]

which means that once performed all possible 1-exchanges the number 1 is no longer in $T$. Or, alternatively, that it is impossible to perform all possible 1-exchanges.

The Potential Infinity Alternative

We will end this chapter by analyzing the above problem of 1-exchanges from the point of view of the potential infinity. Since from this point of view only finite totalities make sense (as large as you wish but always finite), consider any finite number $n$ and the table $T_n$ of the first $n$ natural numbers. 1-Exchanges will be now defined by:

\[
k = 2, 3, 4, \ldots, n \begin{cases} r_{k-1} = k \\ r_k = 1 \end{cases}
\]

and then, only a finite number $n - 1$ of 1-exchanges can be carried out, at the end of which the number 1 will be placed in the last row of $T_n$. 
Thus, for any given natural number \( n \) the 1-exchanges in \( T_n \) are consistent. Only when they take place in the assumed complete lists \( T \) of all natural numbers they become inconsistent. In symbols:

\[
\forall n \in \mathbb{N} : \begin{array}{l}
k = 2, 3, 4, \ldots, n \\
r_{k-1} = k \\
r_k = 1
\end{array}
\]  

(6)

is consistent, while:

\[
k = 2, 3, 4, \ldots \begin{array}{l}
r_{k-1} = k \\
r_k = 1
\end{array}
\]  

(7)

is inconsistent.
10.-Cantor diagonal argument

**INTRODUCTION**

162 Cantor’s diagonal argument makes use of an hypothetical table $T$ that is assumed to contain all real numbers within the real interval $(0, 1)$. That table can be easily redefined in order to ensure it contains at least all rational numbers within $(0, 1)$. In these conditions, could the rows of $T$ be reordered so that a rational antidiagonal can be defined? In that case, and for the same reason as in Cantor original proof, the set of rational numbers would be proved to be non-denumerable. And then we would have a contradiction since Cantor also proved the set of rational numbers is denumerable. Should, therefore, Cantor’s diagonal argument be suspended until it be proved the impossibility of such a reordering? Is that reordering possible? The discussion that follows addresses both questions.

**The $n$-th decimal theorem**

163 We will begin by proving an elementary result on the decimal expansion of rational numbers we will make use of later on. For this, let $M$ be the set of all real numbers within the real interval $(0, 1)$ expressed in decimal notation and completed, in the cases of finitely many decimal digits (decimals hereafter), with infinitely many 0’s in the right side of their decimal expansions, so in the place of 0.25 we will write 0.25000... The subset of all rational numbers in $M$ will be denoted by $M_Q$.

164 Let us prove the following:

**Theorem 164 (of the $n$th decimal).**-For every natural number $n$ there are infinitely many different elements in $M_Q$ with the same decimal $d_n$ in the same $n$th position of its decimal expansion.

Proof.-Let $d_n$ be any decimal digit $(0, 1, 2, \ldots 9)$ and consider any element $r_0$ in $M_Q$ of the form:

$$r_0 = 0.d_1d_2\ldots d_n$$

(1)

where each $d_i$ is any decimal digit. From $r_0$ we define the sequence of rational

---

1It could easily be extended to irrational numbers, although we will not do it here.
numbers:

\[ r_1 = 0.d_1d_2 \ldots d_n1000 \ldots \]  
\[ r_2 = 0.d_1d_2 \ldots d_n11000 \ldots \]  
\[ r_3 = 0.d_1d_2 \ldots d_n111000 \ldots \]  
\[ \ldots \]  
\[ r_k = 0.d_1d_2 \ldots d_n1^{(k)}1000 \ldots \]  
\[ \ldots \]  

The one to one correspondence \( f \) between \( \mathbb{N} \) (the set of natural numbers) and \( M_Q \) defined according to:

\[ f(k) = r_k, \forall k \in \mathbb{N} \]  

proves that, being \( n \) any natural number, there exist a denumerable subset \( f(\mathbb{N}) \) of \( M_Q \), each of whose elements \( r_k \) has a finite decimal expansion of \( n + k \) decimals with the same decimal \( d_n \) in the same \( n \)th position.

**Cantor versus Cantor**

165 Cantor’s set \( M \) is the union of two disjoint sets: the denumerable set \( M_Q \) of all rational numbers in \((0, 1)\) and the set \( M_I \) of all irrationals in the same interval \((0, 1)\). Being \( M_Q \) denumerable, there exists a one to one correspondence \( g \) between \( \mathbb{N} \) and \( M_Q \). On the other hand assume, as Cantor did in 1891 [36], that \( M \) were denumerable. In those conditions it is evident that, being \( M_I \) infinite, it will also be denumerable, otherwise (if it were non-denumerable) its superset \( M \) could not be (only) denumerable. Let then \( h \) be a bijection between \( \mathbb{N} \) and \( M_I \). From \( g \) and \( h \) we define a one to one correspondence or bijection \( f \) between \( \mathbb{N} \) and \( M \) according to:

\[
\begin{align*}
 f(2n - 1) &= g(n) \\
 f(2n) &= h(n)
\end{align*}
\]  

\( \forall n \in \mathbb{N} \)  

We can therefore consider the \( \omega \)-ordered table \( T \) whose successive rows \( r_1, r_2, r_3 \ldots \) are just \( f(1), f(2), f(3) \ldots \). By definition, and being \( M_Q \) (supposedly) denumerable, \( T \) contains a denumerable subtable with all rational numbers in \((0, 1)\).

166 The diagonal of Cantor’s table \( T \) is the real number \( D = 0.d_{11}d_{22}d_{33} \ldots \) whose \( n \)th decimal \( d_{nn} \) is the \( n \)th decimal of the \( n \)th row \( r_n \) of \( T \). From this number Cantor defined another real number in \( M \), the antidiagonal \( D^- \), in the following way: change each decimal \( d_{nn} \) by any other different decimal. This ensures that, being a real number in \( M \), \( D^- \) is different from every row of \( T \): it differs from each row \( r_n \) just in its \( n \)th decimal.

167 In consequence, \( M \) cannot be denumerable as was assumed to be. This
is Cantor’s diagonal argument, an impeccable Modus Tollens (MT)\(^2\) [36]. In effect, let \(p\) and \(q\) be the propositions:

\[
p: M \text{ is denumerable} \quad (8)
q: T \text{ contains all real numbers within } (0, 1) \quad (9)
\]

then, and once proved \(D^-\) is a real number within \(M\) that is not in \(T\), we will have:

\[
p \Rightarrow q \quad (10)
\neg q \quad (11)
\therefore \neg p \quad (12)
\]

168 Now then, since \(D^-\) is a real number in \((0, 1)\), it will be either rational or irrational. But if it were rational then, and for the same reason as in the case of \(M\), the subset \(M_Q\) of all rational numbers in \(M\) would also be non-denumerable. The problem here is that Cantor had already proved the set \(\mathbb{Q}\) of all rational numbers, and therefore \(M_Q\), is denumerable [32].

169 According to 168, if it were possible to reorder the rows of \(T\) in such a way that a rational antidiagonal could be defined, then we would have two contradictory results: the set \(\mathbb{Q}\) of rational numbers would and would not be denumerable. Both results could be considered as proved by Cantor, although the second one only as an unexpected (and so far unknown) consequence of his famous diagonal method. Accordingly, we can state the following:

Conclusion 169.-Cantor’s diagonal argument and all its formal consequences should be suspended until it be proved the impossibility of reordering the rows of \(T\) in such a way that a rational antidiagonal can be defined.

170 Without that demonstration, set theory is under the threat of a fundamental contradiction. It is really shocking that for more than a century no one, including thousands of mathematicians and logicians, has even posed that problem.

RATIONAL ANTIDIAGONALS

171 We will now examine the possibilities and consequences of reordering the rows of \(T\) in the sense indicated in 169.

172 Once assumed the existence of the set of all finite cardinals as a complete totality, Cantor proved the existence of \(\omega\)-ordered collections [38], [40, Theorem 15-A]. In an \(\omega\)-ordered sequence, as the above table \(T\), every element -whatever it be- will always be preceded by a finite number of elements and

\(^2\)The critiques of Cantor’s diagonal argument are invariably related to different aspects which are not pertinent with the formal structure of Cantor’s demonstration.
succeeded by an infinite number of elements. We will see now a conflicting consequence of this immense asymmetry.

173 We will begin by defining the concept of D-modular row. First of all, a row \( r_i \) of \( T \) will be said \( n \)-modular if its \( n \)th decimal is \( (n \mod 10) \). This means that a row is, for instance, 2348-modular if its 2348th decimal is 8; or that it is 453-modular if its 453th decimal is 3. If a row \( r_n \) is \( n \)-modular (being \( n \) in \( n \)-modular the same number as \( n \) in \( r_n \)) it will be said \( D \)-modular. For instance, the rows:

\[
\begin{align*}
    r_1 &= 0.1007647464749943400034577774413 \ldots \quad (13) \\
    r_2 &= 0.220004566778943000000000000000 \ldots \quad (14) \\
    r_3 &= 0.003000000000000000000000000000 \ldots \quad (15) \\
    r_9 &= 0.11122233900000043406666666633 \ldots \quad (16) \\
    r_{13} &= 0.1234567890000300000357585843456931 \ldots \quad (17)
\end{align*}
\]

are all of them D-modular.

174 Consider now the following permutation \( P \) of the rows \( \left< r_n \right> \) of table \( T \). For each successive row \( r_i \):

1. If \( r_i \) is D-modular then let it unchanged.
2. If \( r_i \) is not D-modular then exchange it with any following \( i \)-modular row \( r_j, j > i \), provided that at least one of the following rows \( r_j \) be \( i \)-modular.\(^3\)

The exchange of a non D-modular row \( r_i \) by a following \( i \)-modular row will be referred to as \( D \)-exchange.

175 Notice that, thanks to the condition \( j > i \) (in \( r_{j, j > i} \)), once a non-D-modular row \( r_i \) has been D-exchanged, it becomes D-modular and will remain D-modular and unaffected by the subsequent D-exchanges. And notice also the successive D-exchanges do not alter the \( \omega \)-ordering of table \( T \): \( P \) does not modify the ordering of the \( \omega \)-ordered set \( \mathbb{N} \) of indexes, but the real numbers indexed by the same successive indexes. Or in other words, D-exchanges do not exchange the indexes but the real numbers indexed by them. It exchanges the content of the successive \( T \)’s rows, but not the ordinality of \( T \)’s rows.

176 It is immediate to prove, by Modus Tollens (MT), that all \( T \)'s rows become D-modular as a consequence of permutation \( P \). In fact, let us assume that a row \( r_n \) did not become D-modular as a consequence of \( P \). This means that \( r_n \) is not D-modular and could not be D-exchanged with a following \( n \)-modular row. Now then, all \( n \)-modular rows have the same digit \( (n \mod 10) \) in the same \( n \)th position of its decimal expansion, and according to theorem 164 of the \( n \)th decimal, there are infinitely many rational numbers with the same digit in the same position of its decimal expansion, whatsoever be the digit and the position. Accordingly, since \( n \) is finite, the row \( r_n \) is preceded by a finite

\(^3\)Replace \( r_i \) with \( r_j \) and \( r_j \) with \( r_i \).
number and followed by an infinite number of n-modular rows. Any of these infinitely many succeeding n-modular rows had to be D-exchanged with \( r_n \). It is therefore impossible for \( r_n \) not to be D-modular. In consequence (Modus Tollens), each and every row \( r_n \) becomes D-modular as a consequence of \( P \).

177 It is worth noting the result proved in 176 is a formal consequence of both theorem 164 of the \( n \)th decimal and the fact that every row \( r_n \) of \( T \) is always preceded by a finite number of n-modular rows and followed by an infinite number of n-modular rows. This immense asymmetry is an inevitable side effect of \( \omega \)-order, which, as Cantor proved [40, Theorem 15-A], derives from assuming the existence of the set of all finite cardinals (natural numbers) as a complete totality (in modern terms, hypothesis of the actual infinity subsumed within the Axiom of Infinity).

178 Let us remark the formal structure of proof 176 in order to avoid unnecessary discussions. Consider the following two propositions \( q_1 \) and \( q_2 \) about permutation \( P \):

\( q_1 \): After performing \( P \), not all rows become D-modular.

\( q_2 \): After performing \( P \), at least a non-D-modular row \( r_k \) could not be D-exchanged.

It is quite clear that \( q_1 \) implies \( q_2 \): if not all rows become D-modular then at least a non-D-modular row \( r_k \) could not be D-exchanged. Now then, being \( k \) finite and taking into account the theorem of the \( n \)th decimal, there are infinitely many k-modular rows \( r_{n,n>k} \) following \( r_k \), therefore some of them had to be D-exchanged with \( r_k \). In consequence proposition \( q_2 \) is false and then so will be \( q_1 \). In symbols:

\[
q_1 \Rightarrow q_2 \quad (18)
\]

\[
\neg q_2 \quad (19)
\]

\[
\therefore \neg q_1 \quad (20)
\]

It is quite clear then that, as in the case of Cantor’s diagonal argument, the above proof is also a simple Modus Tollens (see final remark).

179 Let \( T_p \) be the table resulting from permutation \( P \). Since all its rows are D-modular, its diagonal \( D \) will be the rational number 0.1234567890. It is now
immediate to define infinitely many rational antidiagonals from \( D \). Indeed, let us denote by \( p_o \) the period 1234567890 of the diagonal \( D \). We are interested in periods of ten digits none of which coincide in position with the digits of \( p_o \), as is the case, for instance, of 0123456789 or 4545454545 (= 45). The number of those periods is \( 9^{10} \). Among them, let us choose the above two examples and denote them by \( p_1 \) and \( p_2 \) respectively \((p_1 = 0123456789; p_2 = 4545454545)\). Now we define the sequence of rational antidiagonals \( \langle A_n \rangle \) by:

\[
\forall n \in \mathbb{N}: A_n = 0.p_1p_1^n.p_1\hat{p}_2
\]

whose elements cannot be in \( T_p \) for the same reason as Cantor’s antidiagonal: it differs from each row \( r_n \) just in its \( n \)th decimal. Since all of them are rational numbers, we must conclude that \( M_Q \) and its superset \( \mathbb{Q} \) are both non-denumerable.

180 Permutation \( P \) allows us to develop other arguments whose conclusions also suggest the inconsistency of the hypothesis of the actual infinity. For instance, it is clear that rows as 0.\( \hat{2}1 \), and many others, can never become D-modular, and then we would have to admit the absurdity that \( P \) makes all of them disappear from the table. In fact, let \( n \) be any natural number and assume that, for instance, 0.\( \hat{2}1 \) is the \( n \)th row of \( T_p \). Since \( n \) is finite, 0.\( \hat{2}1 \) will be preceded by a finite number of \( n \)-modular rows and followed by an infinite number of \( n \)-modular rows, according to theorem 164 of the \( n \)th decimal. In consequence 0.\( \hat{2}1 \), that is not \( n \)-modular,\(^4\) was D-exchanged with any of those \( n \)-modular rows, and then it cannot be the \( n \)th row of \( T_p \). Thus, and being \( r_n \) any row of \( T_p \), we must conclude 0.\( \hat{2}1 \) has disappeared from the table!

181 The above absurdity 180 is the sort of things you can expect from a list in which each and every element has finitely many predecessors and infinitely many successors. A list in which, in spite of having infinitely many successive elements, it is impossible to reach an element with infinitely many predecessors (what, evidently, makes the above arguments possible). A list, in short, that is simultaneously complete (as the hypothesis of the actual infinity requires) and uncompletable (because no last element completes the list).

182 Permutation \( P \) can even be considered as a case of supertask (hypercomputation). In fact, let \( \langle t_n \rangle \) be an \( \omega \)-ordered and strictly increasing sequence of instants within a finite interval of time \((t_a, t_b)\), being \( t_b \) the limit of the sequence. Assume that \( P \) operates on each row \( r_i \) just at the precise instant \( t_i \) of \( \langle t_n \rangle \). Consequently, \( r_i \) will remain unchanged if it is D-modular (or if it is not D-modular and cannot be D-exchanged) or it will be D-exchanged with any of the following \( i \)-modular rows. At \( t_b \) permutation \( P \) will have been applied to every row of \( T \) as the one to one correspondence \( f(t_i) = r_i \) proves.

183 Assume that at \( t_b \), once accomplished the hypercomputation \( P \), the per-

\(^4\)For each \( n \)th decimal of 0.\( \hat{2}1 \) it holds \((n \mod 10) = 2 \) if \( n \) is odd, or \((n \mod 10) = 1 \) if it is even.
muted table $T_p$ contains a row $r_n$ which is not D-modular. This row, whatsoever it be, will be preceded by a finite number of rows and followed by an infinite number of rows, infinitely many of which are n-modular, and then D-exchangeable with $r_n$. Therefore $r_n$ was D-exchanged. Thus, $r_n$ can only be D-modular in $T_p$.

184 To be simultaneously complete and uncompletable, as is the case of any $\omega$–ordered object, could be, after all, contradictory.

A FINAL REMARK

185 Let me end by recalling that an argument cannot be refuted by another different argument. In W. Hodges words: [101, p. 4]

How does anybody get into a state of mind where they persuade themselves that you can criticize an argument by suggesting a different argument which doesn’t reach the same conclusion?

This inadmissible strategy is frequently used in discussions related to infinity, for instance to refute Cantor’s arguments on the uncountable nature of real numbers. However, to refute an argument means to indicate where and why that argument fails. If two arguments lead to contradictory conclusions, they simply are proving the existence of a contradiction.
Cantor diagonal argument
11.-Not so rational

INTRODUCTION

186 For the purpose of the following discussion, a partition $P_{ab}$ of a right closed interval $(a, b]$ of positive rational numbers will be any set of right-closed intervals of positive rational numbers which are also disjoint and successively adjacent, and whose union is the whole interval:

$$P_{ab} = \{ \ldots (q_i, q_{i+1}], (q_{i+1}, q_{i+2}], (q_{i+2}, q_{i+3}] \ldots \}$$

$$\bigcup_i (q_i, q_{i+1}] = (a, b]$$

As a consequence of their adjacency, the right endpoint of each interval coincides with the left endpoint of the next one, provided that a next interval does exist. As a consequence of their disjointness, each endpoint can only belong to a different interval. Therefore, it holds:

$$(q_i, q_{i+1}] \cup (q_{i+1}, q_{i+2}] = (q_i, q_{i+2}]$$

$$(q_i, q_{i+1}] \cap (q_{j+1}, q_{j+2}] = \emptyset, \forall i, \forall j$$

$$\ldots < q_i < q_{i+1} < q_{i+2} < \ldots$$

where ’$<$’ stands for the natural order of precedence of the rational numbers.

187 As is well known, the set of rational numbers in their natural order of precedence is densely ordered. So, if $a$ and $b$ are any two different rational numbers, then the interval $(a, b)$ contains infinitely many different rational numbers, no matter how close $a$ and $b$ are. Or in other words (and contrary to what happens with any natural number in the sequence of natural numbers 1, 2, 3, . . .), no rational number has an immediate successor in the natural order of precedence of rational numbers. This trivial property of rational numbers will be of capital importance in the following argument.

188 In that argument we will define a partition of a positive rational intervals whose successive parts are defined by means of an $\omega$—ordered sequence of positive rational numbers that contains all positive rational numbers. The successive rationals of this defining sequence define the successive endpoints of
the successive intervals of the partition whenever they belong to the partitioned interval. As we will see, these intervals contain positive rational numbers that are not members of the defining sequence although they would have to be members of the defining sequence, because the defining sequence contains all positive rational numbers.

**A partition a la Cantor**

**189** Let $f$ be a one to one correspondence between the set $\mathbb{N}$ of natural numbers and the denumerable set $\mathbb{Q}^+$ of positive rational numbers, and consider the $\omega-$ordered sequence $(q_n)$ defined by:

$$\forall i \in \mathbb{N} : q_i = f(i) \quad (6)$$

Since $f$ is a one to one correspondence, it is quite clear the $\omega-$ordered sequence $(q_n)$ contains all positive rational numbers. Obviously, $f$ induces an $\omega-$order in $\mathbb{Q}^+$ that makes it possible to consider successively all of their elements: $q_1, q_2, q_3 \ldots$, which in turn makes it possible the next definition.

**190** Let $(a, b]$ be any right closed interval of positive rational numbers. By following the same strategy as in Cantor’ 1874 argument [32], we will now define a partition of $(a, b]$ by means of the successive elements $q_1, q_2, q_3 \ldots$ of the sequence $(q_n)$ in the following way:

- Starting from $q_1$ and following the order $q_1, q_2, q_3 \ldots$ find the first element $q_e$ of $(q_n)$ in $(a, b)$.
  - Define the adjacent and disjoint intervals $(a, q_e], (q_e, b]$. Obviously $(a, q_e] \cup (q_e, b] = (a, b]$.
- Starting from $q_{e+1}$ and following the order $q_{e+1}, q_{e+2}, q_{e+3} \ldots$ find the first element $q_j$ of $(q_n)$ in $(a, b)$.
  - Since the union of the adjacent and disjoint intervals previously defined is $(a, b)$, $q_j$ must belong to one of those intervals. Replace that interval with the intervals $(x, q_j] \text{ and } (q_j, y]$, where $x$ and $y$ are respectively the left and the right endpoint of the replaced interval. Obviously $(x, q_j] \cup (q_j, y] = (x, y]$.
- Starting from $q_{j+1}$ and following the order $q_{j+1}, q_{j+2}, q_{j+3} \ldots$ find the first element $q_m$ of $(q_n)$ in $(a, b)$.
  - Since the union of all the adjacent and disjoint intervals previously defined is $(a, b)$, $q_m$ must belong to one of those intervals. Replace that interval with the intervals $(x, q_m] \text{ and } (q_m, y]$, where $x$ and $y$ are respectively the left and the right endpoint of the replaced interval. Obviously $(x, q_m] \cup (q_m, y] = (x, y]$.
  - And so on.

**191** Once all the successive elements $q_1, q_2, q_3 \ldots$ of $(q_n)$ have been considered, we will have a partition $P_{ab}$ of the rational interval $(a, b]$ because:
- The union of all intervals of the set \( P_{ab} \) is the interval \((a, b]\) since in each replacement (including the first replacement of \((a, b]\)) the replaced interval is the union of the replacing intervals.

- Each replaced interval \((x, y]\) is always replaced by two adjacent and disjoint intervals \((x, z]\) and \((z, y]\), and so that their union \((x, z]\cup(z, y]\) has the same endpoints \(x\) and \(y\) as the replaced interval \((x, y]\). In consequence all replacements maintain the adjacency and disjointness of the replaced interval.

- The intersection of any two elements of \( P_{ab} \) is empty because in each replacement the intersection of the replacing intervals is always empty because the replacing intervals are always disjoint.

192 The partition \( P_{ab} \) will necessarily contain an interval whose left endpoint is \(a\), because all replacements (including the first replacement of \((a, b]\)) maintain the endpoints of the replaced interval in the new replacing intervals. Let \((a, q_s]\) be that interval. Since all rational intervals are densely ordered, between \(a\) and \(q_s\) infinitely many different rationals do exist. Let \(q \neq q_s\) be any element of \((a, q_s]\). Obviously \(q\) can only belong to \((a, q_s]\) because all \( P_{ab} \) intervals are disjoint to each other. It is also clear \(q\) is a positive rational number. But it cannot be an element \(q_v\) of the sequence \(\langle q_n \rangle\) because if that were the case \(q_v\) would be an element of \((a, q_s]\) and only of \((a, q_s]\), and then the interval \((a, q_s]\) would have been replaced with the intervals \((a, q_v], (q_v, q_s]\) when considering \(q_v\) in Definition 190. The same argument can be applied to any other interval of the partition \( P_{ab} \). This proves the sequence \(\langle q_n \rangle\), that contains all positive rational numbers, does not contain all positive rational numbers.

193 Notice, on the other hand, that all rational numbers in \((a, b]\) are elements of \(\langle q_n \rangle\), therefore all of them will have been considered by Definition 190 to define \( P_{ab} \), which means every element in \((a, b]\) is the endpoint of two adjacent and disjoint intervals of \( P_{ab} \). In consequence, all those intervals are empty intervals whose endpoints are two different rational numbers, which is incompatible with the dense ordering of \((a, b]\). This contradiction suggests the impossibility for a denumerable set (i.e. one that can be \(\omega\)–ordered) to be densely ordered.

194 The above cantorian reasoning suggests the following argument. Consider again the sequence of all positive rational numbers defined in (6) and let \(I = (a, b]\) any interval of positive rational numbers. Consider the following sequence of redefinitions of \(I\):

\[
i = 1, 2, 3 \ldots \begin{cases} q_i \in I \Rightarrow I = (a, q_i) \\ q_i \notin I \Rightarrow \text{let } I \text{ unchanged} \end{cases}
\]

195 It is immediate to prove that for each rational number \(v\) it is possible to perform the first \(v\) redefinitions (7) of the interval \(I\). Indeed, it is quite clear the first redefinition (7) can be carried out: Since \(q_1\) is a positive rational number it will belong, or not, to \(I\). In the first case \(I\) is redefined as \((a, q_1)\);
in the second as \((a, b)\). Assume that for any natural number \(n\) it is possible to perform the first \(n\) redefinitions (7) of \(I\), so that \(I = (a, x)\) and \(x\) is one of the first \(n\) elements of \(\langle q_n \rangle\) or \(b\). Since \(q_{n+1}\) is a positive rational number it will belong, or not, to \((a, x)\). In the first case \(I\) is redefined as \((a, q_{n+1})\); in the second as \((a, x)\). This proves that for any natural number \(v\) it is possible to perform the first \(v\) redefinitions (7) of \(I\).

196 Assume now that while the successive redefinitions (7) of \(I\) can be carried out, they are carried out. Since each successive redefinition (7) defines \(I\) as an interval \((a, x), x \in \langle q_n \rangle\), once performed all of those possible redefinitions, \(I\) will defined as an interval \((a, x), x \in \langle q_n \rangle\). Otherwise we would have to accept that the completion of a sequence of definitions has unexpected arbitrary consequences on the defined object, in whose case all definitions and procedures consisting of a sequence of finitely or infinitely many successive steps would be nonsensical and infinitist mathematics would lose all of its meaning. Alternatively, we would have to admit the impossibility of completing any infinite sequence of definitions. In whose case infinitist mathematics would no longer make sense.

197 Therefore, let \(I = (a, x]\) once all possible redefinitions (7) have been carried out. And let \(q\) be any element in \((a, x)\). Obviously \(x\) is a positive rational number, but it cannot be an element \(q_v\) of the sequence \(\langle q_n \rangle\), otherwise the \(v\)th redefinition (7), and then the replacement of \((a, x]\) with \((a, q_v]\) and \((q_v, x]\), would not have been carried out, which, being \(v\) a natural number, is impossible according to 195. Thus, as in the above cantorian argument, \(\langle q_n \rangle\) contains and does not contain all positive rational numbers.

**Discussion**

198 Cantor’s *Beiträge* (‘Contributions’), published in 1895 (Part I, [37]) and 1897 (Part II, [38]) contain the fundamentals of the theory of transfinite cardinals and ordinals. Epigraph 6 of the first article begins by assuming the existence of the set of all finite cardinals as a complete totality (although rather than an assumption it is introduced as an example of ‘transfinite aggregate’ whose existence as a complete totality Cantor took for granted). This implicit assumption (equivalent to our modern Axiom of Infinity) is the only assumption in Cantor’s theory of transfinite numbers. From it, Cantor successfully derived the existence of increasing transfinite ordinals (Theorems §15 A-K) and cardinals (Theorems §16 D-F). The consistency of Cantor theory rests, therefore, on the consistency of that unique foundational assumption.

199 In 1874 Cantor proved for the first time the set of real numbers is not denumerable [32], [33], [44]. Two of the three final alternatives of Cantor’s proof could also be applied to the set of rational numbers. In consequence, it is necessary to prove the conflicting alternatives are never satisfied in the case of the set of rational numbers. Otherwise that set would and would not be

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1English translation [40].
denumerable. Until now, and as far as I know, this problem has not even been raised.

200 In 1891 Cantor proved for the second time the set of real numbers is not denumerable, now by his celebrated diagonal method, an impecable Modus Tollens [36]. Cantor antidiagonal is a real number in the real interval (0, 1), and being real it will be either rational or irrational. If it were rational we would have the same problem as with Cantor’s 1874 argument. So, it should be \textit{formally proved} that no permutation of the $\aleph_0$ rows of Cantor’s table yields a rational diagonal (rational antidiagonals are immediately derived from rational diagonals).

201 The above referred Cantor’s 1874 argument begins by proving the set of algebraic numbers (and then the set of rational numbers) is denumerable. Some years after, in 1885, Cantor published an immediate consequence of this result: non-denumerable partitions of the real line are impossible, for the sole reason that if they were possible the set of rational numbers would be non-denumerable [35]. So, as in the cases of Cantor’s 1874 and 1891 arguments and for the same reasons, we should prove the impossibility of non-countable partitions of the real line by means of an argument independent of Cantor immediate proof.

202 In conclusion, and in order to ensure set theory is free of inconsistencies related to the cardinality of the set of rational numbers, Cantor’s 1784, 1885 and 1891 arguments should be completed in the sense indicated in 199-201.

203 On the other hand, the above argument 189-192 proves the sequence $\langle q_n \rangle$ contains and does not contain all positive rational numbers. Which can only mean the set of rational numbers is and is not denumerable. If that were the case, and in agreement with 198, the assumed existence of the infinite sets as complete totalities would be inconsistent, simply because that assumption is the only assumption of the theory of transfinite numbers.
Not so rational
12.-Non-countable partitions

INTRODUCTION

The set of real numbers was proved to be non denumerable by Cantor’s 1874 argument and Cantor’s diagonal argument. Although the diagonal argument has been contested, I find both arguments are well founded and in fact prove the set of real numbers cannot be denumerable. Both arguments, however, could also be applied to the set of rational numbers, the first of them with certain limitations.

Obviously, if it were possible to apply any of those arguments to the set \( \mathbb{Q} \) of rational numbers, we would be in the face of a fundamental contradiction: that set would and would not be denumerable. And the cause of that contradiction could only be the hypothesis of the actual infinity.

On these Cantor’s arguments we also pointed out two significant details:

1. The hypothesis of the actual infinity will be in question until it is proved the impossibility of satisfying the requirements of both Cantor’s arguments in order to be applied to the set of rational numbers. Notice this is a fact, not a more or less debatable hypothesis.

2. For over a century no one has noted it is, in effect, necessary to prove that impossibility in order to guaranty the consistency of the Axiom of Infinity. This is also a fact. And a shocking one, taking into account the high number of people who have studied both arguments, particularly the diagonal argument.

As we will see in this chapter, there is a third Cantor’s argument involving the cardinality of the set \( \mathbb{Q} \) of rational numbers that could also be used to test the consistency of the actual infinity hypothesis.

CANTOR’S 1885 PROOF

To summarize Cantor’s 1885 argument on the existence of uncountable partitions assume the real line is partitioned in a non-denumerable sequence \( P_\alpha \) of adjacent intervals:

\[
(x_a, y_a] (x_b, y_b] (x_c, y_c] \ldots, \tag{1}
\]

\[
x_b = y_a, x_c = y_b, \ldots \tag{2}
\]
Being each \((x_\alpha, y_\alpha]\) a real interval, it contains infinitely many rational numbers. And being:

\[(x_p, y_p] \cap (x_u, y_u] = \emptyset, \text{ for any pair of } P_\alpha \text{ intervals} \quad (3)\]

we could pick out a rational number \(q_h\) within each interval \((x_h, y_h]\) of \(P_\alpha\) and we will finally have a non-denumerable sequence of different rational numbers, which is impossible because the set of rational number is denumerable.

209 As we have just seen, Cantor’s 1885 proof rests on a previous infinitist result, namely that the set \(\mathbb{Q}\) of rational numbers is denumerable, a result that had been previously proved by Cantor himself \([32]\). Therefore, Cantor’s 1885 proof is not an independent proof in the sense that it does not prove the impossibility of defining a non-countable partition of the real line, what it proves is that such a partition would be in conflict with the countable cardinality of rational numbers. Consequently, if it were possible to define a non-countable partition of the real line we would be in the face of a fundamental contradiction involving again the cardinality of \(\mathbb{Q}\) and then the consistency of the actual infinity hypothesis from which that conclusion derives.

210 Thus, for the third time we face a shocking fact: how is it possible that for over a century no one has tried to define a non-countable partition of the real line, or alternatively to prove that such a partition is impossible independently of Cantor proof of the cardinality of \(\mathbb{Q}\)? As the reader may imagine we will try to define such a partition in the next section.

**Partitions of the real line**

211 Cantor’s ternary set (also known as Cantor dust) is a well known mathematical object we usually discover in introductory courses of calculus, mathematical analysis or fractal geometry \([123]\). The definition of Cantor ternary set is an appropriate example of infinitist procedure of infinitely many successive steps that, in addition, resembles the procedure \(H\) (see 215) we will make use of in the next argument. As we will see, \(H\) allows to define a partition of the real line with the only aid of the elements of the real interval \((0, 1)\).

212 But let’s now recall the way Cantor’s dust is defined. Consider the closed real interval \([0, 1]\). If we remove or delete the open middle third \((1/3, 2/3)\) of this interval we will get two closed intervals:

\[[0, 1/3], \ [2/3, 1]\] \quad (4)

If we now remove the open middle third of each of these intervals, \((1/9, 2/9)\) y \((7/9, 8/9)\), we will get four closed intervals:

\[[0, 1/9], \ [2/9, 1/3], \ [2/3, 7/9], \ [8/9, 1]\] \quad (5)

If we now remove the open middle third of each of these intervals we will get
eight closed intervals, whose open middle third can be removed again, and so on. By continuing this procedure ad infinitum we will get Cantor ternary set (Figure 12.1).

213 Before beginning our discussion it seems convenient to recall the above procedure of infinitely many successive steps is considered as a complete totality of steps whose final result is a completely defined set: Cantor ternary set.

214 In the next argument, and to avoid unnecessary discussions, we will use standard mathematical notation in the place of computer science notation, though this last would be simpler. Let us consider two identical sets $A = B = (0, 1)$ of real numbers, and two identical sets of indexes $I$ and $J$ whose elements will be referred to as $a, b, c, d, e, \ldots$ and whose cardinality is $2^{\aleph_0}$. Being $(0, 1)$ and $I$ of the same cardinality, the elements of $A = B = (0, 1)$ can be indexed as $r_a, r_b, r_c, r_d, \ldots$. Consider also the real variables $u$ and $v$, initially defined as $u = v = 0$.

215 We now define the following procedure $H$ which consists in repeating the same conditional step until the condition is satisfied:

Step: If $A = \emptyset$, or $I = \emptyset$ then end. Else:

Select as $\alpha$ any element of $J$

$I = J - \{\alpha\}$

$J = I$

Select as $r_\alpha$ any element of $B$

$A = B - \{r_\alpha\}$

$B = A$

$v = u + r_\alpha$

$[x_\alpha, y_\alpha] = (u, v)$

$P_\alpha = \{(x_\alpha, y_\alpha)\}$

$u = v$

Next step

216 Each step of $H$ consists in removing any index $\alpha$ from $I$ (via the intermediate set of indexes $J$) in order to index and remove from $A$ any of its elements
$r_\alpha$ (via the intermediate set $B$), which is then used to define a new real interval $(x_\alpha, y_\alpha]$ disjoint and adjacent to the one previously defined, whenever it is not the first defined interval. This new interval defines the set $P_\alpha$, whose only element is that interval.

217 We now define the following partition $P$ of the real line:

$$P = \bigcup_{\alpha} P_\alpha = \bigcup_{\alpha} \{(x_\alpha, y_\alpha]\} = \{(x_a, y_a], (x_b, y_b], (x_c, y_c], (x_d, y_d], \ldots \}$$

whose elements are adjacent and disjoint real intervals since $x_b = y_a$, $x_c = y_b$, $x_d = y_c$, etc. Therefore:

$$(x_h, y_h] \cap (x_s, y_s] = \emptyset, \forall h, s \in I; h \neq s$$

$$(x_h, y_h] \cup (x_i, y_i] = (x_h y_i]$$

being $(x_h, y_h]$ and $(x_i, y_i]$ adjacent. In accordance with their definition, and taking into account each element of $(0, 1)$ is different from each other, the intervals of the partition $P$ also satisfy:

$$\forall (x_h, y_h], (x_s, y_s] \in P:$$

$$y_h - x_h = r_h \in (0, 1)$$

$$y_s - x_s = r_s \in (0, 1)$$

$$r_h \neq r_s$$

which, on the other hand, means each $P$ interval has a different length, greater than zero.

218 Each interval $(x_h, y_h]$ of $P$ defines the real number $y_h - x_h = r_h$ within
the real interval $(0, 1)$, that obviously is the same real number $r_h$ used to define $(x_h, y_h]$, and only $(x_h, y_h]$ because it is removed from $A$ once defined $(x_h, y_h]$. Thus, it is immediate to define a one to one correspondence between $P$ and $(0, 1)$. Indeed, consider the correspondence $f$:

$$f : P \leftrightarrow (0, 1)$$

$$f((x_h, y_h)) = y_h - x_h = r_h$$

Since, according to definition 215, each $y_h - x_h$ is an element of $(0, 1)$, and taking into account (9)-(12), the correspondence $f$ is an injective function (injection). It is also surjective (exhaustive) since in agreement with definition 215 each and every element $r_h$ of $(0, 1)$ was used to define one interval $(x_h, y_h]$ and only one interval, because each $r_h$ was removed from $A$ once defined the corresponding interval. In consequence $f$ is a one to one correspondence (bijection). Therefore the partition $P$ and the set $(0, 1)$ do have the same cardinality: $2^{\aleph_0}$.

219 Now, by following Cantor’s suggestion we would only have to pick out any rational number $q_h$ within each interval of the partition $P$ and we would have a non-denumerable set of rational numbers \( \{q_a, q_b, q_c, \ldots \} \). Consequently, and taking into account the set of rational numbers $\mathbb{Q}$ was also proved to be denumerable [32], we have a new contradiction regarding the cardinality of $\mathbb{Q}$.

220 For the third time, when completing an uncompleted Cantor’s argument, we have found a fundamental contradiction involving the cardinality of the set $\mathbb{Q}$ of rational numbers. As in the precedent cases, this new contradiction points towards the inconsistency the hypothesis of the actual infinity subsumed into the Axiom of Infinity. It is in fact this axiom that legitimizes the existence of infinite sets as complete totalities, and then the completeness of procedures of infinitely many steps as 215 from which the contradiction has been derived.

221 Evidently, the claim that it is actually impossible to complete in physical terms any infinite computation, as the above procedure $H$, has no effect on the argument, mainly for the following two reasons:

1. As most of the infinitist arguments, argument 214-219 is also a conceptual discussion unrelated to the physical world. The formal consistency of the actual infinity hypothesis does not depend upon the actual possibilities of performing this or that procedure, but on the existence of contradictions formally derived from that hypothesis. When formally proved, contradictory results in formal systems depend exclusively on the consistency of the their foundational assumptions, regardless of the possibility of actually performing the finitely or infinitely many steps involved in the corresponding arguments.

2. Infinitist mathematics takes it for granted the completion of all definitions and procedures composed of infinitely many steps and consider the result-

\footnote{Every real interval contains an infinite and densely ordered subset of rational numbers.}
ing objects as complete infinite totalities, as in the introductory example of Cantor ternary set. Argument 214-219 cannot be an exception.
13.-Sets and boxes

INTRODUCTION

222 From the platonic point of view (the dominant perspective in contemporary mathematics), all attempts to define the concept of set have been circular, so that it is now considered as a primitive notion, i.e. as one that cannot be defined in terms of other more basic concepts.

223 From a non-platonic point of view, however, it is possible to define the notion of set as a mental construct. For instance, Charles Dogson (better known as Lewis Carroll) proposed the following concept [45, p. 31]:

Classification, or the formation of Classes, is a Mental Process, in which we imagine that we have put together, in a group, certain Things. Such a group is called a Class.

Carroll’s definition leads immediately to the following one:

A set is a theoretical object that results from a mental process of grouping arbitrary elements previously defined.

It could be proved this definition is not compatible with self-reference one of the sources of inconsistencies in naive set theory. But this type of non-platonic definitions are absolutely unknown in contemporary mathematics. We will introduce some of them in Appendix B.

224 We could imagine a set as a sort of box that contains elements. And while the number of elements is finite the comparison will always be consistent. Notwithstanding, when the number of elements is infinite some differences appear that make both objects radically different. As we will see in this chapter the consideration of an infinite set as a box that contains infinitely many elements leads to certain infinitist infelicities.

EMPTYING SETS AND BOXES

225 Consider a box $BX$ containing an $\omega$—ordered collection of identical balls labeled as $b_1$, $b_2$, $b_3$, ..., and on the other hand an $\omega$—ordered set $B$ whose elements are also a denumerable collection of identical balls labelled as $b_1$, $b_2$, $b_3$,...:

\[ B = \{b_1, b_2, b_3 \ldots \} \]  

(1)
226 From $B$ we define the following $\omega$–ordered sequence of sets $\langle B_n \rangle$:

$$
\begin{align*}
  i = 1, 2, 3, \ldots & : B_i = B - \{b_i\} \\
  i > 1 & : B_i = B_{i-1} - \{b_i\}
\end{align*}
$$

(2)

$\langle B_n \rangle$ is, therefore, the sequence of nested sets:

$$
B_1 \supset B_2 \supset B_3 \supset \ldots
$$

(3)

each of whose members $B_n = \{b_{n+1}, b_{n+2}, b_{n+3}, \ldots\}$ is a denumerable set.

227 Let now $[t_a, t_b]$ be any finite interval of time and $\langle t_n \rangle$ an $\omega$–ordered and strictly increasing sequence of instants within $[t_a, t_b]$ whose limit is $t_b$. Assume that at each instant $t_i$ of $\langle t_n \rangle$ the ball $b_i$ is removed from the box $BX$. Let $BX(t_i)$ be the state of the box (the remaining collection of balls within the box) at instant $t_i$, once removed the ball $b_i$. The successive balls extractions can be symbolically expressed in a form similar to (2):

$$
\begin{align*}
  i = 1, 2, 3, \ldots & : BX(t_i) = BX(t_a) - b_i \\
  i > 1 & : BX(t_i) = BX(t_{i-1}) - b_i
\end{align*}
$$

(4)

228 The one to one correspondence $f(t_i) = b_i$ proves that at $t_b$ all balls will have been removed from the box and $BX$ will be empty. By comparing (2) with (4) we will have:

$$
BX(t_i) = B_i, \forall i \in \mathbb{N}
$$

(5)

229 There is, however, a fundamental difference between the sequence of sets $\langle B_n \rangle$ and the sequence of states $\langle BX(t_i) \rangle$: in each of the successive removal (4) the box $BX$ is always the same box $BX$ while the sets defined by each successive removal (2) are all of them different. As a consequence we will have a final empty box but not a final empty set. We will address this problem in Chapter 19.

230 Meanwhile note that at each instant $t$ in $[t_a, t_b]$ the box contains $\aleph_0$ balls whereas at $t_b$ it is empty. In fact, since $t_b$ is the limit of the sequence $\langle t_n \rangle$, we will have:

$$
\forall t \in [t_a, t_b] : \exists v : t_v \leq t < t_{v+1}
$$

(6)

and then at $t$ only the first $v$ balls $b_1, b_2, \ldots b_v$ have been removed from $BX$, so that $BX$ still contains infinitely many balls $b_{v+1}, b_{v+2}, b_{v+3}, \ldots$ Since this conclusion applies to any $t$ in $[t_a, t_b]$, the only way for the box to become empty at $t_b$ would be by removing simultaneously infinitely many balls just at $t_b$. How is this possible if at $t_b$ no ball is removed from the box? How is it possible if all balls have been removed one by one and with an interval of time greater than zero between any two successive removals? How is it possible that, in those conditions, the box never contains 5, 4, 3, 2, 1, 0 balls?
Although not very usual, it is absolutely legitimate to redefine a set any finite or infinite number of times. No law of logic nor foundational axiom of set theory is violated when redefining a set, in the same way they are not when redefining a variable. So, consider the following sequence of redefinitions of the sets \( X \) and \( Y \) by means of the sequence \( \langle B_n \rangle \):

\[
\begin{align*}
i = 1, 2, 3 \ldots \quad & \begin{cases} X = B_i \\ Y = B_2 \end{cases}
\end{align*}
\]

While the sequence of redefinitions of the set \( Y \) poses no problem and we will finally have \( Y = B_2 \), the successive redefinitions of the set \( X \) poses the following problem: Definitions 7 can only leave \( X \) defined as the empty set,\(^1\) while none of its infinitely many redefinitions defined it as the empty set, since all sets \( B_i \) are denumerable.

In the next chapter we will have the opportunity to analyze another more serious conflict related to a sequence of (finitely or infinitely many) redefinitions of a set.

\[\text{Figure 13.1: Removing balls from the box } BX. \text{ 1.-The box } BX \text{ and its automatic closure at the precise instant } t_1 \text{ of removing the first ball } b_1. \text{ 2.-The box is automatically closed when it contains } k \text{ balls. 3.-The box is not automatically closed because all balls were remove simultaneously.} \]

Catching a fallacy

In the following conceptual argument we will use the same box \( BX \) with the same collection of labelled balls \( \langle b_n \rangle \). Although the box will be now provided with the following:

Closing mechanism \( \text{233.} \)-A mass sensor is regulated so that it will close automatically the box if the box contains \( k \) of balls, being \( k \) any natural number randomly chosen by the closing mechanism once turned on.

We will make use of the same sequence of instants \( \langle t_n \rangle \) and will assume the closing mechanism is regulated before \( t_1 \).

\( ^1 \)Otherwise only a finite number of definitions would have been performed, since if \( b_n \) belong to \( X \) then the \( n \)th redefinition (that define \( X \) as \( \{ b_{n+1}, b_{n+2}, b_{n+3}, \ldots \} \)) would not have been carried out.
Assume now that, while the box is open, at each precise instant \( t_i \) of \( \langle t_n \rangle \) the ball \( b_i \) is removed from the box. It is worth noting of this way of removing the balls from \( BX \), that between the extraction of each ball \( b_i \) and the extraction of the next one \( b_{i+1} \) a time greater than zero \( (t_{i+1} - t_i) \) always goes on. So, the removal of the balls is performed one by one, one after the other and with a non-null interval of time between any two successive removals.

If the closing mechanism 233 works as it has to work then, and taking into account the balls are removed one by one and with a non null interval of time between any two successive removals, at \( t_b \) the box \( BX \) can only be closed with a number \( k \) of balls inside it. Notwithstanding, we will also analyze the possibility that at \( t_b \) the box is empty and not closed.

Let us first analyze the case in which at \( t_b \) the box is closed. This alternative is only possible if the box contains a number \( k \) of balls, which poses the following problems:

1. Taking into account the \( \omega \)-ordered way in which the balls have been successively extracted one by one \( \langle b_1, b_2, b_3, \ldots \rangle \), the remaining \( k \) balls in the box could only be the impossible \( k \) last balls of an \( \omega \)-ordered collection of labelled balls \( \langle b_n \rangle \).

2. The box \( BX \) had to be closed at an instant \( t^* \) before \( t_b \) because at \( t_b \) all balls would have been removed (as the one to one correspondence \( f(t_i) = b_i \) would prove).

3. Being \( t_b \) the limit of the \( \omega \)-ordered sequence \( \langle t_n \rangle \), there exists a natural number \( v \) such that \( t_v \leq t^* < t_{v+1} \). In consequence at \( t^* \) only a finite number \( v \) of balls have been removed and infinitely many of them still remain to be removed.

4. It is therefore impossible for the box \( BX \) to be closed at \( t_b \) with finitely many balls.

Let us now assume that at \( t_b \) the box is empty and open. Taking into account that the number \( k \) used by the closing mechanism to determine when the box must be closed may be any natural number, this alternative is only possible if the box never contains a number \( k \) of balls for all \( k \) in \( \mathbb{N} \). Now then, the least cardinal greater than all finite cardinal is \( \aleph_0 \), which is also the cardinal of the collection \( \langle b_n \rangle \); and the least ordinal greater than all finite ordinal is \( \omega \), just the ordinal of the \( \omega \)-ordered sequence of balls \( \langle b_n \rangle \). Therefore, this alternative can only occur if all balls \( \langle b_n \rangle \) are simultaneously removed from the box, which goes against the fact that all of them have been successively removed one by one and with a non null interval of time between any two successive extractions.

Argument 233-237 seems to put into question the consistency of the actual infinity hypothesis from which it can be inferred that \( \omega \)-ordered sequences or lists exist as complete totalities in spite of the fact that no last element completes the list.
**INFINITIST MAGIC**

239 Consider again the collection of labeled balls $\langle b_n \rangle$ and, in the place of the box $BX$, a hollow cylinder $AB$ placed horizontally and capable of containing the whole collection $\langle b_n \rangle$. Now assume that at each of the successive instants $t_i$ of $\langle t_n \rangle$ each of the successive balls $b_i$ is introduced in $AB$ through its left end $A$ (Figure 13.2).

![Figure 13.2](image-url)

> Figure 13.2: Each of the successive ball $b_i$ of $\langle b_n \rangle$ will be successively introduced within the cylinder $AB$.

240 At $t_b$ the complete collection of balls $\langle b_n \rangle$ will have been introduced inside the cylinder $AB$, as the one to one correspondence $f(t_i) = b_i$ immediately proves.

![Figure 13.3](image-url)

> Figure 13.3: By inclining the cylinder in a direction the successive balls $b_i$ will leave the cylinder through its right end $B$ (top). But what will happen if we incline the cylinder in the opposite direction? (bottom)

241 Assume now that, once completed the above supertask, the left end $A$ of the cylinder is lifted with respect to its right end $B$. The cylinder will be inclined in such a way that all balls $b_i$ of $\langle b_n \rangle$ can roll freely in the direction from $A$ to $B$. As could be expected, in those conditions the successive balls $b_i$ of $\langle b_n \rangle$ will successively leave the cylinder through its right end $B$ (Figure 13.3 top).

242 If, on the contrary, it is the right end $B$ of the cylinder which is lifted with respect to its left end $A$, the balls inside the cylinder will freely roll in the direction from $B$ to $A$. As in 241, a first ball will leave the cylinder. But whatsoever be this ball, it will be a labeled ball $b_v$ proving that only a finite number $v$ of balls were introduced inside the cylinder. The alternative is that no ball leaves the cylinder, in whose case all balls previously introduced in the cylinder would have magically disappeared. The problem is that magic does not belong to formal sciences (Figure 13.3 bottom).
243 The cylinder and the labelled balls \( \langle b_n \rangle \) lead to other infinitist conflicts. For instance, if we introduce a rigid rod through the left end \( A \) we would traverse the entire length of the cylinder without hitting any ball, otherwise we would hit the last ball of an \( \omega \)-ordered collection of balls.
14.-Rationals from irrationals

**n-ExpoFactorial Numbers**

This chapter introduces expofactorial and n-expofactorial numbers, as well as the method of the successive decimal expansions by means of which it is possible to define a different rational number from the infinite decimal expansion of each irrational number within the real interval $(0, 1)$. Evidently, this conclusion goes against other well known results on the cardinality of the set $\mathbb{Q}$ of rational numbers.

Although the method of the successive decimal expansions we will make use of in the next section works with natural numbers of any size, we will use natural numbers unimaginably large: the n-expofactorials numbers defined in 248.

The expofactorial\(^1\) of a natural number $n$, written $n'$ (note the factorial symbol '!' appears as a superscript), is the factorial $n!$ raised to a power tower of order $n!$ of the same exponent $n!$:

$$
n' = n!^{(n!), (n!), \ldots, (n!), n!^n} = n!^{(n!), (n!), \ldots, (n!), n!^n}
$$

Or in Knuth’s notation:

$$
n' = n! \uparrow\uparrow n! \tag{1}
$$

These numbers growth so rapidly that while the expofactorial of 2 (in symbols $2'$) is 16, the expofactorial of 3 (in symbols $3'$) is practically incalculable even with the aid of the most powerful computers:

$$
3' = 66666\ldots = 6666646656 = 666664665646591197721532267796824894043879\ldots
$$

where the incomplete exponent of the last term on the right has nothing less

\(^1\)The first time I considered this type of numbers, I didn’t know they have already been defined by C. A. Pickover ([148] cited in [199]) with the name of superfactorials and the symbols $n\$, the same name and symbols used by Sloane and Plouffe to define $n\$ = $\Pi_{k=1}^{n} k!$ [199]. That said, I will retain my original notation and name.
than 36306 decimals (roughly ten pages of standard text like this one). The
expofactorial of any natural number greater than 2 is so large that it will
probably never be calculated with exactitude (it is not an anodyne power of
ten but a precise sequence of different figures).

248 Expofactorials are insignificant compared with n-expofactorials, recur-
sively defined from expofactorials as follows: the 2-expofactorial of a natural
number \( n \), denoted by \( n!^2 \), is the expofactorial \( n! \) raised to a power tower of
order \( n! \) of the same exponent \( n! \); the 3-expofactorial of \( n \), denoted by \( n!^3 \),
is the 2-expofactorial of \( n \) raised to a power tower of order \( n!^2 \) of the same
exponent \( n!^2 \); the 4-expofactorial of \( n \), denoted by \( n!^4 \), is the 3-expofactorial
of \( n \) raised to a power tower of order \( n!^3 \) of the same exponent \( n!^3 \); and so on:

\[
\begin{align*}
n!^2 &= (n!^1)^{n!^1} \\
n!^3 &= (n!^2)^{n!^2} \\
n!^4 &= (n!^3)^{n!^3} & \ldots
\end{align*}
\]

The grandeur of, for example, \( 9!^9 \) (9-expofactorial of 9) is far beyond human
imagination. Three standard arithmetic symbols, just \( 9!^9 \), is all we need to
define a finite number so large that the standard writing of its precise sequence
of figures would surely require a volume of paper trillions and trillions of times
greater than the volume of the visible universe. If we use the hexadecimal
numeral system, \( F!^F \) would be inconceivable greater.

249 The discussion that follows makes use of the 9-expofactorial of 9. For
simplicity, it will be denoted by the letter ‘k’. So, in what follows k will stand
for \( 9!^9 \).

AN IRRATIONAL SOURCE OF RATIONAL NUMBERS

250 Real numbers within the interval (0, 1) with an infinite decimal expansion
are arithmetically defined as

\[
r = 0.d_1d_2d_3 \ldots = d_1 \times 10^{-1} + d_2 \times 10^{-2} + d_3 \times 10^{-3} + \ldots
\]

where the sequence of decimals digits \( d_1d_2d_3 \ldots \) is \( \omega \)-ordered, as the set \( \mathbb{N} \) of
natural numbers in their natural order of precedence 1, 2, 3, \ldots

251 In accordance with the hypothesis of the actual infinity, subsumed into
the Axiom of Infinity, the infinite decimal expansion \( 0.d_1d_2d_3d_4 \ldots \) of any real
number (with an infinite decimal expansion) within the real interval (0, 1) does
exist as a complete \( \omega \)-ordered totality: it has a first decimal digit (decimal
hereafter), \( d_1 \), and each decimal \( d_n \) (except \( d_1 \)) has an immediate predecessor
\( d_{n-1} \) and an immediate successor \( d_{n+1} \), so that no last decimal exists. Since
the argument that follows deals exclusively with \( \omega \)-ordered infinities, from
now on, and for simplicity, they will be referred to simply as infinities.

252 A point of note is that \( \omega \), the ordinal of the \( \omega \)-ordered sequences, is the smallest infinite ordinal. Therefore, if \( r \) and \( s \) are two real numbers within the real interval \((0, 1)\) and they coincide in their first successive \( \omega \) decimals, then both numbers are identical. On the contrary, and taking into account that between any finite ordinal and \( \omega \) only other finite ordinals do exist, if \( r \) and \( s \) are different then they can only coincide in a finite number of their first successive decimals.

253 Let \( \mathbb{N} \) be the set of natural numbers, \( k \) the 9-expofactorial of 9 (in symbols \( 9^{19} \)), and \( m_\alpha \) any element of the set \( M \) of the irrational numbers within the real interval \((0, 1)\). The exclusive decimal expansion of \( m_\alpha \):

\[
m_\alpha = 0.d_1d_2d_3\ldots
\]

defines the following \( \omega \)-ordered sequence \( \langle q_{\alpha,nk} \rangle \) of rational numbers:

\[
q_{\alpha,k} = 0.d_1d_2\ldots d_k
\]

\[
q_{\alpha,2k} = 0.d_1d_2\ldots d_kd_{k+1}\ldots d_{2k}
\]

\[
q_{\alpha,3k} = 0.d_1d_2\ldots d_kd_{k+1}\ldots d_{2k}d_{2k+1}\ldots d_{3k}
\]

\[
q_{\alpha,nk} = 0.d_1d_2\ldots d_kd_{k+1}\ldots d_{2k}d_{2k+1}\ldots d_{3k}d_{3k+1}\ldots d_{nk}
\]

\[
q_{\alpha,(n+1)k} = 0.d_1d_2\ldots d_kd_{k+1}\ldots d_{2k}d_{2k+1}\ldots d_{3k}d_{3k+1}\ldots d_{nk}
\]

being \( q_{\alpha,nk} \) (for every \( n \) in \( \mathbb{N} \)) the rational number within \((0, 1)\) whose finite decimal expansion \( 0.d_1d_2\ldots d_{nk} \) coincides with the first \( nk \) decimals of \( m_\alpha \). For this reason, \( m_\alpha \) will be said the source of the sequence \( \langle q_{\alpha,nk} \rangle \), and \( \alpha \) will appear as a part of the subindex of each \( q_{\alpha,nk} \). The rational \( q_{\alpha,(n+1)k} \) will be said the k-expansion of the rational \( q_{\alpha,nk} \) because \( q_{\alpha,nk} \) is expanded with the next \( k \) successive decimals (starting from \( d_{nk+1} \)) of the source \( m_\alpha \) in order to define \( q_{\alpha,(n+1)k} \). Don’t forget the unimaginable grandeur of \( k = 9^{19} \).

254 From the perspective of the actual infinity hypothesis, the result of defining the infinitely many natural numbers by adding infinitely many successive times one unit to the first natural number 1, defines infinitely many increasing finite numbers, without ever reaching an infinite number.\(^2\) Consequently, and being \( k \) a natural number, the result of defining the infinitely many elements of \( \langle q_{\alpha,nk} \rangle \) by adding infinitely many successive times \( k \) new decimals to the decimal expansion of \( q_{\alpha,k} \), yield infinitely many finite decimal expansions (rational numbers), explosively increasing but always finite (\( nk \) for each \( n \in \mathbb{N} \)), without ever reaching an infinite decimal expansion.

255 This infinitist assumption will be essential for the next argument since

\(^2\)The recursive definition of natural numbers in set theoretical terms leads to the same conclusion
it legitimates the actual existence of the infinitely many rational numbers in \( \langle q_{\alpha,nk} \rangle \), all of them with finitely many decimals, \( nk \) for each \( n \) in \( \mathbb{N} \). In the same way \( \mathbb{N} \) contains infinitely many finite natural numbers, each of them one unit greater than its immediate predecessor, \( \langle q_{\alpha,nk} \rangle \) contains infinitely many rational numbers with a finite decimal expansion, each with \( k \) decimals more than its immediate predecessor. This is, in fact, infinitist orthodoxy.

256 Let \( P \) be the set of all pairs \((m_\alpha, q_{\alpha,k})\) whose first component is a different element \( m_\alpha \) of the set \( M \) of irrational numbers in \((0, 1)\), and whose second component is the rational number \( q_{\alpha,k} \) within \((0, 1)\) defined by the first \( k \) successive decimals \( d_1, d_2, \ldots, d_k \) of \( m_\alpha \):

\[
(m_\alpha, q_{\alpha,k}) \in P \iff \begin{cases} m_\alpha = 0.d_1d_2 \ldots d_kd_{k+1} \ldots \in M \\
\text{and} \\
q_{\alpha,k} = 0.d_1d_2 \ldots d_k
\end{cases} \quad (10)
\]

Although the first element \( m_\alpha \) of each pair is a different irrational number, the second one \( q_{\alpha,k} \) will be repeated a certain number of times in the different pairs of \( P \).

257 Notice that if there is no irrational number in \((0,1)\) with the same first \( k \) decimals, then the second element of each pair of \( P \) would be a different rational number. In these conditions the discussion that follows would be unnecessary: there would be as many rationals as irrationals within \((0,1)\).

258 Let now \( q_{\alpha,k} \) be any of the repeated rationals in \( P \), and let \( P_\alpha \) be the subset of \( P \) of all pairs \((m_\varphi, q_{\varphi,k})\) whose second rational component \( q_{\varphi,k} \) coincides with \( q_{\alpha,k} \):

\[
P_\alpha = \{(m_\varphi, q_{\varphi,k}) \mid (m_\varphi, q_{\varphi,k}) \in P \wedge q_{\varphi,k} = q_{\alpha,k}\} \quad (11)
\]

For simplicity, the repeated rational numbers in \( P_\alpha \) will be called \( P_\alpha \)-repetitions.

259 By definition, the irrational numbers of all pairs of \( P_\alpha \) are irrationals numbers within \((0,1)\) with the same first \( k \) decimals. Obviously, some of these numbers will also have the first \( 2k \) decimals and some will not.\(^3\) Of the first ones, some will have the first \( 3k \) decimals and some will not. And so on.

260 In accord with 259, if we replace each repeated rational in \( P_\alpha \) with its \( k \)-expansion the number of repeated rationals would decrease. And if we replace the remaining repeated rationals with their corresponding \( k \)-expansion, the number of repeated numbers would decrease again. And so on. The problem is that after each replacement we would have a new set \( P'_\alpha, P''_\alpha, \ldots \) and we could not demonstrate if the repeated rationals disappear or not (see Chapter 19). To avoid this problem we will have to redefine the set \( P_\alpha \) after each replacement.

\(^3\)Change, for instance, any decimal \( d_{(k+1)i} \) in any irrational in \((0,1)\) and you will get an irrational with the same first \( k \) decimals but not with the same \( 2k \) decimals.
Each pair \((m_\varphi, q_\varphi,k)\) of \(P_\alpha\) defines a sequence \(\langle q_\varphi,nk \rangle\) of rational numbers similar to the sequence \(\langle q_\alpha,nk \rangle\) defined in 253, except that the source is now the irrational number \(m_\varphi\). The assumed actual existence, all at once, of the infinitely many decimals of the \(\omega\)-ordered decimal expansion of any irrational number in \((0, 1)\) as a complete totality, legitimates the definitions of the sets \(P, P_\alpha\), as well as the sequences \(\langle q_\varphi,nk \rangle\), all of them as complete totalities.

Let \(A\) be any set of pairs of numbers \((a, b)\) whose first component \(a\) is an irrational number within the real interval \((0, 1)\) and whose second component \(b\) is a rational number within the same real interval \((0, 1)\). Let us define the following two set operators:

1. \(D(A) = \) set of all pairs of \(A\) whose rational components are different, not repeated.
2. \(R(A) = \) set of all pairs of \(A\) whose rational components are repeated.

Evidently \(A = D(A) \cup R(A); D(A) \cap R(A) = \emptyset\).

Consider now the following sequence of redefinitions of the set \(P_\alpha\):

\[
\begin{align*}
    n = 1, 2, 3, \ldots & \quad \text{If } R(P_\alpha) \neq \emptyset \text{ then:} \\
    \quad P^d_\alpha = D(P_\alpha) & \\
    \quad X = \{ (m_\varphi, q_\varphi,(n+1)k) \mid (m_\varphi, q_\varphi,nk) \in R(P_\alpha) \} & \\
    \quad P_\alpha = P^d_\alpha \cup X
\end{align*}
\]

In each redefinition (12) of the set \(P_\alpha\) its repeated rationals are replaced with their corresponding \(k\)-expansions. For this reason definitions (12) will be called \(k\)-replacements. In agreement with 259, in each \(k\)-replacement the number of repeated rationals in \(P_\alpha\) decreases.

We will now try to prove that, by successive \(k\)-replacements, it is possible to replace each repeated rational in \(P_\alpha\) with a different rational within the interval \((0, 1)\).

Let us assume that while \(R(P_\alpha) \neq \emptyset\) and \(P_\alpha\) can be \(k\)-replaced, it is \(k\)-replaced. Once all possible \(k\)-replacements have been carried out, there will be two mutually exclusive alternatives regarding \(R(P_\alpha)\) (the subset of \(P_\alpha\) of all pairs with repeated rationals):

1. \(R(P_\alpha)\) is not empty.
2. \(R(P_\alpha)\) is empty.

Consider the first alternative: \(R(P_\alpha)\) is not empty. We know that for each element \((m_\lambda, q_\lambda,vk)\) in \(R(P_\alpha)\) there is an \(\omega\)-ordered sequence \(\langle q_\lambda,nk \rangle\) of rationals with a finite decimal expansion. So that each \((m_\lambda, q_\lambda,vk)\) in \(R(P_\alpha)\) can be replaced with \((m_\lambda, q_\lambda,(v+1)k)\). Consequently a new \(k\)-replacement of \(P_\alpha\) is possible, which contradict the fact that, being \(R(P_\alpha) \neq \emptyset\), all possible \(k\)-replacements of \(P_\alpha\) have been carried out. Therefore, and by Modus Tollens, The
first alternative is false and then, once performed all possible k-replacements of \( P_\alpha \) the set \( R(P_\alpha) \) is empty.

**Remark 265-1.** Note that argument 265 has nothing to do with constructive reasonings based on the successively performed k-replacements. It is a single Modus Tollens: once performed all possible k-replacements, the hypothesis that \( R(P_\alpha) \) is not empty leads to the contradictory conclusion that not all possible k-replacements have been carried out. That hypothesis must be, therefore, false.

![Figure 14.1: The consequences of being completed without a last completing element.](image)

**Remark 265-2.** Argument 265 takes advantage of the fact that, in accord with the hypothesis of the actual infinity, \( \omega \)-ordered sequences do exist as complete totalities in which each element has infinitely many successors (Figure 14.1). This assumption, makes it possible to ensure that while \( P_\alpha \) contains \( P_\alpha \)-repetitions, i.e. while \( R(P_\alpha) \) is not empty, the repeated numbers can be replaced with their corresponding successive k-expansions by means of successive k-replacements of \( P_\alpha \). And that this sequence of k-replacements can actually be completed because of the actual completeness of each infinite sequence \( \langle q_{\varphi,nk} \rangle \). Consequently, only when \( P_\alpha \) no longer contains \( P_\alpha \)-repetitions, i.e. when \( R(P_\alpha) \) is empty, it will be possible to ensure that all possible k-replacements have been carried out (under penalty of contradiction).

**Remark 265-3.** By contrast, from the potential infinity perspective the existence of completed infinite totalities without a last element that completes them, makes no sense. Thus, from this perspective we are not legitimated to consider the completion of the sequence of k-replacements if this sequence is potentially infinite.

**266** Once removed all \( P_\alpha \)-repetitions, the resulting rational numbers can only have a finite decimal expansion since all elements of all sequences \( \langle q_{\varphi,nk} \rangle \) are rational numbers with a finite expansion.

**267** In accordance with the definition 258 of \( P_\alpha \), the rational numbers result-
ing from the removal of all $P_{\alpha}$-repetitions cannot be repeated in the set $P - P_{\alpha}$ because all rational numbers in this last set differ from the rationals of $P_{\alpha}$ in at least one of their first $k$ decimals.

268 The above argument $258/267$ can be applied to any other repeated rational in the set $P$ of pairs $(m_{\alpha}, q_{\alpha,nk})$. In consequence, all repeated rationals can be replaced with a different rational number derived from the decimal expansion of the first irrational component of the pair. In these conditions each pair of $P$ will be formed by a different irrational number $m_{\alpha}$ and a different rational number $q_{\alpha}$. The one to one correspondence $f$ defined by:

$$f(m_{\alpha}) = q_{\alpha}$$

would be proving the set of rationals and the set of irrationals in $(0, 1)$ have the same cardinality.

**Discussion**

269 The hypothesis of the actual infinity subsumed into the Axiom of Infinity legitimizes the following line of reasoning on which argument $256/268$ is grounded:

269-1 The infinitely many decimals of the decimal expansion of any irrational number within $(0, 1)$ do exist as an actual complete totality.

269-2 The infinite decimal expansions of the irrational numbers in $(0, 1)$ are $\omega$-ordered, being $\omega$ the least infinite ordinal.

269-3 Two different irrational numbers in $(0, 1)$ can only coincide in a finite number of their first successive decimals.

269-4 The infinitely many k-expansions $\langle q_{\varphi,nk} \rangle$ defined from the decimal expansion of each irrational $m_{\varphi}$ in the real interval $(0, 1)$ do exist as an actual complete totality.

269-5 Each of the infinitely many k-expansions $\langle q_{\varphi,nk} \rangle$ is a rational number with finitely many decimals: $nk$ for each $n$ in $\mathbb{N}$.

269-6 In accordance with 269-4 and 269-5, the repeated rationals of $P_{\alpha}$ can be successively replaced with their corresponding successive rational k-expansions any finite or infinite number of times.

269-7 In these conditions, and by Modus Tollens $265$, all $P_{\alpha}$-repetitions can be removed from $P_{\alpha}$, and then from $P$, so that each pair will be formed by a different irrational and a different rational derived from its irrational partner.

269-8 Consequently each irrational number within $(0, 1)$ defines a different rational number within the same interval.

270 Conclusion $269-8$ contradicts other well known results on the cardinality of the set of rational numbers.
To define rational numbers, and $\omega$-ordered sequences of rational numbers, from the decimal expansion of the irrational numbers leads to some other contradictory results we have not dealt with here.

**EPILOG**

As it has been repeatedly said, from the perspective of the actual infinity hypothesis, the infinitely many decimals of a real number with an infinite decimal expansion do exist as a complete $\omega$-ordered totality. In consequence, to consider that a real number *does exist* as the complete totality of its infinitely many decimals, means to consider that number is a mind-independent entity, because human mind cannot embrace the actual infinity (we can not even imagine numbers as $9^{19}$, which are minuscule compared to the actual infinitude of for instance $\aleph_0$). Thus, from the infinitist perspective, all real numbers would be (platonic) mind-independent entities.

From the hypothesis of the potential infinity, however, an irrational number is not a mind-independent entity formed by a complete $\omega$-ordered sequence of decimals that exist all at once and by themselves. From this hypothesis, irrational numbers result from endless processes of calculus that cannot be replaced with a division between two integers, although at each stage of the calculus the number coincides with a rational number of finitely many decimals. In this sense irrational numbers are also definable as (potentially infinite) sequences of rational numbers.

In the case of the rational numbers the processes of calculus can be replaced with a division between two integers, which is not necessarily endless. In its turn, integer numbers would result from the endless process of counting. Naturally, the existence of endless processes of calculus and of counting does not necessarily mean the existence of their corresponding finished results as complete totalities, as is assumed from the infinitist point of view.

We must decide which of the two alternatives is the most appropriate to found a theory of numbers. And the election is not irrelevant: we need mathematics to explain the world. Think, for example, of the problems posed by the actual infinity in certain areas of physics, as quantum electrodynamics (*renormalization*) or quantum gravity [182]. Or the assumed dense ordering of the *continuum* spacetime$^4$ versus the discontinuous nature of ordinary matter, electric charge or energy.

$^4$Founded on the assumed uncountable cardinality $2^{\aleph_0}$ of the real numbers.
15.-Cardinal subtraction

INTRODUCTION

Contrary to what happens with ordinals, the subtraction of cardinals in transfinite arithmetics is not always defined, not even permitted. Notwithstanding, some indirect definitions and results of cardinal subtraction have been given [173, pp. 161-173]. For instance, in ZFC\(^1\) the following results, among others, can be proved:

- If \(a\) and \(b\) are two cardinals, we will say that \(a - b\) exists if there is one, and only one, cardinal \(c\) so that \(a = b + c\). We then write: \(c = a - b\) (Tarski-Bernstein Theorem).

- If \(a\) is an infinite cardinal and \(b\) a (finite or infinite) cardinal then there exist a third cardinal \(c\) such that:

\[
b + c = a \iff b \leq a
\]

If \(b = a\) then \(c\) can take infinitely many values (\(\aleph_0 + n = \aleph_0\) and the like). If not, we will have \(c = a\).

- If \(a\) is an infinite cardinal and \(\aleph_0 \leq a\) then \(2^a - a = 2^a\) (Tarski-Sierpinski Theorem).

- If \(a\) and \(b\) are two cardinals and \(a - b\) does exists, then for any other cardinal \(c\) the difference \((c + a) - b\) also exists and is equal to \(c + (a - b)\)

But, in general, specially if the involved cardinals are alephs, we cannot write things as:

\[
a - c = b
\]

\[
a - a = 0
\]

We have just seen some examples in which the subtraction of transfinite cardinal is allowed, in the last section of this chapter we will see an example in which it is not. Thus, the status of the subtraction of cardinals in transfinite arithmetics is really curious. Although it seems reasonable to declare as undefined the subtraction of two cardinals when we can say nothing on the

\(^1\)In some cases without the aid of the Axiom of Choice.
result of the subtraction, what about the subtraction of cardinals when they lead to contradictory results? To be defined or undefined could be reasonable, but to be defined, undefined or inconsistent as appropriate, seem rather uncomfortable from a formal point of view. How on Earth can be consistent an arithmetic operation that in ‘some cases’ leads to contradictions without having previously determined which are those cases and why they do it?

278 In this chapter we will analyze, at the foundational level of set theory, the reasons for which most of the transfinite cardinal subtractions have to be ignored or prohibited. Obviously, at this foundational level of discussion the only available operation is to pair the elements of two sets. To make use of transfinite arithmetic would be a circular reasoning because transfinite arithmetic just derives from the foundational definitions and assumptions we will concerned with. As we will see, those reasons are immediate consequences of the foundational definition of the infinite sets, which as we know is based on the violation of Axiom of the Whole and the Part. In effect, the subtraction of finite cardinals (all of which observe the old Euclidian axiom) pose no problem, the problem of cardinal subtraction only appears when at least one of the involved cardinals is transfinite.

PROBLEMS WITH CARDINAL SUBTRACTION

279 If \( A \) and \( B \) are any two finite sets such that \( |B| \leq |A| \) and \( f \) is an injective function from \( B \) to \( A \) we will have:

\[
A = (A - f(B)) \cup f(B) \tag{4}
\]

\[
(A - f(B)) \cap f(B) = \emptyset \tag{5}
\]

\[
|A| = |A - f(B)| + |f(B)| \tag{6}
\]

\[
= |A - f(B)| + |B| \tag{7}
\]

So, it could be expected that the subtraction of the cardinals \( |A| \) and \( |B| \) were something similar to:

\[
|A| - |B| = |A - f(B)| \tag{8}
\]

because, being \( B \) and \( f(B) \) equipotent, \( A - f(B) \) is the set that results by taking away (subtracting) from \( A \) as many elements as \( |B| \). It could be proved that Definition (8) always works with finite cardinals.

280 As we will now see, in the case of the actually infinite sets, and due to the violation of the Axiom of the Whole and the Parts, Definition (8) of cardinal subtraction does not work. In fact, let \( A = \{a_1, a_2, a_3, \ldots\} \) and \( B = \{b_1, b_2, b_3, \ldots\} \) be any two denumerable and \( \omega \)-ordered sets. Consider the following injective functions from \( B \) to \( A \):

\[
\forall i \in \mathbb{N} \quad \begin{cases} 
  f_1(b_i) = a_i \\
  f_2(b_i) = a_{i+n}, \ \forall n \in \mathbb{N} \\
  f_3(b_i) = a_{2i}
\end{cases} \tag{9}
\]
where $n$ is any natural number. We would have:

\[
|A| - |B| = |A - f_1(B)| = |\emptyset| = 0
\]  
(10)

\[
|A| - |B| = |A - f_2(B)| = |\{a_1, a_2, \ldots a_n\}| = n, \forall n \in \mathbb{N}
\]  
(11)

\[
|A| - |B| = |A - f_3(B)| = |\{a_1, a_3, a_5, \ldots \}| = \aleph_0
\]  
(12)

Thus, the subtraction of the same two infinite cardinals $|A|$ and $|B|$ yields infinitely many different results, depending on the particular way of pairing the elements of both sets: the elements of $B$ can be paired either with the elements of $A$ ($f_1$, for instance) or with the elements of a proper part of $A$ ($f_2$ or $f_3$), as if the part and the whole were the same thing.

281 We could even prove a set theoretical version of Riemann’s Series Theorem: If $A$ and $B$ are any two $\omega$–ordered sets then the subtraction of their respective cardinals $|A|$ and $|B|$ can be made equal to any given natural number. Indeed, let $A = \{a_1, a_2, a_3, \ldots\}$ and $B = \{b_1, b_2, b_3, \ldots\}$ be any two $\omega$–ordered sets and $n$ any natural number. Now consider the injection $f$ from $B$ to $A$:

\[
f(b_i) = a_{n+i}, \forall i \in \mathbb{N}
\]  
(13)

We will have:

\[
f(B) = \{a_{n+1}, a_{n+2}, a_{n+3}, \ldots\}
\]  
(14)

\[
A - f(B) = \{a_1, a_2, \ldots a_n\}
\]  
(15)

\[
|A| - |B| = |A - f(B)| = |\{a_1, a_2, \ldots a_n\}| = n
\]  
(16)

282 As in the case of Riemann’s Series Theorem we will reinterpret in Chapter 18, the above conclusion can also be reinterpreted as a contradiction derived from the very fundamentals of set theory. In effect, let us denote by:

- $D$: Dedekind’s definition of infinite set.
- $A$: Axiom of Infinity.
- $H_o$: Two sets have the same number of elements if they can be put into a one to one correspondence.

In accord with 281 we can write:

\[
D \land A \land H_o \Rightarrow (|A| - |B| = n) \land (|A| - |B| \neq n)
\]  
(17)

which seems rather contradictory.

283 The possibility to get the same result when operating on different operands (as is the case of the transfinite cardinal addition, multiplication or exponentiation) may be admissible. But the possibility to get infinitely many different results when operating exactly on the same operands (as the above case of cardinal subtraction) seems rather uncomfortable. However, the second possibility is a consequence of the first one. In fact, if we accept that:
\[ b + c = a \]
\[ b + d = a \]
\[ b + e = a \]
\[ \ldots \]

then we should also accept that:
\[ b - a = c \]
\[ b - a = d \]
\[ b - a = e \]
\[ \ldots \]

The preferred solution to this problem has been, notwithstanding, the (more or less explicit) ignorance of cardinal subtraction.

**Faticoni’s argument**

284 In [72], pages 150-51, we can read the following argument on the impossibility of subtracting infinite cardinals (by the way, a typical argument on this issue):

1. \( H1 \): Assume we can define the subtraction \( \aleph_0 - \aleph_0 \) (as the opposite of the addition) so that:
\[ \aleph_0 - \aleph_0 = 0 \] (26)

2. We would have:
\[ 1 + \aleph_0 = \aleph_0 \] (27)
\[ 1 + (\aleph_0 - \aleph_0) = \aleph_0 - \aleph_0 \] (28)
\[ 1 + 0 = 0 \] (29)
\[ 1 = 0 \] (30)

3. In consequence \( H1 \) is impossible.

285 As it could not be otherwise, Faticoni’s argument is grounded on the same basic definitions and assumptions of modern axiomatic set theories. It could be, therefore, completed as:

- **D**: A set is actually infinite if there exists a one to one correspondence between the set and one of its proper subsets.
- **A**: There exist an actual infinite\(^2\) set (Axiom of Infinity).

\(^2\)Notice that \( D \) ans \( A \) state the existence of a set that violates the Euclidean Axiom of the
• **H0**: Two sets have the same number of elements if they can be put into a one to one correspondence.

• **H1**: Assume we can define define the subtraction \( \aleph_0 - \aleph_0 \) (as the opposite of the addition) so that:

\[
\aleph_0 - \aleph_0 = 0 \tag{31}
\]

• We would have:

\[
1 + \aleph_0 = \aleph_0 \tag{32}
\]

\[
1 + (\aleph_0 - \aleph_0) = \aleph_0 - \aleph_0 \tag{33}
\]

\[
1 + 0 = 0 \tag{34}
\]

\[
1 = 0 \tag{35}
\]

• In consequence \( H1 \) is impossible.

It is now evident that absurdity (35) could also be caused by the inconsistency of \( D \) and \( A \), i.e. we could write:

\[
D \land A \land H0 \land H1 \Rightarrow (1 = 0) \tag{36}
\]

286 Perhaps cardinal subtraction is an impossible operation. Let us then consider the possibility of taking away balls from a box that contains balls, that seems to be a little more at hand. Let \( BX \) be a box containing a denumerable collection of red balls. Now add to \( BX \) a denumerable collection of blue balls. At this moment \( BX \) will contain \( \aleph_0 \) red ball plus \( \aleph_0 \) blue balls, i.e. \( \aleph_0 \) balls \((\aleph_0 + \aleph_0 = \aleph_0)\). Now take away from \( BX \) all red balls, i.e. remove \( \aleph_0 \) balls from a box that contains \( \aleph_0 \) balls. The result will be a box that contains \( \aleph_0 \) balls.
(all blue balls). Finally remove all blue balls, i.e. remove $\aleph_0$ balls from a box that contains $\aleph_0$ balls. The result now is a box that contains no balls. Thus, by removing $\aleph_0$ balls from a box that contains $\aleph_0$ balls, we can get either a box that contains $\aleph_0$ balls or a box that contains no balls, a conclusion that is in agreement with 283.
16.-Aleph null

INTRODUCTION

287 To name an object we only need to choose (or invent) an arbitrary term (word(s) or symbol(s)) to denote the object. But to name an object is not the same as to define the object in terms of other previously defined objects. In this last case, we would also have to define those previously defined objects in terms of other previously defined objects and these lasts objects in terms of other previously defined objects, and so on and on. We would finally fall into a potentially infinite regression of definitions.

288 For this reason we are forced to accept primitive concepts we use without having been previously defined. Most basic concepts in both formal and experimental science belong to this category: number, set, space, point, time, mass, etc. In some cases, as for mass or number, an operational definition is possible. In others (set, point, instant, etc) not even that.

289 For the same reason as in the case of primitive concepts, we also need axioms (formal sciences) and fundamental laws (experimental sciences). Although in this case to avoid an infinite regression of arguments. While axioms may be arbitrary, most of the fundamental laws of experimental sciences are inductive conclusions derived from experimental observations and measurements.

290 Euclid’s Elements is perhaps the first axiomatic system in the history of Mathematics. Notwithstanding, the history of mathematics until the beginning of the XX century is full of works no so formalized as it could be expected. This is the case of Cantor’s foundational works on transfinite numbers, his famous Beiträge [37], [38] (English translation [40]).

291 Cantor made no assumption about the existence of infinite sets, he simply took it for granted the existence of ‘transfinite aggregates’. In particular, the existence of the ‘aggregate of all finite cardinals’, whose cardinal is Aleph-null. The next section discusses some inconveniences of Cantor definition of the first transfinite cardinal.

THE LESS TRANSCENDENTAL CARDINAL

292 Beiträge’s Section 6 begins as follows [40, pp. 103-104]:

Aggregates with finite cardinal numbers are called ‘finite aggregates,’ all
other we will call ‘transfinite aggregates’ and their cardinal numbers ‘transfinite cardinal numbers.’ The first example of a transfinite aggregate is given by the totality of finite cardinal numbers \(\nu\); we call its cardinal number ‘Aleph zero’ and denote it by \(\aleph_0\); thus we define:

\[
\aleph_0 = \{\bar{\nu}\}
\]  

(1)

It is then clear that Cantor defined \(\aleph_0\) as the cardinal of the set of all finite cardinal, in modern notation:

\[
\aleph_0 = |\{1, 2, 3, \ldots\}| = |\mathbb{N}|
\]  

(2)

Next Cantor proves \(\aleph_0\) is not a finite cardinal. For this he proves that \(\aleph_0 = \aleph_0 + 1\), while for every finite cardinal \(n\) it holds \(n \neq n + 1\). So, \(\aleph_0\) cannot be a finite cardinal. As could not be otherwise, the proof that \(\aleph_0 = \aleph_0 + 1\) is based on a one to one correspondence. In effect, consider the sets:

\[
\mathbb{N} = \{1, 2, 3, \ldots\} \text{ (Cardinal } \aleph_0) \tag{3}
\]

\[
A = \mathbb{N} \cup \{a\} \text{ (Cardinal } \aleph_0 + 1) \tag{4}
\]

The one to one correspondence \(f\) between \(\mathbb{N}\) and \(A\) defined by:

\[
f(1) = a \tag{5}
\]

\[
f(i + 1) = i, \forall i \in \mathbb{N} \tag{6}
\]

proves that both sets are equipotent, and then that \(\aleph_0 = \aleph_0 + 1\).

By the way, \(n \neq n + 1\) because all finite sets satisfy the Euclidian Axiom of the Whole and the Part. And \(\aleph_0 = \aleph_0 + 1\) because transfinite sets violate, by definition, that Euclidian axiom.

Cantor also proved that:

1. \(\aleph_0\) is greater than all finite cardinals.
2. \(\aleph_0\) is the least transfinite cardinal number.

Thus, these properties of \(\aleph_0\) are formal consequences of having been defined as the cardinal of the set of all finite cardinals. They are not part of the definition of \(\aleph_0\).

We will now examine in which way, if any, the definition of \(\aleph_0\) is related to the operational definition of finite cardinals. Finite cardinals may be operationally defined in different ways, for instance by recursive definitions (see Appendix B), as the following one:

\[
|\{\emptyset\}| = 1 \tag{7}
\]

\[
|\{\emptyset, \{\emptyset\}\}| = 2 \tag{8}
\]

\[
|\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}| = 3 \tag{9}
\]
The less transfinite cardinal —— 91

\[ |\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset, \{\emptyset\}\}\}| = 4 \quad (10) \]

\[
\ldots
\]

or even:

\[ |\{\emptyset\}| = 1 \quad (11) \]
\[ |\{0, 1\}| = 2 \quad (12) \]
\[ |\{0, 1, 2\}| = 3 \quad (13) \]
\[ |\{0, 1, 2, 4\}| = 4 \quad (14) \]

\[
\ldots
\]

298 The sequence of the above recursive definitions, and many similar others, is considered as a complete sequence that originates the complete totality of natural numbers in agreement with the hypothesis of the actual infinity. Notwithstanding, and in spite of the fact that it consists of infinitely many steps and each step defines a number greater than its immediate predecessor, it does not yield an infinite number. It yields an infinite sequence of finite numbers, each one unit greater than its immediate predecessor, but always finite. \( \aleph_0 \) is, therefore, unrelated to this operational sequence. \( \aleph_0 \) is not recursively defined from finite cardinals in operational terms.

299 We lack of a formal definition of number. But we know what we mean when we say the set \( A = \{1, 2, 3, 4, 5\} \) has five elements: we can count them; we can consider them successively; we dispose of operational instruments to identify them. But none of those operational instrument is applicable in the case of \( \aleph_0 \).

300 On the other hand, Cantor’s definition of \( \aleph_0 \) could be equivalent to a circular definition. In effect:

\[ \aleph_0 = |\{1, 2, 3, \ldots\}| \quad (15) \]
\[ = |\{1\} \cup \{2\} \cup \{3\} \cup \ldots| \quad (16) \]
\[ = |\{1\}| + |\{12\}| + |\{3\}| + \ldots \quad (17) \]
\[ = 1 + 1 + 1 + \ldots \quad (18) \]

and the last sum is defined only if we know the number of summands, and that number is just the number being defined by the sum.

301 Let us consider again Cantor original definition of \( \aleph_0 \):

\[ \aleph_0 = |\{1, 2, 3, \ldots\}| \quad \{1, 2, 3, \ldots\} \text{ defining set} \quad (19) \]

and the following conditioned supertask: at each of the successive instants \( t_i \) of the \( \omega \)-ordered sequence of instants \( \langle t_n \rangle \) within the finite real interval \((t_a, t_b)\) and whose limit is \( t_b \), take away the first element of the defining set (19) of \( \aleph_0 \).
if, and only if, the cardinal of the resulting defining set is still $\aleph_0$:

- $t_1$: defining set $\{2, 3, 4, \ldots\} : \aleph_0 = |\{2, 3, 4, \ldots\}|$
- $t_2$: defining set $\{3, 4, 5, \ldots\} : \aleph_0 = |\{3, 4, 5, \ldots\}|$
- $t_3$: defining set $\{4, 5, 6, \ldots\} : \aleph_0 = |\{4, 5, 6, \ldots\}|$

\[ \vdots \]

Let $v$ be any finite cardinal and assume that at $t_b$, once completed the super-task, we have:

- $t_b$: $\aleph_0 = |\{v, v+1, v+2, \ldots\}|$ \hspace{1cm} (20)

Since $\aleph_0 = |\{v+1, v+2, v+3, \ldots\}|$ \hspace{1cm} (21)

number $v$ had to be removed from the defining set at instant $t_v$. So Definition (20) is impossible at $t_b$. And being $v$ any finite cardinal we would have to conclude that, at $t_b$, Definition (20) is impossible for every finite cardinal $v$. So we will have:

- $t_b$: $\aleph_0 = |\emptyset| = 0$ \hspace{1cm} (22)
17.-Arithmetics singularities of $\aleph_0$

**INTRODUCTION**

302 The discussion that follows will be concerned with the elements of the $(\omega + 2)$-ordered set $\mathbb{N}^* = \{1, 2, 3, \ldots, \aleph_0, 2^{\aleph_0}\}$, as well as with the basic arithmetic operations and order relations between finite and infinite cardinals introduced by Cantor in his foundational work on transfinite numbers [40]. Definitions that, essentially, continue to be applicable in modern transfinite mathematics.

303 Once assumed the existence of the set $\mathbb{N}$ of all finite cardinals (natural numbers) as a complete totality, Cantor defined $\aleph_0$ as its cardinal. He then proved $\aleph_0$ is the least cardinal greater than all finite cardinals [40, Theorems 10-A and 10-B].

304 Transfinite arithmetics allows to define arithmetic operations of infinitely many operands. So, not only the operands but also the sequence of operations can be of any finite or infinite length. In the discussion that follows, and for reasons of clarity, we will index the successive operands of arithmetic operations to make it explicit, among other things, the ordering of the involved operands.

**IS $\aleph_0$ A PRIME NUMBER?**

305 Axiomatic set theory, that applies to all finite and infinite sets, allows us to dissociate the set $\mathbb{N}$ of natural numbers in the following way:

$$\{1, 2, 3, \ldots\} = \{1\} \cup \{2, 3, 4, \ldots\}$$  \hspace{1cm} (1)

$$\{1\} \cap \{2, 3, 4, \ldots\} = \emptyset$$  \hspace{1cm} (2)

Aleph-null ($\aleph_0$) is, by definition, the cardinal of $\mathbb{N}$. Taking into account the cardinal of the union of two disjoint sets is the sum of the cardinal of each set, we will have:

$$\aleph_0 = |\{1, 2, 3, \ldots\}|$$  \hspace{1cm} (3)

$$= |\{1\} \cup \{2, 3, 4, \ldots\}|$$  \hspace{1cm} (4)

1In modern terms: actual infinity hypothesis subsumed by the Axiom of Infinity.

2For instance ZFC-axiomatic.

3As is usual, the cardinal of a set $X$ will be denoted by $|X|$. 

93
where the natural number 1 is written as $1_1$ to indicate it stands for the cardinal of the set \{1\}; the same will apply to the successive $1_2$, $1_3$, $1_4$ etc.

**306** By the way, equation (6) $\aleph_0 = 1 + |\{2,3,4,\ldots\}|$ served Cantor to prove $\aleph_0$ is not a natural number (see Chapter 16 on Aleph-null).

**307** By successive dissociations (S-dissociations from now on) of $\mathbb{N}$ we will obtain:

$$\aleph_0 = |\{1,2,3,\ldots\}|$$
$$= |\{1\} \cup \{2,3,4,\ldots\}|$$
$$= |\{1\}| + |\{2,3,4,\ldots\}|$$
$$= 1_1 + |\{2,3,4,\ldots\}|$$
$$= 1_1 + |\{2\} \cup \{3,4,5,\ldots\}|$$
$$= 1_1 + |\{2\}| + |\{3,4,5,\ldots\}|$$
$$= 1_1 + 1_2 + |\{3,4,5,\ldots\}|$$
$$= 1_1 + 1_2 + |\{3\} \cup \{4,5,6,\ldots\}|$$
$$= 1_1 + 1_2 + |\{3\}| + |\{4,5,6,\ldots\}|$$
$$= 1_1 + 1_2 + 1_3 + |\{4,5,6,\ldots\}|$$
$$\ldots$$

It is worth noting that a S-dissociation simply dissociates a set into two disjoint sets (one of them a singleton), so that the cardinal of the original set is the sum of the cardinals of the two dissociated sets.

**308** The successive S-dissociations will be subjected to the following:

Restriction **308**.-A S-dissociation will be carried out if, and only if, the result is a well defined sum of cardinals each of whose summands has an immediate predecessor, except the first one $1_1$.

**309** Transfinite mathematics assumes that procedures of infinitely many steps as the above S-dissociation can in fact be carried out. On the other hand, it can easily be proved, by induction or by Modus Tollens (MT), that for each natural number $v$ it is possible to perform the first $v$ successive S-dissociations.

**310** The MT proof goes as follows: Assume it is false that for every natural number $v$ the first $v$ successive S-dissociations can be carried out. If that were the case, there would exist at least a natural number $n$ such that it is impossible to perform the first $n$ successive S-dissociations. That is to say, there would
exist at least a natural number \( n \) such that:

\[
\aleph_0 = 1_1 + 1_2 + \cdots + 1_{n-1} + |\{n, n+1, n+2, \ldots\}| \tag{17}
\]

cannot be S-dissociated. But this is false because:

\[
\aleph_0 = 1_1 + 1_2 + \cdots + 1_{n-1} + |\{n, n+1, n+2, \ldots\}| \tag{18}
\]

\[
= 1_1 + 1_2 + \cdots + 1_{n-1} + |\{n\} \cup \{n+1, n+2, n+3, \ldots\}| \tag{19}
\]

\[
= 1_1 + 1_2 + \cdots + 1_{n-1} + |\{n\}| + |\{n+1, n+2, n+3, \ldots\}| \tag{20}
\]

\[
= 1_1 + 1_2 + \cdots + 1_{n-1} + 1_n + |\{n+1, n+2, n+3, \ldots\}| \tag{21}
\]

Our initial assumption must therefore be false, and then we can assert that for every natural number \( v \) the first \( v \) successive S-dissociations can be carried out.

**311** The inductive proof is as follows:

1. It is quite clear the first S-dissociation can be carried out because:

\[
\aleph_0 = |\{1, 2, 3, \ldots\}| \tag{22}
\]

\[
= |\{1\} \cup \{2, 3, 4, \ldots\}| \tag{23}
\]

\[
= |\{1\}| + |\{2, 3, 4, \ldots\}| \tag{24}
\]

\[
= 1_1 + |\{2, 3, 4, \ldots\}| \tag{25}
\]

2. Assume that, being \( n \) any natural number, the first \( n \) successive S-dissociations can be carried out. We would have:

\[
\aleph_0 = 1_1 + 1_2 + \cdots + 1_n + |\{n + 1, n + 2, n + 3, \ldots\}| \tag{26}
\]

and then we can write:

\[
\aleph_0 = 1_1 + 1_2 + \cdots + 1_n + |\{n + 1\} \cup \{n + 2, n + 3, \ldots\}| \tag{27}
\]

\[
= 1_1 + 1_2 + \cdots + 1_n + |\{n + 1\}| + |\{n + 2, n + 3, \ldots\}| \tag{28}
\]

\[
= 1_1 + 1_2 + \cdots + 1_n + 1_{n+1} + |\{n + 2, n + 3, \ldots\}| \tag{29}
\]

which means the first \( n + 1 \) successive S-dissociations can also be carried out.

We have then proved the first S-dissociation can be carried out and if, for any \( n \) in \( \mathbb{N} \), the first \( n \) successive S-dissociations can be carried out, then the first \( n + 1 \) successive S-dissociations can also be carried out. This proves that for any \( v \) in \( \mathbb{N} \) the first \( v \) successive S-dissociations can be carried out.

**312** Assume now that while the successive S-dissociations can be carried out, they are carried out. Once performed all possible successive S-dissociations we
would have one of the following two alternatives:

\[ \aleph_0 = 1 + 1 + 2 + \cdots + 1_v + \left| \{v + 1, v + 2, v + 3, \ldots \} \right| \]  
\[ \aleph_0 = 1 + 1 + 1_3 + \ldots \]  

(30)  
(31)

where \( v \) is a certain natural number. Since \( v + 1 \) is also a natural number, the first alternative must be false according to 310 and 311. Consequently, once performed all possible successive S-dissociations we will have:

\[ \aleph_0 = 1 + 1_2 + 1_3 + \ldots \]  

(32)

Let us now prove (32) is an \( \omega \)-ordered sequence of sums. In fact, it cannot be finite because the sum of a finite number of finite cardinals is also finite, while \( \aleph_0 \) is the first infinite cardinal. So, the right side of (32) must have an infinite number of summands. And being infinite, it can only be \( \omega \)-ordered, otherwise it would be at least \( (\omega + 1) \)-ordered and then we would have:

\[ \aleph_0 = 1 + 1_2 + 1_3 + \cdots + 1_\omega + S \]  

(33)

where \( S \) is either 0 or a sum of a finite or infinite number of the same summand 1. In any case, the summand 1_\omega will always be present and it has not immediate predecessor, which violates Restriction 308. In consequence (32) can only be \( \omega \)-ordered.

According to (32), and taking into account the associativity of cardinals addition and the fact that, as Cantor himself proved [40], \( a^x \times a^y = a^{x+y} \) being \( a, x \) and \( y \) any three finite or infinite cardinals, we can write:

\[ 2^{\aleph_0} = 2^{1_1+1_2+1_3+\ldots} \]  
\[ = 2^{1_1+(1_2+1_3+\ldots)} \]  
\[ = 2^{1_1 \times 2^{1_2+1_3+1_4+\ldots}} \]  

(34)  
(35)  
(36)

where all 1_1, 1_2, 1_3 \ldots stand for the first cardinal 1. Here (and hereafter), the subindexes simply denote the order of the corresponding summands.

The successive power dissociations of \( 2^{\aleph_0} \) (P-dissociations hereafter) would be:

\[ 2^{\aleph_0} = 2^{1_1+1_2+1_3+\ldots} \]  
\[ = 2^{1_1+(1_2+1_3+\ldots)} \]  
\[ = 2^{1_1 \times 2^{1_2+1_3+1_4+\ldots}} \]  
\[ = 2^{1_1 \times 2^{1_2+(1_3+1_4+\ldots)}} \]  

(37)  
(38)  
(39)  
(40)

\( ^4 \omega + 1 \) is the less infinite ordinal greater than \( \omega \).

\( ^5 \)It is the limit of all finite ordinals.
Is \( \aleph_0 \) a prime number? —— 97

\[
= 2^{11} \times 2^{12} \times 2^{13+14+15+...} 
\]

\[
= 2^{11} \times 2^{12} \times 2^{13+(14+15+...)} 
\]

\[
= 2^{11} \times 2^{12} \times 2^{13} \times 2^{14+15+16+...} 
\]

\[
= 2^{11} \times 2^{12} \times 2^{13} \times 2^{14+(15+16+...)} 
\]

\[...
\]

Notice a P-dissociation is a simple application of a standard property of the product of powers.

316 The successive P-dissociations will be subjected to the following:

Restriction 316.-A P-dissociation will be carried out if, and only if, the result is a well defined product of powers each of whose factors has an immediate predecessor, except the first of them \( 2^{11} \).

317 Let us prove by MT (an inductive proof is also possible) that for every natural number \( v \) the first \( v \) successive P-dissociations can be carried out. Assume it is false that for every natural number \( v \) the first \( v \) successive P-dissociations can be carried out. In such a case there would exist at least a natural number \( n \) such that:

\[
2^{\aleph_0} = 2^{11} \times 2^{12} \times \ldots \times 2^{1n-1} \times 2^{1n+1n+1n+2+...} 
\]

cannot be P-dissociated. But this false because:

\[
2^{\aleph_0} = 2^{11} \times 2^{12} \times \ldots \times 2^{1n-1} \times 2^{1n+1n+1n+2+...} 
\]

\[
= 2^{11} \times 2^{12} \times \ldots \times 2^{1n-1} \times 2^{1n+(1n+1n+2+...)} 
\]

\[
= 2^{11} \times 2^{12} \times \ldots \times 2^{1n-1} \times 2^{1n} \times 2^{1n+1n+2+1n+3+...} 
\]

Therefore our initial assumption must be false and we can assert that for every natural number \( v \) the first \( v \) successive P-dissociations can be carried out.

318 Assume that while the successive P-dissociations can be carried out, they are carried out. Once performed all possible successive P-dissociations we will have one of the following two alternatives:

\[
2^{\aleph_0} = 2^{11} \times 2^{12} \times \ldots \times 2^{1v-1} \times 2^{1v+1v+1n+3+...} 
\]

\[
2^{\aleph_0} = 2^{11} \times 2^{12} \times 2^{13} \times \ldots 
\]

where \( v \) is a certain natural number. According to 317, and being \( v \) a natural number, the first alternative must be false. Consequently, once performed all possible successive P-dissociations we will have:

\[
2^{\aleph_0} = 2^{11} \times 2^{12} \times 2^{13} \times \ldots 
\]
that, obviously, can also be written as:

\[ 2^{\alpha_0} = 2_1 \times 2_2 \times 2_3 \times \ldots \]  

(52)

Let us now prove (52) is an \( \omega \)-ordered sequence of multiplications. In fact, it cannot be finite because the product of a finite number of finite factors is also a finite number, while \( 2^{\alpha_0} \) is uncountably infinite. The right side of (52) can only have an infinite number of factors. Furthermore, it must be an \( \omega \)-ordered sequence of factors otherwise it would be at least \((\omega + 1)\)-ordered and then we could write:

\[ 2^{\alpha_0} = 2_1 \times 2_2 \times 2_3 \times \ldots \times 2_\omega \times P \]  

(53)

where \( P \) is either 1 or a product of a finite or infinite number of the same factor 2. In any case, the factor \( 2_\omega \) will always be present and has not immediate predecessor,\(^6\) which violates Restriction 316. In consequence the right side of (52) can only be \( \omega \)-ordered.

Equation (52), on the other hand, is taken for granted and, as Cantor did, can be immediately derived from Cantor’s definition of cardinal exponentiation through the notion of covering [40, § 4].

Cantor’s order relations ‘greater’ and ‘less with powers’ [40] will be used now to prove that \( 2^{\alpha_0} \) is the less transfinite cardinal that can be expressed as a product of finite cardinals greater than 1. Obviously the number of factors cannot be finite since \( 2^{\alpha_0} \) is uncountably infinite. Thus it must be infinite. Let \( \alpha \) be any transfinite ordinal greater than \( \omega \), and let \( d = a_1 \times a_2 \times a_3 \times \ldots \) be the product of any \( \alpha \)-ordered sequence of finite cardinals \( a_1, a_2, a_3, \ldots, a_\omega, \ldots \) all of them greater than 1. By transfinite induction and taking into account that \( \omega < \alpha \) and \( 2_k \leq a_k \), for every \( k \), we will have:

\[ \forall k < \omega : 2_1 \times 2_2 \times \ldots \times 2_k \leq a_1 \times a_2 \times \ldots \times a_k \]  

and then:

\[ 2_1 \times 2_2 \times 2_3 \times \ldots \leq a_1 \times a_2 \times a_3 \times \ldots \]  

(55)

\[ \leq a_1 \times a_2 \times a_3 \times \ldots \times a_\omega \times \ldots \]  

(56)

Thus we can write:

\[ 2^{\alpha_0} = 2_1 \times 2_2 \times 2_3 \times \ldots < a_1 \times a_2 \times a_3 \times \ldots \times a_\omega \times \ldots \]  

(57)

This proves that \( 2^{\alpha_0} \) is the least transfinite cardinal that can be expressed as the product of an infinite sequence of finite factors greater than 1, i.e equal or greater than 2.

\(^6\) It is the limit of all \( 2_n \)
An immediate consequence of \( \aleph_0 \) is that it cannot be expressed as a product of finite cardinals greater than 1. In fact, if the number of factors is finite the product will also be finite; and if the number of factors is infinite it will be equal or greater than \( 2^{\aleph_0} \), which in turn is greater than \( \aleph_0 \). Thus, as in the case of prime numbers, \( \aleph_0 \) must always form part of its own factorizations.

**Aleph-null and the power of the continuum**

Let us write the first factor \( 2_1 \) in (52) as \( 1_1 + 2_1 \). We will have:

\[
2^{\aleph_0} = (1_1 + 2_1) \times 2_2 \times 2_3 \times 2_4 \times \ldots 
\]  
(58)

Taking into account the associativity of cardinal multiplications as well as the distributive property of cardinal multiplication over cardinal addition, we can successively duplicate the number of summands in the first factor of (58) by multiplying it by the successive second factors of (58):

\[
2^{\aleph_0} = (1_1 + 2_1) \times 2_2 \times 2_3 \times 2_4 \times \ldots 
\]  
(59)

\[
= (1_1 + 2_1) \times (2_2) \times (2_3 \times 2_4 \times \ldots) 
\]  
(60)

\[
= (1_1 + 2_1 + 1_3 + 1_4) \times 2_3 \times 2_4 \times 2_5 \times \ldots 
\]  
(61)

\[
= (1_1 + 2_1 + 1_3 + 1_4) \times (2_3) \times (2_4 \times 2_5 \times \ldots) 
\]  
(62)

\[
= (1_1 + 2_1 + \ldots + 1_8) \times 2_4 \times 2_5 \times 2_6 \times \ldots 
\]  
(63)

\[
= (1_1 + 2_1 + \ldots + 1_8) \times (2_4) \times (2_5 \times 2_6 \times \ldots) 
\]  
(64)

\[
= (1_1 + 2_1 + \ldots + 1_{16}) \times 2_5 \times 2_6 \times 2_7 \times \ldots 
\]  
(65)

\[
= (1_1 + 2_1 + \ldots + 1_{16}) \times (2_5) \times (2_6 \times 2_7 \times \ldots) 
\]  
(66)

\[
= (1_1 + 2_1 + \ldots + 1_{32}) \times 2_6 \times 2_7 \times 2_8 \times \ldots 
\]  
(67)

These successive duplications of the first factor of (58) will be referred to as F-duplications. They will be subjected to following:

**Restriction 324.** An F-duplication will be carried out if, and only if, each summand \( 1_n \) of the resulting sum has an immediate predecessor \( 1_{n-1} \), except the first one \( 1_1 \).

Let us prove, by MT (an inductive proof is also possible), that for every natural number \( v \) the first \( v \) successive F-duplications can be carried out. For this, assume it is false that for every natural number \( v \) the first \( v \) successive F-duplications can be carried out. There would exist at least a natural number \( n \) such that it is impossible to perform the first \( n \) successive F-duplications.
That is to say, there would exist at least a natural number $n$ such that:

$$2^\aleph_0 = (1_1 + 1_2 + \cdots + 1_{2^{n-1}}) \times (2_n \times 2_{n+1} \times 2_{n+2} \times \cdots)$$  \hspace{1cm} (68)

cannot be F-duplicated. It is immediate to prove this is false because:

$$2^\aleph_0 = (1_1 + 1_2 + \cdots + 1_{2^{n-1}}) \times (2_n \times 2_{n+1} \times 2_{n+2} \times \cdots)$$  \hspace{1cm} (69)

$$= (1_1 + 1_2 + \cdots + 1_{2^n}) \times (2_{n+1} \times 2_{n+2} \times 2_{n+3} \times \cdots)$$  \hspace{1cm} (70)

Our initial assumption must therefore be false and then we can assert that for every natural number $v$ the first $v$ successive F-duplications can be carried out.

326 Assume now that while the successive F-duplications can be carried out, they are carried out. Once performed all possible successive F-duplications we would have one of the following two alternatives:

$$2^\aleph_0 = (1_1 + 1_2 + \cdots + 1_{2^{v-1}}) \times (2_v \times 2_{v+1} \times 2_{v+2} \times \cdots)$$  \hspace{1cm} (72)

$$2^\aleph_0 = 1_1 + 1_2 + 1_3 + \cdots$$  \hspace{1cm} (73)

where $v$ is a certain natural number. Being $v$ a natural number, the first alternative must be false according to 325. Consequently, once performed all possible successive F-duplications we will have:

$$2^\aleph_0 = 1_1 + 1_2 + 1_3 + \cdots$$  \hspace{1cm} (74)

327 Let us now prove (74) is an $\omega$-ordered sequence of sums. In fact, it cannot be finite because the sum of a finite number of finite summands is also a finite number, while $2^\aleph_0$ is uncountably infinite. Therefore, the right side of (74) will have an infinite number of summands. In addition, it can only be an $\omega$-ordered sequence of summands, otherwise it would be at least $(\omega + 1)$-ordered, and then we would have:

$$2^\aleph_0 = 1_1 + 1_2 + 1_3 + \cdots + 1_\omega + S$$  \hspace{1cm} (75)

where $S$ is either 0 or the sum of a finite or infinite number of the same summand 1. In any case, the summand $1_\omega$ would always be present and has not immediate predecessor, which violates Restriction 324. In consequence the right side of (74) can only be $\omega$-ordered.

328 Taking into account (74) and (32) we can write:

$$2^\aleph_0 = 2_1 \times 2_2 \times 2_3 \times \cdots = 1_1 + 1_2 + 1_3 + \cdots = \aleph_0$$  \hspace{1cm} (76)
which contradicts Cantor’s theorem:

\[ \aleph_0 < 2^{\aleph_0} \]  

(77)

**Remark 328-1.** It seems convenient to recall that argument 323-328 is exclusively based on well established definitions, operations and properties of transfinite arithmetics. It simply takes advantage of a consequence of the hypothesis of the actual infinity: the existence of \( \omega \)-ordered sequences as complete totalities, in spite of the fact that no last element completes them. The argument is, therefore, a formal consequence of assuming the *completion of uncompletable*. This infinitist assumption makes it possible to complete any definition or procedure composed of an \( \omega \)-ordered sequence of steps in which no last step completes the sequence.
Arithmetics singularities of $\mathbb{N}_o$
18.-Reinterpreting Riemann series theorem

DEFINITIONS

329 Riemann series theorem (see below) states that it is possible to reorder the summands of a conditionally convergent series in such a way that it converges to any desired number or to infinity. As we will see, the theorem only applies if infinitely many terms are involved in the rearrangement. In those conditions to converge and not to converge to a given number could be reinterpreted as a contradiction derived from the inconsistency of the actual infinity.

330 A series \( \sum_{i=0}^{\infty} a_i \) is conditionally convergent if, and only if:

1. The series converges to a finite number \( L \):

\[
\lim_{n \to \infty} \sum_{i=0}^{\infty} a_i = L \tag{1}
\]

2. The series of its positive (negative) terms diverges to positive (negative) infinite.

\[
\lim_{n \to \infty} \sum_{i=0}^{\infty} |a_i| = \infty \tag{2}
\]

331 Riemann series theorem states that by the appropriate rearrangement of its terms, any conditionally convergent series can be made to converge to any given finite number or to infinity.

DISCUSSION

332 We will exclusively deal with conditionally convergent series of real numbers that may converge to different finite numbers by rearrangements based on the application of the commutative, associative and distributive properties of the elementary arithmetic operations in the field of real numbers.

333 Let \( S = \sum_{i=1}^{\infty} a_i \) be any conditional convergent series and \( v \) any natural number. Consider the sum of the first \( v \) terms of \( S \). Since the number \( v \) of summands is finite, the number of its possible permutations is also finite. Let \( \langle P_i \rangle_{1 \leq i \leq n} \) be any finite sequence of \( n \) permutations of the first \( v \) summands of \( S \), being each permutation \( P_i \) of \( \langle P_i \rangle_{1 \leq i \leq n} \) the result of applying one of the
properties associative, commutative or distributive to its immediate predecessor $P_{i-1}$. Let $\langle S_{v,i} \rangle_{1 \leq i \leq n}$ be the sequence of the corresponding sums, i.e. each $S_{v,i}$ is the sum of the first $v$ summands of $S$ reordered as $P_i$.

334 If for a certain index $i$ we would have:

$$S_{v,i-1} \neq S_{v,i}$$

we would have to conclude that a single application of the commutative, associative or distributive properties changes the result of the sum, which is impossible because no simple application of the commutative, associative or distributive properties can alter the result of a sum if those properties hold as they must hold in the field of the real numbers. Inequality 3 is then impossible for any natural, and then finite, number $v$.

335 It holds the following:

**Theorem of the consistent reorderings** For any $v$ in $\mathbb{N}$ the sum of first $v$ terms of any conditionally convergent series is always the same, irrespective of the rearrangement of the involved summands.

We can therefore assert that only when the number of summands is infinite the sum depends on the summand rearrangement. We must conclude it is the infinite number of summands the cause of that dependence.

336 According to Riemann series theorem, if $S$ is any conditionally convergent series and $r$ any real number the sum of its infinitely many terms is and is not equal to $r$, depending on the order the terms of the series are summed. This is the type of result one can expect if the hypothesis of the actual infinity were inconsistent. Riemann series theorem could, therefore, be reinterpreted as a proof of the inconsistency of the actual infinity hypothesis. And that possibility, as legitimate as any other, should be explicitly declared in the theorem statement.
19.-Nested set inconsistency

A DENUMERABLE VERSION OF THE NESTED-SET THEOREM

Let \( A_1 = \{a_1, a_2, a_3, \ldots \} \) be any \( \omega \)-ordered set and consider the following recursive definition:

\[
A_{i+1} = A_i - \{a_i\}; \quad i = 1, 2, 3, \ldots
\]

that yields the \( \omega \)-ordered sequence of nested sets:

\[
S = \langle A_n \rangle = A_1 \supset A_2 \supset A_3 \supset \ldots
\]

being each set \( A_n = \{a_n, a_{n+1}, a_{n+2}, \ldots \} \) a denumerable proper subset of all its predecessors, as well as a superset of all its successors.

The following theorem is a denumerable version of the so called Nested Set Theorem.\(^1\)

**Empty intersection theorem.** - The sequence \( S \) of sets \( \langle A_n \rangle \) defined in 337 satisfies:

\[
\bigcap A_i = \emptyset
\]

The proof is immediate: if an element \( a_k \) would belong to the intersection then only a finite number (equal or less than \( k \)) of sets would have been defined by (1), since \( a_k \) does not belong to \( A_{k+1}, A_{k+2}, A_{k+3}, \ldots \).

Empty intersection theorem (EIT for short) is a trivial result in modern infinitist mathematics. As far as I know, it has never been involved in any discussion on the formal nature of infinity. The theorem simply states the sets \( \langle A_n \rangle \) have no common element. The implications of the fact that each \( A_i \) is a denumerable proper subset of all its predecessors have never been examined. In the next discussion we will have the opportunity to examine some of those implications.

\(^1\) The original version, also called Cantor’s intersection theorem, deals with compact sets, and the conclusion is exactly the contrary, i.e. that the intersection is nonempty.
Before beginning our discussion, let us examine an elementary physical version of EIT. Let $BX$ be a box containing a denumerable collection of balls labeled as $b_1, b_2, b_3, \ldots$, and let $\langle t_n \rangle$ be a strictly increasing $\omega$-ordered sequence of instants within the real interval $(t_a, t_b)$ whose limit is just $t_b$. Now consider the following supertask: at each instant $t_i$ remove the ball $b_i$, and only it, from the box. The one to one correspondence $f$ between $\langle t_n \rangle$ and $\langle b_n \rangle$ defined by $f(t_i) = b_i$ proves that at $t_b$ all balls will have been removed from $BX$.

In accordance with the way of removing the balls, one by one and in such a way that between the removal of a ball $b_n$ and the removal of the next one $b_{n+1}$ an interval of time $t_{n+1} - t_n$ greater than zero always elapses, it could be expected that just before completing the removal of all balls from the box, the box will contain \ldots 5, 4, 3, 2, 1, 0 balls. Nothing further from the (infinitist) truth: before being empty, the box will never contain a finite number $n$ of balls, whatever be $n$, simply because those balls would be the impossible last $n$ balls of an $\omega$-ordered collection of labeled balls; and the successive instants at which the successive balls were successively removed would be the impossible last $n$ instants of an $\omega$-ordered sequence of instants.

Let $f(t)$ be the number of balls within the box at any instant $t$ in $[t_a, t_b]$, i.e. the number of balls to be removed at the precise instant $t$. As a consequence of $\omega$-order, we will have the following inevitable dichotomy:

$$\forall t \in [t_a, t_b] : f(t) = \begin{cases} \aleph_0 & \text{if } t \in [t_a, t_b) \\ 0 & \text{if } t = t_b \end{cases}$$

(4)

Otherwise, if for a $t$ in $[t_a, t_b)$ we would have $f(t) = n$, being $n$ any natural number, then there would exist the impossible last $n$ terms of an $\omega$-ordered sequence.

Taking into account the one to one correspondence $f(t_i) = b_i$, all balls $\langle b_n \rangle$ are removed one by one from the box $BX$, one after the other and in such a way that an interval of time $\Delta t = t_{i+1} - t_i$ greater than zero always elapses between the removal of two successive balls $b_i, b_{i+1} \forall i \in \mathbb{N}$. But according to the above $\aleph_0$ or 0 dichotomy (4), this is impossible because the
number of balls to be removed from the box has to change directly\(^2\) from \(\aleph_0\) to 0, and this is only possible by removing simultaneously \(\aleph_0\) balls.

344 Evidently, the box \(BX\) plays the role of the set \(A_1\) and the successive removal of balls represent the successive steps of the recursive definition \(A_{i+1} = A_i - \{a_i\}\). Since the successive elements \(a_1, a_2, a_3, \ldots\) of \(A_1\) are successively removed in order to define the successive terms \(A_1, A_2, A_3, \ldots\) of the sequence \(S\), we could write:

\[
\begin{align*}
i = 1, 2, 3, \ldots & \quad \begin{cases} A_{i+1} = A_i - \{a_i\} \\ A_1 = \{a_1, a_2, \ldots, a_i, a_{i+1}, a_{i+2}, \ldots\} \end{cases}
\end{align*}
\]

where \(a_1, a_2, \ldots, a_i\) simply indicate the successive elements \(a_1, a_2, \ldots, a_i\) of \(A_1\) that have been used in order to define the successive members \(A_2, A_3, \ldots A_i\) of the sequence \(S\).

345 As in the case of the box \(BX\), and for the same reasons, if we focus our attention on the number of elements that remain unmarked in (5) as the recursive definition (1) progresses, then we will immediately come to the conclusion that that number can only take two values: \(\aleph_0\) and 0.

346 The \(\aleph_0\) or 0 dichotomy implies the number of unmarked elements in (5) changes directly from \(\aleph_0\) to 0, and this is only possible by marking \(\aleph_0\) elements at once, i.e. by defining simultaneously \(\aleph_0\) sets of the sequence \(S\), which evidently is not compatible with the recursiveness of that definition, in the same way that to remove simultaneously \(\aleph_0\) balls from the box is not compatible with the successiveness of the extractions.

347 There is, however, a significant difference between taking away the balls from \(BX\) and the recursive definition (1): while the box \(BX\) is always the same box \(BX\) as the balls are successively removed from it (which makes it evident the fallacy of the removal), the set \(A_1\) originates a sequence of sets: starting from \(A_1\), each set \(A_i\) originates a new set \(A_{i+1}\) when the element \(a_i\) is removed from it in order to define the next term of the sequence. Thus, \(A_1\) dissolves in a complete infinite sequence of sets without a last set completing the sequence, which hides the fallacy of removing one by one all elements of a collection without ever resting . . . three, two, one, elements to be removed.

348 Faced with the evidence of the fact that by removing one by one the finitely or infinitely many balls within a box you will inevitably get a box that will successively contain . . . , 5, 4, 3, 2, 1, 0 balls, some infinitists claim that while you can add one by one infinitely many balls to an initially empty box, you cannot remove one by one those balls from the box simply because the subtraction between transfinite cardinals is not always defined.\(^3\)

\(^2\)Without intermediate finite states at which only a finite number of balls remain to be removed.

\(^3\)The subtraction between cardinals is not always defined because it leads to contradictions.
It is quite clear, however, here we are not subtracting cardinals, we are not performing arithmetics operations but removing balls from a box. What to think about a formal theory that prohibits removing balls from a box because the removal puts the theory into question? It is hard to believe that the same theorists that allows you to remove any element of any set prohibits to remove a ball from a box.

**Nested-set inconsistency**

The above discussion on EIT suggests this theorem is not as trivial as it seems to be. It, in fact, motivates the short discussion that follows, whose main objective is to put into question the formal consistency of the the actual infinity hypothesis.

It seems convenient at this point to recall that Cantor took it for granted the existence of the set of all finite cardinals as a complete infinite totality (Axiom of Infinity in modern terms), and that from that initial assumption he successfully derived the infinite sequence of the transfinite ordinals of denumerable well-ordered sets built on $\omega$, the smallest of them [40, Theorem 15-K]. Thus any result affecting the formal consistency of $\omega$ will affect the whole sequence of transfinite ordinals of the second class as well as the formal consistency of the actual infinity hypothesis subsumed into the Axiom of Infinity.

Let us just begin by assuming the Axiom of Infinity and then the existence of $\omega$-ordered sets and $\omega$-ordered sequences as complete infinite totalities.

Consider again the above sequence of sets $S = A_1, A_2, A_3, \ldots$. From $S$ we will define the sequence $S^*$ of sets according to:

$$
n = 1, 2, 3, \ldots \quad \begin{cases} 
n = 1 : S^* = A_1 \\
n > 1 : \bigcap_{i=1}^{i=n} A_i \neq \emptyset \Rightarrow \text{Add } A_n \text{ to } S^* 
\end{cases}
$$

As in previous arguments in this book, it could easily be proved by induction or by Modus Tollens that for any natural number $v$ the first $v$ successive definitions (6) can be carried out. The inductive proof is as follow. It is quite clear the first definition (6) $S^* = A_1$ can be carried out. Assume that, being $n$ any natural number, the first $n$ successive definitions (6) can be carried out, leaving $S^*$ defined as the sequence $A_1, A_1, \ldots A_n$. Since $A_{n+1}$ is a well defined set the sequence:

$$A_1, A_2, \ldots A_n, A_{n+1}$$

is a well defined sequence of sets, and then the intersection:

$$A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}$$
is also a well defined set, which is all we need to carry out the \((n+1)\)th definition. Consequently the first \((n+1)\) successive definitions (6) can be carried out. We have then proved the first definition (6) can be carried out, and that if for any natural number \(n\) the first \(n\) successive definitions (6) can be carried out, then the first \((n+1)\) successive definitions (6) can also be carried out. This proves that for any natural number \(v\), the first \(v\) successive definitions (6) can be carried out.

355 Assume that while the successive definitions (6) can be carried out they are carried out. Once all possible successive definitions (6) have been carried out, the sequence \(S^*\) will be formed by a certain (finite or infinite) number of sets that by definition have a nonempty intersection. Let, therefore, \(a_{v-1}\) be any element of that intersection. Evidently we have:

\[
a_{v-1} \notin A_v
\]

And in consequence \(A_v\) is not a member of the sequence \(S^*\).

356 It is immediate to prove, however, \(A_v\) is a member of \(S^*\):

1. The subindex \(v\) in \(A_v\) is a natural number.
2. According to 354, for each natural number \(v\) the first \(v\) successive definitions (6) can be carried out.
3. All possible successive definitions (6) have been carried out.
4. The first \(v\) successive definitions have been carried out.
5. The \(v\)th definition (6) adds \(A_v\) to \(S^*\) because:

\[
A_1 \cap A_2 \cap \cdots \cap A_v = A_v \neq \emptyset
\]

6. In consequence \(A_v\) is a member of \(S^*\).

357 We have, therefore, derived a contradiction from our initial assumption: the set \(A_v\) is and is not in the sequence \(S^*\).

358 The alternative to the above contradiction is another contradiction even more elemental: after having performed all possible successive definitions (6), not all possible successive definitions (6) have been performed.

359 It could also be argued that \(S^*\) is defined infinitely many times and that although each and every definition (6) defines \(S^*\) as a sequence of sets whose intersection is not empty, the completion of the sequence of successive definitions (6) converts \(S^*\) into a sequence of sets whose intersection is empty; or leaves \(S^*\) undefined. As if the completion of an \(\omega\)−ordered sequence of definitions, as such a completion, have additional arbitrary consequences on the defined object. The same arbitrary consequences could be expected in any other definition of procedure consisting of an \(\omega\)−ordered sequence of steps. In those conditions any thing could be expected in infinitist mathematics.
Nested set inconsistency
20.-Zeno Dichotomies

INTRODUCTORY DEFINITIONS

360 This chapter introduces a formalized version of Zeno’s dichotomies I and II based on the successiveness of $\omega$—order and $\omega^*$—order respectively. Each of these formalized versions leads to a contradiction.

361 In the second half of the XX century, several solutions to some of Zeno’s paradoxes were proposed with the aid of Cantor’s transfinite arithmetic, topology, measure theory and more recently internal set theory\(^1\) (a branch of non-standard analysis). It is also worth noting the solutions proposed by P. Lynds\(^2\) within classical and quantum mechanics frameworks. Some of these solutions, however, have been contested. And in most cases, the proposed solutions do not explain where Zeno’s arguments fail. Moreover, some of the proposed solutions gave rise to a new collection of problems so exciting as Zeno’s paradoxes.\(^3\) In the discussion that follows I propose a new way of discussing Zeno’s Dichotomies based on the notion of $\omega$—order, the order induced by $\omega$, the first transfinite ordinal.

362 As is well known, an $\omega$—ordered sequence is one in which there is a first element and each element has an immediate successor and an immediate predecessor, except the first one. According to the hypothesis of the actual infinity subsumed into the Axiom of Infinity, an $\omega$—ordered sequence is a complete totality despite the fact that no last element completes it\(^4\). The sequence of natural numbers in their natural order of precedence is an example of $\omega$—ordered sequence.

363 An $\omega^*$—ordered sequence is one in which there exists a last element and each element has an immediate predecessor and an immediate successor, except the last one. From the same infinitist perspective, $\omega^*$—ordered sequences are complete totalities in spite of the fact that there is not a first element to begin with. The increasing sequence of negative integers, $\ldots, -3, -2, -1$, is an example of $\omega^*$—ordered sequence.

\(^1\)\([90], [91], [206], [92], [94], [93], [131], [130]\)
\(^2\)\([116], [117]\)
\(^3\)\([145], [4], [154], [165], [106] [174]\)
\(^4\)Cantor, in fact, proved the existence of $\omega$—ordered sequences by assuming the existence of the set of all finite cardinals as a complete totality [40, Teorem 15-A],

111
Let us consider a particle $P$ moving through the $X$ axis from point -1 to point 2 at a constant finite velocity $v$ (Figure 20.1). Assume $P$ is at point 0 just at the precise instant $t_0$. At instant $t_1 = t_0 + 1/v$ it will be exactly at point 1. Consider now the following $\omega-$ordered sequence of Z-points [194] within the real interval $(0, 1)$, defined by

$$z_n = \frac{2^n - 1}{2^n}, \; \forall n \in \mathbb{N}$$

and the $\omega^*$-ordered sequence of Z*-points within the same interval defined by:

$$z^*_{n*} = \frac{1}{2^n}, \; \forall n \in \mathbb{N}$$

where $z^*_{n*}$ stands for the last but $n - 1$ element of the $\omega^*$-ordered sequence of Z*-points; $\omega^*$-order implies that between any two successive Z*-points no other Z*-point exists. The same applies to the $\omega$-ordering of Z-points.

Although the points of the $X$ axis are densely ordered, Z*-points and Z-points are not. Because of their successiveness, they can only be traversed in a successive way, one at a time, one after the other. And in such a way that between any two successive Z*-points, or Z-points, a distance greater than zero must always be traversed. This type of successiveness will be of paramount importance in the argument that follows.

As $P$ passes over the points of the real interval $[0, 1]$ it must traverse the successive Z*-points and the successive Z-Points. It makes no sense to wonder about the instant at which the successive Z*-points begin to be traversed because there is not a first Z*-point to be traversed. The same could be said on the instant at which the traversal of the Z-points ends, in this case because there is not a last Z-point to be traversed. For this reason we will focus our attention on the number of Z*-points that have already been traversed and on the number of Z-points that still remain to be traversed at any instant $t$ within the interval $[t_0, t_1]$.

In this sense, and being $t$ any instant in $[t_0, t_1]$, let $Z^*(t)$ be the number of Z*-points $P$ has traversed just at instant $t$. And let $Z(t)$ be the number of Z-points to be traversed by $P$ at instant $t$. The discussion that follows examines the evolution of $Z^*(t)$ and $Z(t)$ as $P$ moves from point 0 to point 1. Both
discussions are formalized versions of Zeno’s Dichotomy II and I respectively.\textsuperscript{5}

368 The strategy of pairing off the $Z^*$-points (or the $Z$-points) with the successive instants of an strictly increasing infinite sequence of instants was firstly used by Aristotle [11] when trying to solve Zeno’s dichotomies. Although Aristotle ended up by rejecting his original strategy, it is still the preferred to solve both paradoxes. As we will see, however, the successiveness of $Z^*$-points and $Z$-points leads to a conflicting conclusion.

Zeno’s Dichotomy II

369 Let us begin by analyzing the way $P$ passes over the $Z^*$-points. Since the sequence of $Z^*$-points is $\omega^*$—ordered the first element does not exist, and consequently the first $n$ elements do not exist either, for any finite number $n$. Thus, and taking into account that $P$ is at point 0 at $t_0$ and at point 1 at $t_1$, we will have:

$$\forall t \in [t_0, t_1] \begin{cases} 
 t = t_0 : & Z^*(t) = 0 \\
 t > t_0 : & Z^*(t) = \aleph_0
\end{cases}$$

Therefore, no instant $t$ exists within $[t_0, t_1]$ at which $Z^*(t) = n$, whatever be the finite number $n$, otherwise there would exist the first $n$ elements of an $\omega^*$—ordered sequence. Notice $Z^*(t)$ is well defined in the whole interval $[0, 1]$. Thus, equation (3) expresses a dichotomy: $Z^*(t)$ can only take two values along the whole interval $[t_0, t_1]$: either 0 or $\aleph_0$.

370 In agreement with 369 and regarding the number of traversed $Z^*$-points, $P$ can only exhibit two successive states: the state $P^*(0)$ at which it has traversed zero $Z^*$-points, and the state $P^*(\aleph_0)$ at which it has traversed aleph-null $Z^*$-points. Now then, taking into account the successiveness of $Z^*$-points and the fact that between any two successive $Z^*$-points a distance greater than zero always exists, to traverse aleph-null $Z^*$-points, whatsoever they be, means to traverse a distance greater than zero. And, evidently, to traverse a distance greater than zero at the finite velocity $v$ of $P$ means the traversal has to last a time greater than zero.

371 Although it is impossible to calculate neither the exact duration of the transition $P^*(0)-P^*(\aleph_0)$ nor the distance $P$ must traverse to reach the state $P^*(\aleph_0)$ (there is neither a first instant nor a first point at which the transition begins), we have proved in 370 that, undetermined as it might be, that duration must be greater than zero. We will now prove no real number greater than zero exists for that duration. The same applies to the distance $P$ must traverse within $[0, 1]$ to become $P^*(\aleph_0)$ from $P^*(0)$.

372 Let $d$ be any real number greater than 0 and consider the real interval $(0, d)$, which contains $2^{\aleph_0}$ points densely ordered. According to the above dichotomy $P^*(0)-P^*(\aleph_0)$, at any point $x$ within $(0, d)$ our particle $P$ have already

\textsuperscript{5}See, for instance, [30], [31], [195], [165], [106], [197], [52], [129].
traversed aleph-null $Z^*$-points. In consequence $d$ is not the distance $P$ must traverse to become $P^*(\aleph_0)$ from $P(0)$. Now then, since $d$ is any real number greater than zero, we must conclude that no real number greater than zero exists such that it could be the distance $P$ must traverse to become $P^*(\aleph_0)$ from $P(0)$. The same conclusion, and for the same reasons, can be deduced for the amount of time $P$ must spend to become $P^*(\aleph_0)$ from $P^*(0)$.

373 In line with 370 and 372, $P$ needs to traverse a distance greater than zero during a time greater than zero to become $P^*(\aleph_0)$ from $P^*(0)$, but neither that distance nor that time can take a value greater than zero. Note this is not a question of indeterminacy but of impossibility: no positive real number exists so that it could be the distance or the time $P$ needs to become $P^*(\aleph_0)$ from $P^*(0)$. None. If it were a question of indeterminacy there would exist a set of possible numbers for that duration and for that distance, although we could not determine which of them would be the solution. In our case that set is simply empty.

Zeno’s Dichotomy I

374 We will now examine the way $P$ traverses the $Z$-points between point 0 and point 1. Being $Z(t)$ the number of $Z$-points to be traversed by $P$ at the precise instant $t$ in $[t_0, t_1]$, that number can only take two values: either $\aleph_0$ or 0. In fact, assume that at any instant $t$ within $[t_0, t_1]$ the number of $Z$-points to be traversed by $P$ is a finite number $n > 0$. This would imply the impossible existence of the last $n$ points of an $\omega$-ordered sequence of points. Thus, we have a new dichotomy:

$$\forall t \in [t_0, t_1] \begin{cases} t < t_1 : & Z(t) = \aleph_0 \\ t = t_1 : & Z(t) = 0 \end{cases} \quad (4)$$

Therefore, no instant $t$ exist at which $Z(t) = n$, whatever be the finite number $n$. Notice $Z(t)$ is well defined in the whole interval $[0, 1]$. Thus, equation (4) expresses a new dichotomy: $Z(t)$ can only take two values: either $\aleph_0$ or 0.

375 In accord with 374 and regarding the number of $Z$-points to be traversed, $P$ can only exhibit two successive states: the state $P(\aleph_0)$ at which that number is $\aleph_0$, and the state $P(0)$ at which that number is 0. The number of $Z$-points to be traversed by $P$ decreases directly from $\aleph_0$ to 0, without finite intermediate states at which only a finite number of $Z$-points remain to be traversed.

376 Taking into account the successiveness of $Z$-points and the fact that between any two successive $Z$-points a distance greater than zero always exists, to traverse aleph-null $Z$-points, whatsoever they be, means to traverse a distance greater than zero. And, evidently, to traverse a distance greater than zero at the finite velocity $v$ of $P$ means the traversal has to last a time greater than zero.
Although it is impossible to calculate the exact duration of the transition $P(\aleph_0) \rightarrow P(0)$ (there is not a last instant at which the transition ends), we have proved in 376 that, undetermined as it might be, that duration must be greater than zero. We will now prove no real number greater than zero exists for that duration. The same applies to the distance $P$ must traverse within $[0, 1]$ to become $P(0)$ from $P(\aleph_0)$.

Let $\tau$ be any real number greater than zero, consider the real interval of time $(0, \tau)$ and any instant $\tau'$ within $(0, \tau)$. The number of Z-points to be traversed at instant $t_1 - \tau'$ has not still begin to decrease because that number at $t_1 - \tau'$ is still $\aleph_0$, if not it would be a finite number $n$ and then the impossible last $n$ points of an $\omega$-ordered sequence of points would exist. Consequently, $\tau$, which is any positive real number, is not the time during which the number of Z-points to be traversed by $P$ decreases from $\aleph_0$ to 0.

In compliance with 376 and 378, the time during which the number of Z-points to be traversed by $P$ decreases from $\aleph_0$ to 0 must be greater than zero, but cannot take any value greater than zero. A symmetric reasoning now regarding the distance along which the number of Z-points to be traversed by $P$ decreases from $\aleph_0$ to 0, proves that distance, that must be greater than zero, cannot take any value greater than zero either. Note again this is not a question of indeterminacy but of impossibility: no positive real number exists so that it could be the distance or the time $P$ needs to become $P(0)$ from $P(\aleph_0)$. None. If it were a question of indeterminacy there would exist a set of possible numbers for that duration and for that distance, although we could not determine which of them would be the solution. In our case that set is simply empty.

**Conclusion**

The transitions from $P^*(0)$ to $P^*(\aleph_0)$ and from $P(\aleph_0)$ to $P(0)$ can only take place along a distance and a time greater than zero, but cannot take place along a distance and a time greater than zero because no positive real number exist neither for those distances nor for those times.

The above contradictions are direct consequences of the $\omega$—order and the $\omega^*$—order which, in turn, are direct consequences of assuming the existence of actual infinite totalities. It is then this assumption, the assumption of the actual infinity, the ultimate cause of both contradictions.

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6Recall $\aleph_0$ is the smallest infinite cardinal greater than all finite cardinal.
21.-Hilbert’s machine

HILBERT’S HOTEL

382 In the next discussion we will make use of a supermachine inspired by the emblematic Hilbert’s Hotel. But before beginning, let us relate some of the prodigious, and suspicious, abilities of the illustrious Hotel.

![Hilbert Hotel](image)

Figure 21.1: The power of the ellipsis: An infinitist way of making money.

383 Its director, for instance, has discovered a fantastic way of getting rich: he demands one euro to $R_1$ (the guest of the room 1); $R_1$ recover his euro by demanding one euro to $R_2$ (the guest of the room 2); $R_2$ recover his euro by demanding one euro to $R_3$ (the guest of the room 3); and so on. Finally all guests recover his euro, and then our crafty director demands a second euro to $R_1$ which recover again his euro by demanding one euro to $R_2$, which recover again his euro by demanding one euro to $R_3$, and so on and on. Thousands of euros coming from the (infinitist) nothingness to the pocket of the fortunate director.

384 Hilbert’s Hotel is even capable of violating the laws of thermodynamics by making it possible the functioning of a perpetuum mobile: in fact we would only have to power the appropriate machine with the calories obtained from the successive rooms of the prodigious hotel in the same way its director gets his euros.

385 Unbelievable as it may seem, infinitists justify all those absurd pathologies, and many others, in behalf of the peculiarities of the actual infinity. They prefer to assume any pathological behaviour of the world before examining the consistency of the pathogene. In the next discussion, however, we will come to a contradiction that cannot be easily subsumed in the picturesque nature of
the actual infinity.

**Definitions**

386 In the following conceptual discussion we will make use of a theoretical device, inspired by the emblematic Hilbert Hotel, that will be referred to as *Hilbert machine*, composed of the following elements (see Figure 21.2):

1. An infinite horizontal wire divided into two infinite parts, the left and the right side:
   
   (a) The right side in turn is divided into an $\omega$–ordered sequence of adjacent sections $\langle S_n \rangle$ of equal length labeled from left to right as $S_1$, $S_2$, $S_3$, . . . . They will be referred to as right sections.
   
   (b) The left side is also divided into an $\omega$–ordered sequence of adjacent sections $\langle S'_n \rangle$ of all them of the same length as the right sections, and labeled now from right to left as . . . , $S'_4$, $S'_2$, $S'_1$; being $S'_1$ adjacent to $S_1$. They will be referred to as left sections.

2. An $\omega$–ordered sequence of beads $\langle b_n \rangle$ strung on the wire, so that they can slide on the wire as the beads of an abacus, being each bead $b_i$ initially strung on center of the right section $S_i$.

3. All beads are mechanically linked by an sliding mechanism that slides simultaneously all beads the same distance along the wire.

4. The sliding mechanism is adjusted in such a way that it slides simultaneously each bead exactly one section to the left (L-slidings).

5. The first bead $b_1$ is equipped with a sensor $E$ to detect empty left sections. L-slidings will be performed if, and only if, this sensor detects the left section adjacent to $b_1$ is empty.

387 Before performing an L-sliding, the sensor $E$ detects if the left section adjacent to $b_1$ is empty, so that $b_1$ can slide to it and each bead $b_{i,i>1}$ to the section previously occupied by $b_{i-1}$. This way of functioning allows us to impose the following restriction to L-slidings:

Restriction 387.-An L-sliding will be carried out if, and only if, all beads remain strung on the wire.
Since the sections $\langle S'_n \rangle$ of the left side of the wire are $\omega$-ordered each section $S'_n$ has an immediate successor $S'_{n+1}$ just on its left. In accord with the hypothesis of the actual infinity all those infinitely many left sections exist as a complete totality in spite of the fact that there is no last section completing the sequence.

Let us begin by proving that for each natural number $v$ the first $v$ L-slidings can be carried out. It is quite clear the first L-sliding can be carried out: the sensor $E$ detects the empty left section $S'_1$ so that $b_1$ can slide to $S'_1$ and each $b_{i, i>1}$ to $S_{i-1}$. Assume that, being $n$ any natural number, the first $n$ L-slidings can be carried out. As a consequence of those first $n$ L-slidings, and taking into account that each of them moves all beads exactly one section to the left, the ball $b_1$ will be placed exactly $n$ sections to the left from its original position $S_1$, i.e. in the left section $S'_n$. For the same reason, each ball $b_{i, i>1}$ will be placed $n$ section to the left from its original position. In these conditions $E$ detects the empty left section $S'_{n+1}$ so that $b_1$ can slide to $S'_{n+1}$ and each bead $b_{i, i>1}$ to the section previously occupied by $b_{i-1}$. Consequently the $(n + 1)$-th L-sliding can also be carried out. We have then proved the first L-sliding can be carried out, and if for any natural number $n$ the first $n$ L-slidings can be carried out, then the first $(n + 1)$ L-slidings can also be carried out. This finally proves that for each natural number $v$ the first $v$ L-slidings can be carried out.

According to restriction 387, the successive L-slidings can be performed if, and only if, all beads remain strung on the wire. Assume then that all L-slidings that observe Restriction 387, and only them, are successively carried out. In these conditions we will prove the two following contradictory results.

The first result is the following:

Theorem 391.-Once performed all possible L-slidings that observe Restriction 387, and only them, at least one bead remains strung on the wire.

Proof.-Since the only L-slidings that have been performed are those that observe Restriction 387 and this restrictions states that an L-sliding is carried out if, and only if, all beads remain strung on the wire, it immediately follows that after performing all those L-slidings, and only them, all beads remain strung on the wire. Assume, on the contrary, that once performed all possible L-slidings that observe Restriction 387, and only them, no ball remains strung on the wire. Since none of the performed L-slidings removes a ball from the wire, only the completion of all of them, as such a completion, could have removed all the infinitely many balls from the wire. We would have to accept that the completion of infinitely many successive actions, as such a completion, has arbitrary additional effects, in whose case any thing could be expected from any procedure or definition consisting of infinitely many successive steps. Evidently, this ad hoc arbitrariness is incompatible with formal sciences.
The second result is the following:

**Theorem 392.** Once performed all possible L-slidings that observe Restriction 387, and only them, no beads remains strung on the wire.

**Proof.** Let \( b_v \) be any bead and assume that once performed all possible L-slidings that observe Restriction 387, and only them, it is strung on the right section \( S_k \). It must be \( k < v \) because all L-slidings are towards the left, the direction towards which the indexes of \( (S_n) \) decreases. Since \( b_v \) was initially on \( S_v \) only a finite number \( v - k \) of L-slidings would have been performed, and then it would not have been possible to perform the the first \( v - k + 1 \) L-slidings, which goes against 389 because \( v - k + 1 \) is a natural number. A similar reasoning can be applied if \( b_v \) were finally placed on a left section \( S'_{n+1} \), being now the number of performed L-slidings exactly \( v + n - 1 \) and then it would not have been possible to perform the first \( v + n \) L-slidings, which also goes against 389. Thus, since \( b_v \) is any bead, if all possible L-slidings that observe Restriction 387, and only them, have been performed, then no bead remain strung on the wire.

**DISCUSSION**

**393** Let us compare the functioning of the above Hilbert machine (\( H_\omega \) from now on) with the functioning of a finite version of the machine (symbolically \( H_n \)). This finite machine has a finite number \( n \) of both right and left sections (Figure 21.3). A finite sequence of \( n \) beads are initially placed in the right side of the wire, each bead \( b_i \) strung on the center of the right section \( S_i \). It is immediate to prove that \( H_n \) can only perform \( n \) L-slidings because not having a left section \( S'_{n+1} \), the sensor \( E \) will stop the machine after the first \( n \) L-slidings. \( H_n \) halts with each left section \( S'_i \) occupied by the bead \( b_{n-i+1} \) and all right sections empty, and this is all. No contradiction is derived from the functioning of \( H_n \). Thus for any natural number \( n \), the corresponding machine \( H_n \) is a consistent theoretical artifact. Only the infinite Hilbert’s machine \( H_\omega \) is inconsistent.

**394** What contradiction 391-392 proves is not the inconsistent functioning of a supermachine. What it proves is the inconsistency of \( \omega \)–order itself. Perhaps we should not be surprised by this conclusion. After all, an \( \omega \)–ordered sequence is one which is both complete (as the actual infinity requires) and uncompletable (there is not a last element that completes it). On the other hand, and as Cantor proved [38], [40], \( \omega \)–order is an inevitable consequence of assuming the existence of infinite sets as complete totalities. An existence axiomatically stated in our days by the Axiom of Infinity, in all axiomatic set
theories including its most popular versions ZFC and BNG [188], [186]. It is therefore that axiom the ultimate cause of contradiction 391-392.
22.-Jordan curves of infinite length

INTRODUCTION

The $\omega$-ordered sequence $\langle z_n \rangle$ of Z-points within the real interval $(0, 1)$ defined by:

$$z_n = \frac{2^n - 1}{2^n}$$  \hspace{1cm} (1)

is an example of $\omega$-partition of a finite line segment. Each pair of successive points $x_n, x_{n+1}$ defines a part of the partition. The successive parts are disjoint and adjacent, so that the right end of any of them coincides with the left end of the following one:

$$(x_1, x_2], (x_2, x_3], (x_3, x_4], \ldots$$  \hspace{1cm} (2)

As is well known, at least since the 18th century, $\omega$-partitions of finite line segments are only possible if the successive adjacent parts of the $\omega$-partition are of a decreasing length, otherwise the length of the line would have to be infinite [16]. This inevitable restriction originates a huge asymmetry in the partition. Indeed, whatever be the length of the $\omega$-partitioned segment $AB$ and whatever be the $\omega$-partition, all its parts, except a finite number of them, will necessarily lie within a final interval $CB$ arbitrarily small.

For the sake of illustration, consider an $\omega$-partition of a straight line segment $AB$ whose length is $10^{30}$ light years, the assumed diameter of the visible universe. Whatever be the $\omega$-partition of this enormous segment all its infinitely many parts, except a finite number of them, will inevitably lie within a final interval $CB$ inconceivable less than, for instance, Planck length ($\sim 10^{-33}$ cm). There is no way of performing a less unbiased partition if the partition has to be $\omega$-ordered, the smaller of the infinite partitions (Figure 22.2). Thus, $\omega$-partitions are $\omega$-asymmetrical. And being $\omega$ the less infinite ordinal, any transfinite partition has to contain at least one $\omega$-ordered partition.

The unaesthetic consequence of the above asymmetry becomes a little
more controversial if the partitioned object is a closed line as a Jordan curve. The objective of the following short discussion is just to examine such a partition.

INFINITE PARTITION OF A JORDAN CURVE

399 Let $f(x)$ be a real valued function whose graph is a Jordan Curve\(^1\) $J$ in the Euclidian plane $\mathbb{R}^2$. Let $a$ and $b$ be the endpoints of the arc $\bar{a}b$ in $J$. We will write $L(a, b)$ to denote the length of $\bar{a}b$:

$$L(a, b) = \int_{a}^{b} \sqrt{1 + (f(x)')^2} dx$$ (3)

400 Assume $J$ has an infinite length. In these conditions, let $r$ be any proper real number greater than 0 and assume $J$ is divided clockwise from any point $x_1$ into a certain number of adjacent parts $\bar{x}_1x_2$, $\bar{x}_2x_3$, $\bar{x}_3x_4$ ... so that each part $\bar{x}_i\bar{x}_{i+1}$ has a finite length equal or greater than $r$:

$$L(x_i, x_{i+1}) \geq r, \ \forall i \in I$$ (4)

where $I$ is the partition’s set of indexes.

401 The parts of a partition are successive and adjacent, so that the end of any of them coincides with the beginning of the next one. In these conditions each part has an immediate successor (except the last one, if a last one does exist) and an immediate predecessor (except the first one, if a first one does exist). Therefore, partitions do have ordinality. They are $\alpha$-ordered, being $\alpha$ a finite or infinite ordinal.

402 Evidently, the partition $\langle x_i \rangle_{i \in I}$ must be infinite otherwise, and being finite the length of every part, $J$ would have a finite length. In addition,

\[^1\]A Jordan curve is a simple closed line that is topologically equivalent to the unit circle, i.e. one that does not intersect itself.
and according to Cantor [35], $\langle x_i \rangle_{i \in I}$ cannot be uncountably infinite. In fact, consider the sequence of real numbers $\langle r_i \rangle_{i \in I}$ defined as:

\[
\begin{align*}
    r_1 &= x_1 \\
    r_{i+1} &= r_i + L(x_i, x_{i+1}), \forall i \in I
\end{align*}
\]

The one to one correspondence $f$ between $\langle x_i \rangle_{i \in I}$ and $\langle r_i \rangle_{i \in I}$ defined by $f(x_i) = r_i$ proves that both sequences have the same cardinality. Thus, if the first one were uncountably infinite so would be the second. But $\langle r_i \rangle_{i \in I}$ cannot be uncountably infinite because if that were the case we could pick out a different rational number $q_i$ in each real interval $[r_i, r_{i+1})$ and then we would have an uncountable set of different rational numbers, which, according to Cantor, is impossible.

403 In Chapter 12 we examined the possibilities of non-countable partitions and proved that Cantor conclusion could not be the right conclusion. In any case, and being $\omega$ the smallest infinite ordinal, the ordinal of any infinite partition must be either $\omega$ or greater than $\omega$, which means that it contains at least an $\omega$—ordered subpartition.

404 Therefore, the ordinal of the partition $\langle x_i \rangle_{i \in I}$ will be $\omega$ or greater than $\omega$. It is immediate to prove, however, that ordinal cannot be neither $\omega$ nor greater than $\omega$.

405 Consider a point $y$ anticlockwise from $x_1$ and such that:

\[ L(y, x_1) = r/2 \]

From (4) we infer that $y$ can only belong to the last part of $\langle x_i \rangle_{i \in I}$. So, this partition cannot be $\omega$—ordered because $\omega$—ordered partitions have not a last part.

406 Assume then, the ordinal of $\langle x_i \rangle_{i \in I}$ is greater than $\omega$. In these conditions there will exist a part $\overline{x_\omega x_{\omega+1}}$. Now then, according again to (4), the point $z$
anticlockwise from $x_\omega$ and such that $L(z, x_\omega) = r/2$ can only belong to a part immediately preceding $\vec{x_\omega x_{\omega + 1}}$, which is impossible. This proves the ordinal of $\langle x_i \rangle_{i \in I}$ cannot be greater than $\omega$.

We have just proved the ordinal of the partition $\langle x_i \rangle_{i \in I}$ must be either $\omega$ or greater than $\omega$, but it can be neither $\omega$ nor greater than $\omega$. In consequence, Jordan curves of infinite length are inconsistent objects.
THE UNARY NUMERAL SYSTEM

Perhaps the most primitive way of representing numbers is what we now call the unary numeral system (UNS). As its name suggests, only a numeral is needed to represent any natural number. Here we will use the numeral '1'. The successive natural numbers will then be written as:

$$1, 11, 111, 1111, 11111, \ldots$$

Although, for obvious reasons, the UNS is not the most appropriate for advanced calculus, it is the system that best represents the essential nature of natural numbers: each natural number is exactly one unit greater than its immediate predecessor. In consequence, the unary expression of each natural number has exactly one numeral more than the unary expression of its immediate predecessor. In addition, the UNS suggests a recursive arithmetic definition of natural numbers: starting from the first of them, the number 1, add one unit to define the next one.

In conformity with the hypothesis of the actual infinity, the infinitely many natural numbers, all of them finite, do exist all at once, as a complete totality. The result of defining the infinitely many finite natural numbers by adding infinitely many successive units to the first natural number 1 is an actual infinitude of increasing finite numbers, each one unit greater than its immediate predecessor, in spite of which no infinite number is reached. Or in terms of the UNS, according to the infinitist orthodoxy it is possible to define infinitely many finite strings of '1's each with one numeral '1' more than its immediate predecessor, without ever reaching a string with infinitely many '1's.

Let us put to the test the above hypothesis on the existence of an actual infinitude of finite numbers, each one unit greater than its immediate predecessor. For this, consider a special unary writing machine (UWM) capable of writing horizontal strings of '1's of any finite length. Now let UWM work according to the following conditions:

1. On an empty tape, UWM writes a first numeral '1'.

The numeral of a number is not a number but the symbol we use to refer to the number. Thus, the numeral '5' is the symbol for the number 5 in the usual decimal numeral system.
2. UWM writes a new numeral ‘1’ on the right side of the last previously written ‘1’ if, and only if, the result is a finite string of ‘1’s, i.e. the unary expression of a natural number. Otherwise UWM stops.

![Writing head](image)

Figure 23.1: The unary writing machine on the point of writing the fifth numeral.

412 It is immediate to prove by induction the following:

Theorem 412.- For any natural number $v$, UWM can write a finite string $S_v = 11^{(v)}1$ of $v$ numerals ‘1’.

Proof.- Given that the string 1 is finite, UWM can write the first finite string $S_1 = 1$. Assume UWM can write the string $S_n = 11^{(n)}1$ of $n$ numerals, being $n$ any finite natural number. Since $n + 1$ is also finite, UWM can write a new numeral ‘1’ on the right end of $S_n$, i.e. a finite string $S_{n+1} = 11^{(n+1)}1$ of $n+1$ numerals. So, UWM can write the first string $S_1$ and if it can write a string $S_n = 11^{(n)}1$, being $n$ any natural number, then it can also write a string of $S_{n+1} = 11^{(n+1)}1$ of $n+1$ numerals. This proves that for any natural number $v$, UWM can write a string $S_v = 11^{(v)}1$ of $v$ numerals ‘1’.

413 Assume now that while UWM can write a new numeral ‘1’ on the right of the last previously written ‘1’, it does it. Let $S$ be the resulting string once all possible numerals have been written. First of all, notice we assume it is possible to carry out all possible actions of a sequence of successive actions, just because they are possible. Otherwise we would be in the face of a basic contradiction, that of an impossible possibility. We are, therefore, assuming an immediate consequence of the First Law of logic: if something is possible then it is possible.

414 The string $S$ cannot have an infinite number of numerals because UWM writes a new numeral if, and only if, the resulting string has a finite number of numerals. But $S$ cannot have a finite number of numerals either. In fact, assume $S$ has $v$ numerals, being $v$ any natural number. This would imply UWM did not write the $(v + 1)$th numeral, which, according to Theorem 412 and being $v + 1$ a finite number, is impossible if all possible ‘1’’s have been written. The alternative to this argument would be that it is impossible to write all possible numerals, which is an even more basic contradiction.

415 As we will see, the above argument can easily be converted in a supertask argument. Indeed, let $(t_n)$ be an $\omega$-ordered and strictly increasing sequence of instants within the finite interval of time $(t_a, t_b)$ whose limit is $t_b$. Assume that at $t_1$ UWM writes a first numeral ‘1’ on an empty tape and then at each successive $t_{i,i>1}$ of $(t_n)$ it writes a new numeral just on the right of the one
previously written if, and only if, the resulting string of 1s is finite. At \( t_b \) we will have a string \( S \) of numerals that cannot be finite nor infinite.

416 In fact, if it were finite it would have a finite number \( n \) of numerals and then UWM would have been halted before writing the \((n+1)\)th numeral, which is impossible because \( n + 1 \) is also finite and therefore UWM can also write the \((n+1)\)th numeral '1'.

417 If \( S \) is infinite then, apart from violating the condition under which the supertask has to be performed, infinitely many numerals had to be written at \( t_b \), once the supertask had already finished. In effect, at any instant \( t \) within \((t_a, t_b)\) UWM has written only a finite number of numerals: taking into account that \( t_b \) is the limit of \( \langle t_n \rangle \) we will have:

\[
\exists v \in \mathbb{N} : t_v \leq t < t_{v+1}
\]  

and then at instant \( t \) only a finite number \( v \) of numerals have been written. And so for any \( t \) in \((t_a, t_b)\). Therefore, no instant exists within \((t_a, t_b)\) at which UWM has written an infinite string of 1s. Consequently, if \( S \) is infinite, at \( t_b \), the first instant at which UWM is halted, UWM would still have to write infinitely many numerals.

418 Infinitist claim that all natural numbers can be counted in a finite interval of time: by counting each natural number \( n \) at the precise instant \( t_n \) of the sequence of instants \( \langle t_n \rangle \) (Chapter 2, 12). It is curious that all natural numbers can be counted but not written in the UNS, being both processes absolutely equivalent in formal terms. Except in that the writing leaves an uncomfortable final output in the form of a string of numerals \( (S) \) that cannot be neither finite nor infinite.

419 This is the type of result one can expect when it is assumed the possibility to append infinitely many successive '1's to an initial string \( S_1 \) = 1 without making the string infinite. Or, what is the same, when it is assumed the possibility to add infinitely many successive units to a first unit (the first natural number) without ever reaching an infinite number.

420 There is other more explicit way of making it evident the fallacy of adding infinitely many successive units to a first unit without ever reaching an infinite number. Or alternatively, the fallacy of writing infinitely many times a numeral '1' on the right of a first numeral '1' without ever reaching an infinite string of numerals '1'. Is the following conditioned supertask.

421 Let \( BX \) be an empty box, \( \langle b_n \rangle \) an \( \omega \)-ordered collection of labelled balls, and \( \langle t_n \rangle \) an strictly increasing sequence of instants within the interval \((t_a, t_b)\) whose limit is \( t_b \). Now consider the following conditioned supertask: At each of the successive instants \( t_i \) of \( \langle t_n \rangle \) add the ball \( b_i \) if, and only if, the number of balls in the box \( BX \) is finite.

422 At instant \( t_b \) our supertask will have finished and \( BX \) will contain a
certain number of balls. According to the hypothesis of the actual infinity subsumed into the Axiom of Infinity, there exist a complete totality of infinitely many finite natural numbers, 1, 2, 3, \ldots, each one unit greater than its immediate predecessor. The collection of balls $\langle b_n \rangle$ and the sequence of instants $\langle t_n \rangle$ also exist as complete totalities. All those complete collections of natural numbers, balls and instants are legitimized by the Axiom of Infinity. In turn, those collections and sequences legitimize the completion of our supertask within the interval of time $(t_a, t_b)$.

423 Consider the following two exhaustive and mutually exclusive alternatives regarding the number of balls in the box $BX$ at instant $t_b$:

1. The box $BX$ contains finitely many balls.
2. The box $BX$ contains infinitely many balls.

424 Let $v$ be any natural number and assume that at $t_b$ the box $BX$ contains $v$ balls. Since $v + 1$ is also a finite natural number the ball $b_{v+1}$ was also added to $BX$ at instant $t_{v+1}$. So, at $t_b$ the box $BX$ cannot contain $v$ balls, and being $v$ any finite natural number we can conclude that at $t_b$ the box $BX$ cannot contain a finite number of balls.

425 At $t_b$ the box $BX$ cannot contain infinitely many balls either. In fact:

1. $BX$ will contain infinitely many balls only if the condition, *add a ball to the box if, and only if, the number of balls in the box is finite*, has been arbitrarily violated. Or if there exists a finite number $v$ such that $v + 1$ is infinite, which obviously is not the case.
2. Being $t_b$ the limit of the sequence $\langle t_n \rangle$, at each instant $t$ within $(t_a, t_b)$ the box $BX$ contain a finite number $v$ of balls:

$$\exists v \in \mathbb{N} : \ t_v \leq t < t_{v+1}$$

So, the only way that $BX$ contain infinitely many balls at $t_b$ is by adding infinitely many balls just at $t_b$, which is impossible because at $t_b$ all balls have already been added to the box, at $t_b$ no ball is added to the box $BX$.

426 The conclusion about the box and the balls is, therefore, quite clear: If the list of the finite natural numbers, each one unit greater than its immediate predecessor, exist as an infinite and complete totality, then the box $BX$ can contain neither a finite nor an infinite number of balls.
The unary table of natural numbers

Consider now the following ω-ordered table $T$ of natural numbers in their natural order of precedence and written in the UNS:

- Row $r_1$: 1
- Row $r_2$: 11
- Row $r_3$: 111
- Row $r_4$: 1111
- Row $r_5$: 11111

... 

The $n$th row of $T$, symbolically $r_n$, corresponds to the unary representation of the natural number $n$, being, therefore, composed of exactly $n$ numerals ‘1’. According to the hypothesis of the actual infinity, the infinitely many rows of $T$, one for each natural number, do exist all at once, as a complete totality.

The number of rows of table $T$ is the same as the number of natural numbers, i.e. $\aleph_0$, the cardinal of the set of natural numbers. According to the infinitist orthodoxy, $\aleph_0$ is the less infinite cardinal, the less number greater than all finite natural numbers (see Chapters 2 and 16 on the actual infinity and aleph null).

The first column of $T$ has $\aleph_0$ elements, one for each row, one for each natural number. Since each element of this column belongs to a different row and no other column has more elements than this first column (it could easily be proved that each column of $T$ has $\aleph_0$ elements), we can say this first column defines the number of rows of $T$, in the sense that the first element of each row is a different element of this first column, and then a one to one correspondence $f$ between the rows $\langle r_i \rangle$ of $T$ and the elements $\langle c_{1i} \rangle$ of its first column can be defined:

$$f(r_i) = c_{1i}, \ \forall r_i \in T \quad (4)$$

However, while the number of $T$'s rows is completely defined by the number of '1's of the first column, the number of its columns is highly problematic, as we will immediately see.

Being each row $r_n$ composed of exactly $n$ numerals ‘1’, and being each of those numerals an element of a different column, that row ensures the existence of at least $n$ columns in $T$. It is in this sense that we will say that $r_n$ defines exactly $n$ columns:

- $r_4 = 1111$ defines 4 columns
- $r_9 = 111111111$ defines 9 columns

...
Let’s begin by proving the number of columns of table $T$ cannot be finite. In effect, let $n$ be any natural number. $T$ cannot have $n$ columns because in that case the number $n + 1$ would not belong to the table; the unary representation of that number is a string of $n + 1$ numerals ‘1’ and then a row of $T$ that defines $n + 1$ columns. Thus, whatsoever be the finite number $n$, $T$ cannot have $n$ columns.

And now we will prove the number of columns of table $T$ cannot be infinite either. Since each row is the unary expression of a natural number and all natural numbers are finite, each row $r_n$ will consist of a finite string of $n$ numerals ‘1’. So, every row of $T$ defines a finite number of columns. Or in other words, since no natural number is infinite, no row defines infinitely many columns. But if no row defines an infinite number of columns, $T$ cannot have an infinite number of columns, unless the number of its columns is defined not by one row but by a certain number of rows. We will examine now this possibility.

Assume the infinite number of columns ($C$ from now on) of table $T$ is not defined by a particular row but by a group of rows, even by the whole table. Evidently, if a group of rows (or the whole table) is needed in order to define $C$, then at least two rows of the group will contribute together to the definition. Where ‘contribute together’ means that each row define certain columns that the other does not. Let $r_k$ and $r_n$ be any two of those contributing rows. If $r_k$ and $r_n$ contribute together to define $C$, then $r_k$ will define certain columns that $r_n$ does not, and vice versa. Otherwise only one of them would be necessary in order to define $C$.

Now then, since $k$ and $n$ are natural numbers we will have either $k < n$ or $k > n$. Assume $k < n$, in this case $r_k$ defines the firsts $k$ columns of $T$ and $r_n$ the firsts $n$ columns of $T$, so that, although $r_n$ defines $(n - k)$ columns that $r_k$ does not, all columns defined by $r_k$ are also defined by $r_n$. This proves the impossibility that any two different rows of a group of rows (including the whole table) contribute together to define $C$.

And things can get worse with respect to the definition of $C$. In effect, let $\langle t_n \rangle$ be any $\omega$-ordered strictly increasing sequence of instants within the real interval $(t_a, t_b)$ whose limit is $t_b$ and consider the following conditional supertask:

Supertask 435.-At each instant $t_i$ of $\langle t_n \rangle$ remove from $T$ the row $r_i$ if, and only if, the remaining rows define the same number of columns as if $r_i$ were not removed. Otherwise end the supertask.

In any case, at instant $t_b$ supertask 435 would have been performed and we will have the following two mutually exclusive alternatives:

1. At $t_b$ not all rows have been removed.
2. At $t_b$ all rows have been removed.

In accord with the first alternative, and taking into account the successive way the rows have been removed, there will be a first row $r_n$ that was not removed
because its removal would have changed the number of columns of \( T \). But this is impossible because all columns defined by \( r_n \) are also defined by the next row \( r_{n+1} \). The first alternative is then false. We must therefore conclude the second alternative is true, which means \( T \) has the same number of columns as an empty table! Again a consequence of being complete and uncompletable as the list of natural numbers is assumed to be from the perspective of the actual infinity hypothesis.

437 While, in accordance with the hypothesis of the actual infinity subsumed within the Axiom of Infinity, \( T \) is a complete and well defined totality composed of infinitely many rows, argument 431/435 proves the number of its columns can be neither finite nor infinite, which sounds rather contradictory.
Infinity one by one
24.-Timetabling the infinite

INTRODUCTION

438 Mathematics is not usually concerned with the way the infinitely many successive steps of, for instance, a recursive \(\omega\)-ordered definition could in fact be carried out. It simply assumes they are carried out in their complete totalities. But the finitely or infinitely many successive steps of any definition or procedure could easily be timetabled by any sequence of instants of the same ordinality as the sequence of steps, and a one to one correspondence between both sequences. Evidently, the correspondence between instants and steps has no effect on the result of the timetabled definition or procedure. It simply states the successive instants at which each of its successive steps could take place.

439 In the next two sections we will timetabled an \(\omega\)-ordered sequence of definitions as a result of which we will find a new infinitist infelicity, now one of a temporal nature.

RECURSIVE DEFINITIONS

440 Let \(\langle a_n \rangle\) be any \(\omega\)-ordered sequence \(a_1, a_2, a_3, \ldots\) and consider the following \(\omega\)-ordered sequence \(\langle D_n \rangle\) of recursive definitions:

\[
\begin{align*}
D_{n, n=1} &: A_n = \{a_n\} \\
D_{n, n>1} &: A_n = A_{n-1} \cup \{a_n\}
\end{align*}
\]  

(1)

The result of the sequence of definitions \(\langle D_n \rangle\) is assumed to be an \(\omega\)-ordered sequence \(\langle A_n \rangle\) of nested sets \(A_1 \subset A_2 \subset A_3 \subset \ldots\) that, according to the hypothesis of the actual infinity, exists as a complete totality.

441 Let now \((t_a, t_b)\) be any finite interval of time and let \(\langle t_n \rangle\) be an \(\omega\)-ordered and strictly increasing sequence of instants within \((t_a, t_b)\) whose limit is \(t_b\), as is the case of, for example, the classic sequence defined by:

\[
t_n = t_a + (t_b - t_a) \times \frac{2^n - 1}{2^n}
\]  

(2)
Definition (2) assumes time is infinitely divisible, what may, or may not, be the case in the physical world. This is not, however, an impediment to infinitist formal theories because they could be assumed to be developed in a conceptual universe in which time is arbitrarily defined as infinitely divisible.

The sequence of definitions \( \langle D_n \rangle \) can be timetabled by the sequence \( \langle t_n \rangle \) in an elementary way: by assuming that each \( n \)th step takes places at the precise instant \( t_n \). The one to one correspondence \( f \) defined by:

\[
f : \langle t_i \rangle \leftrightarrow \langle D_i \rangle \quad (3)
\]

\[
f(t_i) = D_i, \ \forall i \in \mathbb{N} \quad (4)
\]

proves that at \( t_b \) we will have the same \( \omega \)-ordered totality \( \langle A_n \rangle \) defined in (1).

A CONFLICTING DEFINITION

443 Timetabling mathematical definitions composed of infinitely many steps reveals some significant insufficiencies on the assumed completeness of the involved \( \omega \)-ordered totalities. We will now examine one of them.

444 Let \( x \) and \( y \) be two natural variables (whose domain is the set of natural numbers) and consider the following \( \omega \)-ordered sequence of (re)definitions \( \langle D_n \rangle \) of both variables, where \( D_n(x) \) and \( D_n(y) \) stand for the \( n \)th definition of \( x \) and \( y \) respectively:

\[
\text{At each successive instant } t_n \text{ of } \langle t_n \rangle \begin{cases} 
D_n(y) = 1 \\
D_n(x) = n
\end{cases} \quad (5)
\]

where \( n \) in \( t_n \) is the same as in \( D_n(x) = n \). Since \( t_b \) is the limit of \( \langle t_n \rangle \), at \( t_b \) the sequence of definitions \( \langle D_n \rangle \) will have been completed. Thus, \( t_b \) is the first instant at which the variables \( x \) and \( y \) are no longer redefined.

445 We will now prove that \( x \) and \( y \) remain well defined along the whole interval \([t_1, t_b]\). In fact, let \( t \) be any instant within \([t_1, t_b]\). Evidently, it holds \( t_1 \leq t < t_b \). So, if \( t = t_1 \) we will have \( x = 1; y = 1 \). And if \( t_1 < t \), there will be an index \( v \) such that \( t_v \leq t < t_{v+1} \) because \( \langle t_n \rangle \) is an \( \omega \)-ordered and strictly increasing sequence whose limit is \( t_b \). In this case we will have \( x = v; y = 1 \). This proves that both variables remain well defined along the whole interval \([t_1, t_b]\).

446 Since \( x \) and \( y \) remain well defined along the whole interval \([t_1, t_b]\) and no other definition takes place neither at \( t_b \) nor after \( t_b \), we can conclude both variables remain well defined in the whole closed interval \([t_1, t_b]\).

447 It is immediate to prove, however, that \( x \) is not defined at \( t_b \). Although it was always defined as a natural number, its current value at \( t_b \) cannot be
a natural number, otherwise, and taking into account that it was successively defined as the successive natural numbers in their natural order of precedence, that number would be the impossible last natural number or, alternatively, only a finite number of definitions would have been carried out. Notice this is not a question of indeterminacy but of impossibility: no natural number \( v \) exists such that the value of \( x \) at \( t_b \) could be \( v \). None. So, we know nothing on the current value of \( x \) at \( t_b \). After infinitely many correct definitions it successfully get undefined just at the precise instant \( t_b \). The problem is that nothing happens at \( t_b \) that can undefine \( x \).

448 In agreement with 446 and 447, we must conclude that, as a consequence of having being defined infinitely many successive times, at \( t_b \) the variable \( x \) is and is not defined.
25.-Spacetime divisibility

THE LEAST INFINITE ORDINAL

449 The first transfinite ordinal\(^1\) \(\omega\) is the least ordinal greater than all finite ordinals. It is the limit of the sequence of all finite ordinals 1, 2, 3, \ldots. The ordinal \(\omega\) defines a type of well order called \(\omega\)-order:\(^2\) a set or sequence is \(\omega\)-ordered if it has a first element and every element has an immediate successor\(^3\) and an immediate predecessor, except the first one, that has no predecessor. In consequence there is not a last element in an \(\omega\)-ordered set or sequence. The set \(\mathbb{N}\) of natural numbers in their natural order of precedence \(\{1, 2, 3, \ldots\}\) is a well known example of \(\omega\)-ordered set.

450 \(\omega^*\)-Order is the symmetrical reflection of \(\omega\)-order: a set or sequence is \(\omega^*\)-ordered if it has a last element and each element has an immediate predecessor and an immediate successor, except the last one, that has no successor. In consequence there is not a first element:

\[
\underbrace{\ldots t_3^*, t_2^*, t_1^*}, \quad \underbrace{t_1, t_2, t_3, \ldots}
\]

(1)

where \(1^*, 2^*, 3^*, \ldots\) means last, last but one, last but two, etc. The set \(\mathbb{Z}^-\) of negative integers in their natural order of precedence \(\{\ldots, -3, -2, -1\}\) is a well known example of \(\omega^*\)-ordered set.

451 In accordance with the definition of \(\omega\)-order given in 449, every element of an \(\omega\)-ordered set has a finite number of predecessors and an infinite number of successors. In the case of \(\omega^*\)-order every element of an \(\omega^*\)-ordered set has finitely many successors and infinitely many predecessors. This immense asymmetry in the number of predecessors and successors (\(\omega\)-asymmetry) is a well known fact, although it is usually ignored in infinitist literature.

452 Most of the arguments we will make use of in this chapter are similar

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\(^1\)According to Cantor’s classical terminology [40], finite ordinals as 1, 2, 3, \ldots are said of the first class while transfinite ordinals, as \(\omega\), \(\omega +1\), \(\omega +2\), \ldots, are said of the second class. An ordinal of the second class is of the second kind if, as \(\omega\), it is the limit of an infinite sequence of ordinals; it is of the first kind if it is of the form \(\alpha + n\), where \(\alpha\) is an ordinal of the second class second kind and \(n\) a finite ordinal.

\(^2\)In formal terms, a set is \(\omega\)-ordered if it is well ordered and its ordinal is \(\omega\).

\(^3\)Between an element and its immediate successor no other element of the sequence exists.
to the arguments about dichotomies of Chapter 20, although now the main objective is to analyze the assumed infinite divisibility of space and time. The discussion could be developed in terms of points, in terms of instants, and in terms of both points and instants. Being the three of them similar, we will only discuss the case of instants. But before doing it, let us examine an extravagant infinitist asymmetry.

Let $\langle t_n \rangle$ be any strictly increasing and $\omega$—ordered sequence of instants defined within the finite real interval $(t_a, t_b)$ whose limit is $t_b$. And let $S$ be a supertask whose infinitely many actions $\langle a_n \rangle$ are performed at the infinitely many successive instants of $\langle t_n \rangle$, each action $a_i$ performed at the precise instant $t_i$. In consequence at $t_b$ all the infinitely many actions $\langle a_n \rangle$ will have been performed. Note there is not an instant at which the supertask ends but a first instant $t_b$ at which the supertask already has ended.

Thus, the limit $t_b$ is not the instant at which $S$ ends but the first instant after $S$ has ended, the first instant after performing all the infinitely many actions $\langle a_n \rangle$. Being $t_b$ the limit of $\langle t_n \rangle$, at any instant $t$ before $t_b$ and arbitrarily close to it, only a finite number of tasks will have been performed and infinitely many of them still remain to be performed ($\omega$-asymmetry).

To grasp the colossal magnitude of the above $\omega$-asymmetry, assume the interval $[t_a, t_b]$ is trillions of times greater than the age of the universe and consider, on the other hand, an interval of time $\tau = 0.000\ldots001$ seconds so small that we would need trillions and trillions of pages of standard text to write all its zeroes between the decimal point and the final decimal 1, a number of pages so huge that the whole visible universe would not have sufficient room for them. Well, only a finite number of tasks will have been performed during the trillions of years elapsed between $t_a$ and $t_b - \tau$ while infinitely many tasks, practically all of them, will have to be performed just in our unimaginably small interval of time $\tau$. Thus, rather than unesthetic, $\omega$-asymmetry is repulsive.

And things can get worse. Assume we remove from $[t_a, t_b]$ all instants at which infinitely many tasks of the supertask $S$ still remain to be performed. We would have to remove all instants within $[t_a, t_b)$, except $t_b$. In fact, let $t$ be any instant within $[t_a, t_b)$ different from $t_b$. Since $t_b$ is the limit of $\langle t_n \rangle$, we will have:

$$\exists v \in \mathbb{N} : t_v \leq t < t_{v+1}$$

Therefore, at $t$ only a finite number $v$ of tasks have been carried out and then infinitely many tasks still have to be performed. In consequence $t$ must be removed from $[t_a, t_b)$. Therefore, and being $t$ any instant within $[t_a, t_b)$ different from $t_b$, all instants within $[t_a, t_b)$, except $t_b$, have to be removed from that interval. So at $t_b$, the first instant after completing the supertask, there still

---

4 There is not an instant at which $S$ ends because $\langle a_n \rangle$ is $\omega$—ordered and $\omega$—ordered sequences have not last element.

5 A sphere of 93000 billion light years.
Spacetime dichotomies —— 141

remains infinitely many tasks to be performed.

Spacetime dichotomies

Consider any finite interval of time \([t_a, t_b]\) and within it two sequences of instants, the \(\omega\)—ordered sequence of \(t\)-instants:

\[
\langle t_i \rangle : t_i = t_a + \frac{2^i - 1}{2^i} (t_b - t_a), \; \forall i \in \mathbb{N}
\] (3)

whose limit superior is \(t_b\), and the \(\omega^*\)—ordered sequence of \(t^*\)-instants:

\[
\langle t^*_{i^*} \rangle : t^*_{i^*} = t_a + \frac{1}{2^i}, \; \forall i \in \mathbb{N}
\] (4)

whose limit inferior is \(t_a\) and where \(i^*\) stands for the last but \((i - 1)\) element of the \(\omega^*\)—ordered sequence \(\langle t^*_{i^*} \rangle_{i \in \mathbb{N}}\).

We will examine the way the successive \(t^*\)-instants of \(\langle t^*_{n^*} \rangle\) and the successive \(t\)-instants of \(\langle t_n \rangle\) elapse as time passes from \(t_a\) to \(t_b\), for this we will make use of the two following functions:

\[
f^*(t) = \text{number of elapsed } t^*\text{-instants at } t, \forall t \in [t_a, t_b]
\] (5)

\[
f(t) = \text{number of } t\text{-instants not yet elapsed at } t, \forall t \in [t_a, t_b]
\] (6)

In accordance with the definitions of \(\omega^*\)—order and \(\omega\)—order we can write:

\[
f^*(t) = \begin{cases} 0 & \text{if } t = t_a \\
\aleph_0 & \text{if } t > t_a \end{cases}
\]

\[
f(t) = \begin{cases} \aleph_0 & \text{if } t < t_b \\
0 & \text{if } t = t_b \end{cases}
\] (7)

Otherwise, if there would exist an instant \(t\) such that \(f^*(t) = n\) or \(f(t) = n\), being \(n\) a natural number, then there would also exist the impossible firsts \(n\) elements of an \(\omega^*\)—ordered sequence, or the impossible lasts \(n\) elements of an \(\omega\)—ordered sequence.

According to 459, the functions \(f^*\) and \(f\) are well defined for every \(t\) in \([t_a, t_b]\); they map the interval \([t_a, t_b]\) onto the set of two elements \(\{0, \aleph_0\}\):

\[
f^* : [t_a, t_b] \mapsto \{0, \aleph_0\}
\] (8)

\[
f : [t_a, t_b] \mapsto \{0, \aleph_0\}
\] (9)

The function \(f^*\) defines, therefore, a dichotomy \((t^*\text{-dichotomy})\):

- Regarding the number of elapsed \(t^*\)-instants when time passes from \(t_a\) to \(t_b\) only two values are possible: 0 and \(\aleph_0\).

The function \(f\) also defines a dichotomy, \((t\text{-dichotomy})\):

- Regarding the number of \(t\)-instants to elapse as time passes from \(t_a\) to \(t_b\) only two values are possible: \(\aleph_0\) and 0.
With respect to the number of $t^*$-instants elapsed from $t_a$, the passing of time from $t_a$ to $t_b$ can only exhibit two states: the state $S^*(0)$ at which no $t^*$-instant has elapsed, and the state $S^*(\aleph_0)$ at which infinitely many $t^*$-instants have already elapsed. Intermediate finite states $S^*(n)$ at which only a finite number $n$ of $t^*$-instants have elapsed are not possible. The passing of time becomes $S^*(\aleph_0)$ directly from $S^*(0)$. Similarly, with respect to the number of $t$-instants not yet elapsed, the passing of time from $t_a$ to $t_b$ can only exhibit two states: $S(\aleph_0)$ and $S(0)$; without intermediate finite states $S(n)$ at which only a finite number $n$ of $t$-instants have yet to elapse. The passing of time becomes $S(0)$ directly from $S(\aleph_0)$.

While the transition $S^*(0) \rightarrow S^*(\aleph_0)$ poses no additional problem, it could be argued the transition $S(\aleph_0) \rightarrow S(0)$ makes no sense because the subtraction of infinite cardinals is not always permitted in transfinite arithmetic. This is so absurd as to say that time does not pass because transfinite cardinals cannot always be subtracted.\(^6\) In any case, what would make no sense would be the arithmetic analysis of that transition. But arithmetic analysis has nothing to do with the type of reasoning we have made use of and we will make use of in the next discussion. A reasoning that is exclusively based on a consequence of $\omega$—order and $\omega^*$—order: that being $n$ any natural number, neither the first elements of an $\omega^*$—ordered sequence nor the last elements of an $\omega$—ordered sequence do exist.

If time passes from $t_a$ to $t_b$, as it has to pass, then the successive $t$-instants will also pass and at $t_b$ all of them will have passed. The transition will take place whether or not we have an appropriate definition for cardinal subtraction. And if it takes place, it will last for a time equal to or greater than zero. This is all we need to know in order to perform the next logical analysis of the transition $S(\aleph_0) \rightarrow S(0)$.

\(^6\)The subtraction of transfinite cardinals may lead to contradictions.
We will now examine the duration of the transitions:

\[
S^*(0) \rightarrow S^*(\aleph_0) \tag{10}
\]

\[
S(\aleph_0) \rightarrow S(0) \tag{11}
\]

According to (7) the number of \(t^*-\)instants elapsed from \(t_a\) and the number of \(t\)-instants not yet elapsed from \(t_a\) are well defined along the whole interval \([t_a, t_b]\). On the other hand, both transitions must take place within the same time interval \([t_a, t_b]\).

Although the real interval \([t_a, t_b]\) is densely ordered, the sequences \(\langle t^*_{ia} \rangle\) and \(\langle t_i \rangle\) within \([t_a, t_b]\), are not. These sequences are \(\omega^*\)-ordered and \(\omega\)-ordered respectively, which means that \(t^*-\)instants and \(t\)-instants are strictly successive, i.e. between any \(t^*-\)instant and its immediate successor no other \(t^*-\)instant exists; and the same applies to \(t\)-instants. Thus, \(t^*-\)instants and \(t\)-instants can only elapse successively, one at a time, and in such a way that between any two of those successive instants a time greater than zero always goes on. In consequence, the number of \(t^*-\)instants elapsed from \(t_a\) can only increase one by one, from 0 to \(\aleph_0\). The same applies to the way the number of \(t\)-instants not yet elapsed from \(t_a\) decreases from \(\aleph_0\) to 0. This successiveness will play an important role in the next discussion.

As a consequence of \(t^*-\)dichotomy, the number of \(t^*-\)instants elapsed from \(t_a\) has to increase one by one from 0 to \(\aleph_0\) without traversing the increasing sequence of natural numbers 1, 2, 3, \ldots. Analogously, the number of \(t\)-instants to elapse has to decrease one by one from \(\aleph_0\) to 0 without traversing the decreasing sequence of natural numbers \ldots, 3, 2, 1 (see Figure 25.2).

![Figure 25.2](image-url)
interval of time within \([t_a, t_b]\) during which the number of \(t^*\)-instants elapsed from \(t_a\) increases, \textit{one by one and with a non-zero time interval between each increment}, from zero to \(\aleph_o\). Similarly, the duration of the transition \(S(\aleph_o) \to S(0)\) is the interval of time within \([t_a, t_b]\) during which the number of \(t\)-instants to elapse decreases, \textit{one by one and with a non-zero time interval between each decrement}, from \(\aleph_o\) to zero.

469 Since a time greater than zero always elapses between any two successive \(t^*\)-instants, a time greater than zero must necessarily elapse between an infinite number of successive \(t^*\)-instants. This is why the transition \(S^*(0) \to S^*(\aleph_o)\) will necessarily take a time greater than zero. The same conclusion, and for the same reason, applies to the transition \(S(\aleph_o) \to S(0)\).

470 It is worth noting we are not calculating the exact duration of the transitions \(S^*(0) \to S^*(\aleph_o)\) and \(S(\aleph_o) \to S(0)\) but demonstrating both of them must of necessity last a timer greater than zero. The exact duration of those transitions cannot be calculated because there is neither a first instant at which the transition \(S^*(0) \to S^*(\aleph_o)\) begins, nor a last instant at which the transition \(S(\aleph_o) \to S(0)\) ends. But, indeterminable as they may be, both durations have to be greater than zero for the reason given in 469. We will next prove, however, that no positive real number can be the duration of those transitions.

471 Assume the transition \(S^*(0) \to S^*(\aleph_o)\) lasts a time \(\tau\), being \(\tau\) any positive real number. Let \(\tau^*\) be any instant within the real interval \((0, \tau)\). According to \(t^*\)-dichotomy, the number of elapsed \(t^*\)-instants at \(t_a + \tau^*\) is \(\aleph_o\), and then the transition \(S^*(0) \to S^*(\aleph_o)\) already has finished. Consequently the transition \(S^*(0) \to S^*(\aleph_o)\) lasts a time less than \(\tau\). And being \(\tau\) any real number greater than 0, we must conclude the duration of \(S^*(0) \to S^*(\aleph_o)\) is less than any real number greater than zero. And this is possible only if that duration is null.

472 An argument similar to 471 proves the transition \(S(\aleph_o) \to S(0)\) has also to be instantaneous. It could be argued the transition \(S(\aleph_o) \to S(0)\) lasts a time \(t_b - t_a\), but this is impossible because at \(t_a + \tau\), being \(\tau\) any positive real number less than \(t_b - t_a\), the number of \(t\)-instants not yet elapsed is \(\aleph_o\), and then the transition \(S(\aleph_o) \to S(0)\) has not begun. In consequence it lasts a time less than \(t_b - t_a\).

473 According to 471 and 472, infinitely many successive \(t^*\)-instants and infinitely many successive \(t\)-instants have to elapse simultaneously. But this is impossible because the successive \(t^*\)-instants and \(t\)-instants cannot elapse simultaneously: between any two of those successive \(t^*\)-instants \(t^*_{n, t^*_{n+1}}\) (or \(t\)-instants \(t_n, t_{n+1}\)) a finite interval of time greater than zero always passes: just the interval \([t^*_{n, t^*_{n+1}}]\) (or the interval \([t_n, t_{n+1}]\) in the case of \(t\)-instants). The transitions \(S^*(0) \to S^*(\aleph_o)\) and \(S(\aleph_o) \to S(0)\) have to last a time greater than zero, but they cannot last a time greater than zero (471-472). We have therefore two contradictions proving the impossibility of dividing any finite interval of time into an actual infinitude of \(\omega^*\)-ordered parts and into an actual infinitude of \(\omega\)-ordered parts (see Z-Clock in Figure 25.2).
As a last resort, some infinitists claim that it make no sense to try to calculate the duration of the transitions \( S^*(0) \rightarrow S^*(\aleph_0) \) and \( S(\aleph_0) \rightarrow S(0) \) just because there are neither first element in \( \omega^* \)-ordered sequences nor last element in \( \omega \)-ordered sequences. But here we have not been trying to calculate the duration of those transitions, we have simply trying to prove they have to last a time greater than zero but cannot last a time greater than zero.

Any denumerable infinite partition of time must be \( \alpha \)-ordered or \( \alpha^* \)-ordered, being \( \alpha \) an ordinal of the second class (first or second kind). Thus, we will have:

\[
\alpha = \omega
\]  

or:

\[
\alpha = \omega + \beta
\]

where \( \beta \) is an ordinal or the second class (first or second kind). In consequence, any transfinite partition of time has to contain at least an impossible \( \omega \)-ordered (or \( \omega^* \)-ordered ) partition. Denumerable partitions of time are therefore impossible. And since any non-countable division contains infinitely many denumerable divisions we must conclude time is not infinitely divisible in consistent terms.

If in the place of the passage of time and the sequences of \( t^* \)-instants and \( t \)-instants we were considered the uniform linear motion of a particle traversing the \( Z^* \) points \( \langle z^*_i \rangle \) and \( Z \)-points \( \langle z_n \rangle \) defined within the real interval \([0, 1]\) of the real line as:

\[
\langle z^*_i \rangle : \quad z^*_i = \frac{1}{2^i}, \quad \forall i \in \mathbb{N}
\]

\[
\langle z_i \rangle : \quad z_i = \frac{2^i - 1}{2^i}, \quad \forall i \in \mathbb{N}
\]

we would have come to the same conclusion, and for the same reasons, on the infinite divisibility of space we have come on the infinite divisibility of time.

The above conclusions on the divisibility of space and time not only apply to space and time but to the very notion of densely ordered continuum.
Appendix A

The problem of change

INTRODUCTION

478 Change is the most pervasive characteristic of our continuously evolving universe. But change is also the most elusive and difficult question we have ever been faced with.\(^1\) So elusive that it could be inconsistent, as it has been claimed at least from pre-Socratic times.\(^2\) Evidently, if that were the case the task of explaining nature in consistent terms would be impossible. In this appendix we will prove that change is inconsistent in the spacetime continuum, although it could find a solution in certain discrete spacetimes as those of cellular automata.

479 For the sake of simplicity, and in order to avoid unnecessary complications, we will discuss here the problem of causal changes in physical macroscopic objects. So, if Ob is one of those macroscopic objects, we will say Ob changes causally from the state \(S_a\) to the state \(S_b\) if there exist a set of (physical) laws \(L\) such that, under the same conditions \(C\) and as a consequence of those laws and conditions, the state of Ob is \(S_a\) at instant \(t_a\) and \(S_b\) at an ulterior instant \(t_b\). In symbols:

\[
\begin{align*}
\text{Causal change} & \quad \{S_a \mapsto S_b \}\hspace{1cm} (1) \\
& \quad L(S_a, C, t_a) = (S_b, t_b)
\end{align*}
\]

Since we will only deal with causal changes (1), from now on they will be referred to simply as changes.

480 The change \(S_a \mapsto S_b\) can be direct, without intermediate states, in which case it will be referred to as canonical change. It can also be the result of an ordered sequence of canonical changes:

\[
\{S_i\} : S_a \mapsto S_1 \mapsto S_2 \mapsto S_3 \mapsto \ldots \mapsto S_b \hspace{1cm} (2)
\]

Notice that every element \(S_n\) of \(\{S_i\}\) must have an immediate predecessor \(S_{n-1}\).

\(^1\)For a general background see [139], [168] and the particular view of H. Bergson in [20], [21].

\(^2\)Not only pre-Socratic authors as Parmenides or Zeno of Elea claimed the impossibility of change, modern authors as J.E. McTaggart also defended that impossibility [132]
(except the first of them \(S_a\)) so that \(S_n\) can be causally derived from \(S_{n-1}\):

\[
\forall S_{n\neq a} \in \{S_i\} : \mathcal{L}(S_{n-1}, C_{n-1}, t_{n-1}) = (S_n, t_n)
\]  

(3)

The objective of our discussion will exclusively be canonical changes, be them or not forming part of a sequence of canonical changes. But before focusing our attention exclusively on canonical changes we must analyze a second possibility for a change to occur.

481 Indeed, infinitists claim that a change can also be the result of completing a densely ordered sequence of non canonical changes (one in which between any two changes infinitely many other changes do occur\(^3\)). For this reason, and before discussing the problem of canonical change (the classical problem of change) we will prove the impossibility for a change to occur as a consequence of completing a densely ordered sequence of non canonical changes. Recall that the infinitud of a densely ordered sequence may be numerable (as in the case of rational numbers) or non-denumerable (as in the case of real numbers). The distinction is irrelevant to our discussion. For the sake of simplicity we will choose the case of denumerable dense order, whose cardinal is \(\aleph_0\).

482 In the first place, it is quite clear that in a densely ordered sequence of changes no change can be canonical. In fact, if \([S_a, S_b]\) is a densely ordered sequence of changes and \(S_\lambda\) is any element of the sequence then it is impossible that \(S_\lambda\) results from a canonical change of an immediate predecessor \(S_\mu\), simply because in a densely ordered sequence no element has an immediate predecessor. Therefore, \(S_\mu\) cannot immediately precede \(S_\lambda\) and then the canonical change:

\[
\mathcal{L}(S_\mu, C_\mu, t_\mu) = (S_\lambda, t_\lambda)
\]

is impossible

483 Assume \(S_a \rightarrow S_b\) takes place through a densely ordered sequence of non canonical changes \([S_a, S_b]\). The state \(S_b\) results, therefore, from the completion of a denumerable and densely ordered sequence of changes. Thus, the state of our object \(Ob\) will be \(S_a\) at a certain instant \(t_a\), and \(S_b\) at another posterior instant \(t_b\). In those conditions, let \(f(t)\), for each \(t\) in \([t_a, t_b]\), be the number of those changes that still have to be performed at instant \(t\) in order to reach \(S_b\). It is immediate that \(f(t)\) can only take two values: either \(\aleph_0\) or 0. In effect, if \(f(t)\) could take a finite value \(n\) then there would exist the impossible lasts \(n\) changes of a densely ordered sequence of changes.

484 According to 483, \(f(t)\) defines a dichotomy: the number of changes to be performed at each instant \(t\) in \([t_a, t_b]\) to reach \(S_b\), can only be either \(\aleph_0\) or 0. In consequence, there is no instant within \([t_a, t_b]\) at which only a finite number of changes remain to be performed in order to reach \(S_b\). Or in other words, that number has to change directly from \(\aleph_0\) to 0 without intermediate states at

\(^3\)It is hard to explain in physical terms what on earth a sequence of non canonical changes could really be.
which only a finite number of changes remain to be performed. In consequence infinitely many changes have to occur simultaneously.

485 If infinitely many changes take place simultaneously then infinitely many states will also exist simultaneously and then none of those states could be the cause of any other of those states. In consequence causal changes, in the sense given by (1), are not possible through a densely ordered sequence of changes. In addition, instantaneous changes could never take place in the continuum spacetime (see 488).

THE PROBLEM OF CHANGE

486 Consider any canonical change $S_a \mapsto S_b$ of any object $Ob$. We will begin by proving that change must be instantaneous, i.e. of a null duration. In fact, assume its duration is $\tau > 0$, being $\tau$ any positive real number. For every $\tau'$ in the real interval $(0, \tau)$, the state of our object $Ob$ will be either $S_a$ or $S_b$. If it were $S_a$ then the change would not yet have begun and its duration would be equal or less than $\tau - \tau'$ in the place of $\tau$. If it were $S_b$ then the change would have already finished and its duration would be equal or less than $\tau'$ in the place of $\tau$. But $Ob$ must be in one of those two states because $S_a \mapsto S_b$ is a canonical change. Consequently, the duration of the canonical change $S_a \mapsto S_b$ is less than any real number greater than zero. Therefore, it must be instantaneous.

487 We will now prove that instantaneous changes (of a null duration) are impossible in a spacetime continuum. As we will see, the reason for that impossibility is that if $t$ is any instant of a densely ordered sequence of instants then $t$ has neither immediate predecessor $p(t)$ nor immediate successor $s(t)$, such that no time elapses neither from $p(t)$ to $t$ nor from $t$ to $s(t)$, in the same way no natural number exists between 4 and 5, or between 5 and 6.

488 Assume the instantaneous canonical change $S_a \mapsto S_b$ takes place at a certain instant $t$ of the spacetime continuum. The change would be instantaneous if the state of $Ob$ were $S_a$ at instant $t$ and $S_b$ at an hypothetical immediate successor $s(t)$ of $t$, being null the time elapsed between $t$ and $s(t)$. But in the spacetime continuum this is impossible because $t$ has not immediate successor $s(t)$, so that between any two different instants of the spacetime continuum a time greater than zero always passes.

489 To propose the coexistence of $S_a$ and $S_b$ at a certain instant as a solution to the problem of change $S_a \mapsto S_b$ means to pose the problem of that change in terms of the change $S_a \mapsto (S_aS_b)$, where $(S_aS_b)$ stands for that supposed coexistence of states. And the same would apply to the changes $S_a \mapsto (S_a(S_aS_b))$, $S_a \mapsto (S_a(S_a(S_aS_b)))$, etc.

490 We have then proved that:

1. Causal changes cannot take place through a densely ordered sequence of changes (see 484-485).
2. Canonical changes take place instantaneously (see 486).
3. Instantaneous changes are impossible in the spacetime continuum (see 488).

We must, therefore, conclude that it holds the following:

**Theorem of change.**--Change is impossible in the spacetime continuum.

491 Being change so pervasive in our current universe, theorem of change could be indicating the spacetime continuum is not the most appropriate representation of space and time. Space and time could, in fact, be of a discrete nature. In the next section we will analyze the possibility that change may occur in discrete spacetimes.

492 Before analysing the possibility of change in discrete spacetimes, let us summarize the above argument 481-490 in spatial terms (space is involved in many physical changes, for instance in motion or change of position). In the space continuum, points have not immediate successor and this pose an additional difficulty to the problem of change. Indeed, argument 481-490 can be entirely rewritten in geometrical terms by replacing the concept of instant with the concept of point and the concept of temporal instantaneousness with the concept of spatial null-extension.

493 Consider a change of position performed at a finite velocity \(v\) from a point \(a\) to another point \(b\) through the real interval \([a, b]\) in the space continuum. Since no point within \([a, b]\) has immediate successor, the trip from \(a\) to \(b\) can only take place through a densely ordered (and now uncountable) sequence of points. So, let \(f(x)\) be the number of points to be traversed at any point \(x\) within \([a, b]\) in order to reach \(b\). This function can only take two values:

\[
f(x) = 2^{\aleph_0} \text{ for any } x \text{ within } [a, b] \quad (5)
\]

\[
f(x) = 0 \text{ just at point } b \quad (6)
\]
According to 493, \( f(x) \) is well defined along the whole interval \( [a, b] \) and then it defines a dichotomy: the number of points to be traversed at each point \( x \) in \( [a, b] \) to reach point \( b \), can only be either \( 2^{\aleph_0} \) or 0. In consequence, there is no point \( x \) within \( [a, b] \) at which only a finite, or denumerable infinite, number of points remains to be traversed in order to reach \( b \). Or in other words, that number has to change directly from \( 2^{\aleph_0} \) to 0.

The fact that the number of points to be traversed in order to reach \( b \) changes directly from \( 2^{\aleph_0} \) to 0 is only possible if the traversal is performed at an infinite velocity, which is not the case, or if infinitely many points occupy a null extension.

In fact, assume the transition from \( 2^{\aleph_0} \) to 0 in the number of points to be traversed in order to reach \( b \) takes place through a certain traversal \( \xi \) of length \( \epsilon \), being \( \epsilon \) any positive real number, including \( b - a \). At any \( x \) within the interval \((0, \epsilon)\) the number of points to be traversed in order to reach \( b \) can only be \( 2^{\aleph_0} \) (dichotomy of \( f(x) \)) and then the traversal \( \xi \) has not yet begun. Therefore, the length of \( \xi \) is less than \( \epsilon \), for any positive real number \( \epsilon \). In consequence, no real number greater than zero exists so that it could be the length of \( \xi \). Therefore, \( \xi \) can only have a null length.

For the same lack of successiveness as in the case of time intervals, all space intervals between any two different points do have a length greater than zero. So intervals of a null length between two different points are impossible in the space continuum. Consequently, the change of position at a finite velocity from point \( a \) to point \( b \) is consistently impossible in the space continuum.

Notice we have not been trying to calculate the exact distance along which the change from \( 2^{\aleph_0} \) to 0 in the number of points to be traversed in order to reach \( b \) takes place. We have been simply trying to prove, on the one hand, that that distance can only be null (\( 2^{\aleph_0} \) or 0 dichotomy); and on the other, that null distances between two different points are impossible in the space continuum.

**A discrete model: cellular automata**

Cellular automata like models (CALM) provide a new interesting perspective to analyze the way the universe could be evolving. It provides a discrete spacetime framework that makes it possible a new analysis of some of the apparently unsolvable problems and paradoxical situations in modern physics. As we will see in the next short discussion, twenty seven centuries after it was posed, the old problem of change could find a first consistent solution in the discrete spacetime of CALMs.

In CALMs, space is exclusively composed of indivisible minimal units: qukits (quantum space units). Time is also composed of a sequence of successive indivisible units: quuits (quantum time units). No extension exists between a qukit and its immediate successor in any spatial direction. Similarly, no time
elapses between a qutit and its immediate successor. Each qusit can exhibit

different states, each defined by a certain set of variables. The states of all

qusits change simultaneously at each successive qutit in accordance with the

laws driving the evolution of the automaton. Once changed, the state of each

qusit remains unchanged for a qutit. In what follows we will assume this is

the case, although in the place of one qutit, the state of each qusit could also

remain unchanged for a certain (integer) numbers of qutits.

152 — A.-The problem of change
instantaneous changes: the state $A_n$ at qubit $\tau_n$ changes to $A_{n+1}$ at the next qubit $\tau_{n+1}$. And this is possible because the state of each qubit is updated at each qubit and maintained just for one qubit. At least, we could say that in Cellular Automata-like models, the problem of change does not arise.

504 The case of the quantum jump, or quantum leap, may be an appropriate example of canonical change. An electron, for instance, is in the state $S_1$ at a certain instant $t_1$ and in the state $S_2$ at another posterior instant $t_2$, without being in any intermediate state between $S_1$ and $S_2$ because of the quantum nature of the jump. It is therefore a canonical change. In the spacetime continuum the interval $(t_1, t_2)$ must always be greater than zero and during that time the electron cannot be neither at $S_1$ nor at $S_2$. During that time the electron simply cannot exist. It must disappear at $t_1$ and reappear at $t_2$. In the digital spacetime of a CALM all we have to do is to consider two successive qubits, $\tau_1$ and $\tau_2$. At $\tau_1$ our electron would be in the state $S_1$ and at $\tau_2$ in the state $S_2$.

505 By way of example, assume that:

- The universe has $2.66 \times 10^{185}$ qubits.
- The universe contains $10^{80}$ elementary particles.
- Each particle is defined by $p$ variables
- Each particle is, somehow, present at each qubit.

Let U-CALM be a 3D-CALM of $2.66 \times 10^{185}$ qubits in which the state of each qubit is defined by $p \times 10^{80}$ state variables. If it were possible to build U-CALM, perhaps we could observe the self-organizing and evolution of an object similar to our universe.

506 The problem is that U-CALM could never be built within the universe not even by making use of all its elementary particles. Other thing would be its theoretical analysis. U-CALM would be incomparably less complex than, for instance, any matrix of infinite elements (which are usual in mathematics and theoretical physics). We could model the universe, provided we know the basic laws that make it self-organize and evolve. In this circumstances, to simulate does not mean to reproduce the exact history of the universe: recursive qubit interactions and the resulting non-linear dynamics open the door of unexpectedness and creativity, as in the case of the terrestrial biosphere.

507 In any case, and as noted in 506, we could theorize on U-CALM, we could use it as a theoretical reference to grasp the essence, magnitude and possibilities of real universes. Colosal as it may seem, U-CALM would be a finite object and then composed of a number of elements incomparably less than the number of points ($2^{\aleph_0}$) of the simple interval (0, 1) in the spacetime continuum.
In addition, while the points within (0,1) have no physical significance, each element of U-CALM would be plenty of physical meaning.

508 To conclude this chapter, let us imagine we build a very advanced computer game in which its characters evolve until they become aware of their own intelligence. When trying to explain their digital universe, they would surely have the same type of problems we have when trying to explain the incessant changes we observe in our digital universe.
Appendix B
Suggestions for a natural theory of sets

INTRODUCTION
509 In my opinion, modern set theories are excessively tortuous and complicated thanks to the following three reasons:

1. The platonic scenario where they all have been formally founded and developed, which means that sets are considered as platonic objects whose existence is mind independent.

2. The restrictions to avoid self-reference, another platonic assumption of presocratic origins. To assume semantic self-reference means to assume the existence of autolanguages, of languages with the ability to refer to themselves. From a non-platonic perspective, on the other hand, only men have the ability to refer to other objects, including himself. From this perspective, self-reference is an exclusive ability of men, and then only language and the successive metalanguages are considered.

3. The hypothesis of the actual infinity subsumed into the Axiom of Infinity according to which the infinite sets do exist as complete totalities.

This appendix suggests another foundational alternative far away from the platonic scenario: the natural scenario of mind intentional activities.

510 The discussion that follows is in fact founded on a natural (non platonic) definition of set. It also introduces the concept of uncompletable sequences, via the successor set definition. Uncompletable sequences of successor sets are then used to define the uncompletable sequence of finite cardinals and then the concept of potentially infinite set.

A NATURAL DEFINITION OF SET
511 We assume here sets and natural numbers are elementary theoretical objects resulting from our intentional mind activity. Consequently they don’t have mind independent existence.

512 Perhaps the most basic mind intentional process consists in considering (focusing our attention on) any object or group of objects. There are, in turn, two basic ways to consider objects, either successively or simultaneously. The first leads to the concept of natural number; the second to the concept of set.
When we consider successively different objects we are in a certain way counting them. A natural number is a sort of measurement of the amount of successively considered objects (see below). On the other hand, if we consider simultaneously different objects we are grouping them into a totality that is a new object different from each of the considered objects. Accordingly, let us propose the following natural definition of set suggested by Lewis Carroll [46]:

**Definition of set.**—A set is the theoretical object that results from a mental process of grouping arbitrary elements previously defined.

Obviously the physical world is plenty of natural groups of objects, for instance the set of all ions within a certain crystal of pyrite. Human mind has the ability to recognize those natural groups but also the ability to define many other arbitrary sets, that may include abstract and imaginary objects.

Obviously, Definition 513 is of a constructive nature: it only indicates the way sets are constructed: by mental processes of grouping. Being constructive, it is not a circular semantic definition. Sets are defined as theoretical objects because human mind can only construct theoretical objects. Furthermore, Definition (513) requires the elements to be grouped have to be previously defined (either by enumeration or by comprehension). This seems a reasonable requirement, otherwise we would not know what we are grouping, what we are defining.

On the other hand, that simple requirement (to be defined before to be grouped) invalidates self-referring sets. In fact, according to it, a set cannot belong to itself because it does not exist as an element that may be grouped until the set has been defined. Paradoxes as those of Cantor (set of all cardinals), Burali-Forti (set of all ordinals) and Russell (set of all sets that do not belong to themselves) are immediately ruled out.

Let us now compare the above constructive definition of set with the following two platonic attempts due to G. Cantor:

By a 'manifold' or 'aggregate' I generally understand every multiplicity which can be thought of as a one, i.e. any totality of definite elements which by means of a law can be bound up into a whole, and I believe that in this I am defining something which is related to the Platonic eidos or idea ([41, page 93]).

By an 'aggregate' (Menge) we are to understand any collection into a whole \( M \) of definite and separate objects \( m \) of our intuition or our thought. ([37, p. 481], [40, 85])

Since 'multiplicity' and 'collection' are synonymous of 'set' both definitions are circular. Circularity could not be avoided in all subsequent attempts to define the notion of platonic set, and it was finally declared as undefinable, i.e. as a primitive concept that cannot be defined in terms of other more basic concepts. The impossibility to define platonic sets probably indicates that sets are not the platonic objects they were assumed to be but products of our intentional mind activity.
Fortunately, most of the symbols, conventions and operations of classic axiomatic set theories can be preserved in non platonic set theories. Particularly the notions of membership, subset, empty set, union, intersection, correspondences and the like. By contrast, most of the axioms needed in platonic set theories become unnecessary in non-platonic scenarios.

As we will see in this appendix, one of the most significant notions in a constructive theory of sets is that of successor set, which follows immediately from Definition 513. Indeed, it is immediate to prove the following:

**Theorem of the successor set.**—Each set \(A\) defines a new set, its successor set \(s(A)\), of which it is an element

Proof.—Once defined a set \(A\), we will have at our disposal a new object, the set \(A\), and according to Definition 513, we can group it with any other arbitrary elements previously defined. For instance with the elements just used to define \(A\). So we can define a new set \(S(A)\) as:

\[
s(A) = A \cup \{A\} \tag{1}
\]

\(s(A)\) is said the successor set of the set \(A\). As we will see the concept of successor may be used to define, also in constructive terms, the successive natural numbers.

By uncompletable we mean here something that not only is uncomplete but also that cannot be completed. In line with this idea, we will define the notion of uncompletable sequence as:

**Definition 520.**—An uncompletable sequence is one whose elements can never be considered as a complete totality, in the sense that we can always increase the sequence of considered elements by considering new elements.

The notion of successor set allows to define uncompletable sequences of sets. Assume that through successive definitions of successor sets of an initial set \(A\):

\[
A, \ s(A), \ s(s(A)), \ldots \ s(s(s(s(A)))) \tag{2}
\]

we get a final set \(X\):

\[
X = s(s(s(s(s(A)))))) \tag{3}
\]

whose successor set cannot be defined. Whatevsoever be the set \(X\), it will be a well defined object and then, according to Definition 513, we can group it with any arbitrary elements previously defined, including \(X\)’s elements, to form a new set. So we can define:

\[
s(X) = X \cup \{X\} = s(s(s(s(s(s(s(s(s(s(s(s(s(s(A)))))))))))))), \tag{4}
\]

Consequently it is false that the successor set of \(X\) cannot be defined. Thus
the sequence of successive successor sets of $A$ is in fact uncompletable because we can always increase the sequence of considered sets by considering a new element, namely the successor sets of the last set just defined. We can therefore assert the following:

**Theorem of the sequence of successors.**—The sequence of successor sets of any set is uncompletable.

**Sets and Numbers**

522 Although several constructive and formal attempts to define the concept of number have been carried out, this concept could in fact be primitive, non-definable in terms of other more basic concepts. In any case we can assume that two sets have the same *number* of elements if they can be put into a one to one correspondence. All sets that can be put into a one to one correspondence among each other define a class of sets, and then a number: the cardinal of all sets of that class. The cardinal of a set $A$ is usually denoted by $|A|$.

523 To count the elements of a set $A$ means finally to consider successively each one of its elements. We could define a number (name, numeral and properties) each time we consider a new element of $A$ as an indication of the quantity of the considered elements, as an indication of the size of the set. Though in this context number, quantity and size are semantically indistinguishable and then the attempt of definition is also circular. After all, perhaps only operative definitions of the concept of number are possible. We will immediately introduce one of them.

524 One of the best known uncompletable sequence of successor sets is the following one based on the notion of *empty set* $\emptyset$:

\[
\begin{align*}
\emptyset &= \emptyset \\
s(\emptyset) &= \emptyset \cup \{\emptyset\} = \{\emptyset\} \\
s(s(\emptyset)) &= \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\} \\
s(s(s(\emptyset))) &= \{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \\
&\ldots
\end{align*}
\]

525 We call finite cardinals, or natural numbers, just to the cardinals of the above successive sets (Von Neumann definition of 1923 [142]):

\[
\begin{align*}
|\emptyset| &= 0 \\
|\{\emptyset\}| &= 1 \\
|\{\emptyset, \{\emptyset\}\}| &= 1 + 1 = 2 \\
|\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}| &= 2 + 1 = 3 \\
|\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}| &= 3 + 1 = 4 \\
&\ldots
\end{align*}
\]
where we write +1 to indicate a new element has been added to the precedent set in order to define the new set and its corresponding new finite cardinal. For the sake of clarity, we will write the above sequence of finite cardinals as:

\[
\begin{align*}
|\emptyset| &= 0 \\
|\{\emptyset\}| &= |\{0\}| = 1 \\
|\{\emptyset, s^1(\emptyset)\}| &= |\{0, 1\}| = 1 + 1 = 2 \\
|\{\emptyset, s^1(\emptyset), s^2(\emptyset)\}| &= |\{0, 1, 2\}| = 2 + 1 = 3 \\
|\{\emptyset, s^1(\emptyset), s^2(\emptyset), s^3(\emptyset)\}| &= |\{0, 1, 2, 3\}| = 3 + 1 = 4 \\
&\ldots
\end{align*}
\]

where \(s^2(\emptyset)\) is \(s(s(\emptyset))\), \(s^3(\emptyset)\) is \(s(s(s(\emptyset)))\) and so on. Note each cardinal \(n\) is recursively defined in terms of the previously defined \(n - 1\), except the first of them 0.

526 Notice also the above definition of the successive finite cardinals or natural numbers is only an operational definition. Ultimately we lack of an appropriate definition of number. So, to say the cardinal of a set is the number of its elements is to say nothing from a strictly formal point of view. But we need to define the cardinal of a set as the number of its elements even if the concept of number is not properly defined but accepted as a primitive concept.

527 According to 520 the above sequence (14)-(19) is uncompletable so that no last finite cardinal exists. In fact, whatsoever be the finite cardinal \(n\) we consider, we will have:

\[
n = |\{\emptyset, s^1(\emptyset), s^2(\emptyset), \ldots s^{n-1}(\emptyset)\}|
\]

and since the sequence of successor sets is uncompletable in accord with 521, the successor set of \(s^{n-1}(\emptyset)\) does exists, and then we can write:

\[
s^n(\emptyset) = s^{n-1}(\emptyset) \cup \{s^{n-1}(\emptyset)\} \\
= \{\emptyset, s^1(\emptyset), s^2(\emptyset), \ldots s^{n-1}(\emptyset), s^n(\emptyset)\}
\]

In accordance with (14)-(19) the set \(s^n(\emptyset)\) defines the finite cardinal \(n + 1\):

\[
|s^n(\emptyset)| = |\{\emptyset, s^1(\emptyset), s^2(\emptyset), \ldots s^n(\emptyset)\}| = |\{0, 1, 2, \ldots n\}| = n + 1
\]

We can therefore assert that being \(n\) a finite cardinal (natural number) of the uncompletable sequence (14)-(19), \(n + 1\) is also a finite cardinal or natural number of the uncompletable sequence (14)-(19). Thus, we can write:

**Theorem of the sequence of cardinals.** If \(n\) is a finite natural number, and then the cardinal of a member of the uncompletable sequence of successor sets of the empty set, then \(n + 1\) is also a
finite natural number and then the cardinal of a set of the same uncompletable sequence.

528 It is worth noting this constructive way of defining natural numbers based on definition 513 does not pose any problem of existence and consequently goes without the aid of auxiliary axioms (as Peano’s axioms). This is so because we are not trying to define the set of natural numbers as a complete mind-independent totality, but as an uncompletable and operational sequence of successive terms.

529 Since all sets of the same cardinality are equipotent we can say that a natural number \(n\) is the immediate successor of other natural number \(m\) (or that \(m\) is the immediate predecessor of \(n\)) if \(n\) is the cardinal of the successor set of any set of cardinal \(m\). Or in other words, if \(n = m + 1\). Evidently if \(n\) is the immediate successor of \(m\) then it is also a successor (though not immediate) of all predecessors of \(m\). To be a successor induces an order relation \(<\) in the set of finite cardinals that coincides with the natural order of precedence of natural numbers (the natural order of counting): \(m < n\) if and only if \(n\) is a successor of \(m\).

530 Let us now consider the set \(N_n\) of the first \(n\) natural numbers:

\[
N_n = \{1, 2, 3, \ldots, n\}
\]  
(24)

We will prove the following:

**Theorem 530.** The cardinal of the set \(N_n\) of the firsts \(n\) natural numbers is just \(n\).

**Proof.** By definition, \(n\) is the cardinal of the set:

\[
A = \{\emptyset, s^1(\emptyset), s^2(\emptyset), \ldots, s^{n-1}(\emptyset)\}
\]  
(25)

Let \(f\) be a function from \(N_n\) to \(A\) defined as:

\[
f(i) = \begin{cases} 
\emptyset & \text{if } i = 1 \\
\text{s}^{i-1}(\emptyset) & \text{if } i > 1 
\end{cases}
\]  
(26)

It is clear that \(f\) is a one to one correspondence. Therefore \(N_n\) and \(A\) are equipotent, i.e. the cardinal of \(N_n\) is \(n\).

531 As a consequence of the recursive way they are defined, the elements of \(N_n\) exhibit a type of ordering we will call *natural order* and denote by \(n\)-order, whose main characteristics are:

1. There is a first element: the only one without predecessors (1).
2. There is a last element: the only one without successors (\(n\)).
3. Each element \(k\) has an immediate successor \(k + 1\), except the last one.
4. Each element has \(k\) an immediate predecessor \(k - 1\), except the first one.
Note that \( n \)-order is the same as \( \omega \)-order except that in \( \omega \)-order there is not a last element. Thus, \( \omega \)-ordered sets are complete totalities (as the actual infinity requires) although no last element completes them. Evidently, this is not the case of \( n \)-ordered sets.

**Finite sets**

532 As is well known, the hypothesis of the actual infinity subsumed into the Axiom of Infinity states the existence of a set equipotent with the set of all finite cardinals (and then with that of natural numbers) considered as a complete totality, as if the above sequence (5)-(8) could in fact be actually completed. By contrast, in a non-platonic theory of sets that sequence is uncompletable and then cannot be considered as a complete totality. That sequence is an example of potentially infinite object. In the next section we will introduce them in a form a little more detailed. In this one we will focus our attention on finite sets. To begin with, consider the following elementary definition based on the above sequences of successor sets and finite cardinals:

**Definition 532.**-A set is finite if, and only if, it has a finite cardinal.

The above theorems and definitions allow to prove the following results on the finite sets.

533 **Theorem of the finite ordering 1.**- Every finite set can be \( n \)-ordered.

Proof.-Let \( A \) be any finite set. According to Definition 532, there will be a finite cardinal \( n \) such as \(|A| = n\). Being \( A \) equipotent with all sets of the same cardinality it will equipotent to the \( n \)-ordered set \( \mathbb{N}_n \) of the first \( n \) finite cardinals whose cardinal is \( n \) in accord with 530. So a one to one correspondence \( f \) exists between \( \mathbb{N}_n \) and \( A \). Accordingly, we can write:

\[
A^* = \{f(1), f(2), f(3), \ldots, f(n)\}
\]  

which is the \( n \)-ordered version of the set \( A \).

534 **Theorem of the finite ordering 2.**-Every finite set is \( n \)-ordered.

*Proof.* In agreement with the theorem of the finite ordering 1, any finite set \( A \) of cardinal \( n \) can be \( n \)-ordered by means of a one to one correspondence \( f \) between \( \mathbb{N}_n \) and \( A \), so that we can write:

\[
A^* = \{f(1), f(2), \ldots, f(n)\} \quad (n\text{-ordered version of } A)
\]  

Since the sequence \( f(1), f(2), \ldots f(n) \) contains all elements of the set \( A \), the original ordering of this set can only be one of the finitely many reorderings of \( f(1), f(2), \ldots f(n) \), i.e. one of the \( n! \) permutations of \( f(1), f(2), \ldots f(n) \). Since each permutation of \( f(1), f(2), \ldots f(n) \) changes the indexed elements but not the \( n \)-ordered set of indexes \( \{1, 2, \ldots n\} \), each permutation is \( n \)-ordered. Therefore \( A \) is \( n \)-ordered.
Theorem of the next cardinal.-If $A$ is a finite set of cardinal $n$ then its successor set $S(A) = A \cup \{A\}$ is a finite set of cardinal $n + 1$.

Proof.-Since the cardinal of the set $A$ is $n$ and, according to 530, the cardinal of $N_n$ is also $n$ there will be a one to one correspondence $f$ between $A$ and the set $N_n = \{1, 2, 3, \ldots n\}$. The one by one correspondence $g$ defined by:

\[
g : A \cup \{A\} \mapsto \{1, 2, 3, \ldots n, n + 1\}
\]

\[
\forall a \in A : g(a) = f(a)
\]

\[
g(A) = n + 1
\]

proves $S(A)$ is a finite set whose cardinal $n + 1$.

Theorem of the finite extension.-If $A$ is a finite set and $b$ an element which does not belong to $A$ then the set $A \cup \{b\}$ is also finite.

Proof.-Let $f$ be a correspondence between the sets $A \cup \{b\}$ and $s(A)$ defined by:

\[
f(a) = a, \forall a \in A
\]

\[
f(b) = A
\]

Evidently $f$ is a bijection between $A \cup \{b\}$ and $s(A)$. So these sets have the same cardinality. Let $n$ be the cardinal of $A$, according to theorem 535 of the next cardinal, the cardinal of $s(A)$ is the finite cardinal $n + 1$. Thus the cardinal of $A \cup \{b\}$ is also $n + 1$. Consequently $A \cup \{b\}$ is a finite set.

Theorem of the finite union.-If $A$ and $B$ are any two finite sets then the set $A \cup B$ is also finite.

Proof.-Being $B$ finite it is $n$-ordered and its elements can be represented as $b_1, b_2, \ldots b_k$. According to theorem 536 of the finite extension, the successive sets:

\[
A \cup \{b_1\}
\]

\[
A \cup \{b_1\} \cup \{b_2\}
\]

\[
\vdots
\]

\[
A \cup \{b_1\} \cup \{b_2\} \cdots \cup \{b_k\} = A \cup B
\]

are all them finite.

Theorem of infinity.-The set $N$ of finite cardinals defined in accordance with (14)-(19) is not finite.

Proof.-Let us assume it is finite. According to Definition 532 it will have a finite cardinal $n$, which is also the cardinal of the $(n - 1)$th successive successor set of $\emptyset$ in (5)-(8). According to 527 this sequence is uncompletable so that the $n$th term, and then the finite cardinal $n + 1$, also exists. Therefore $n$ is not the cardinal of $N$. This proves that no finite cardinal $n$ can be the cardinal of $N$. Therefore $N$ is not finite.
Potentially infinite sets

As far as I know, potentially infinite sets have never deserved the attention of mathematicians. Probably because set theories are infinitist theories founded and developed by infinitists that assume the hypothesis of the actual infinity. From the above constructive perspective we can only consider the ability of our minds to perform endless (uncompletable) process as that of counting or defining in recursive terms. The objects resulting from those uncompletable processes could be used to define sets in the sense of Definition 513. But those sets could never be considered as complete totalities, as in the case of finite sets. Those uncompletable totalities would represent the set theoretical version of the potential infinity introduced by Aristotle twenty four centuries ago [11, Book VIII].

The following one, could be an operative definition of potentially infinite sets:

**Definition 540.**-A set $X$ is potentially infinite if for any finite set $A$ there exist a finite subset $B$ of $X$ such that $|A| < |B|$.

An immediate consequence of Definition 540 is the following:

**Theorem 541.**-Potentially infinite sets do not have definite cardinals.

Proof.-Let $X$ be any potentially infinite set. Assume it has a finite cardinal $n$. Consider the set $N_n$ of the first $n$ finite cardinals, whose cardinal is just $n$ according to 530. In accord with Definition 540 and being $N_n$ finite, there exists a finite subset $B$ of $X$ such that $|N_n| < |B|$. Therefore, $n$ cannot be the cardinal of the set $X$. Consequently no finite cardinal $n$ is the cardinal of the potentially infinite set $X$.

In the universe of non platonic sets, a set can only be either finite (with a finite cardinal) or potentially infinite (without a definite cardinal). The above theorem of infinity probes the set of finite cardinals is not finite. We will now see it is potentially infinite.

**Theorem 542.**-The set $\mathbb{N}$ of finite cardinals is potentially infinite.

Proof.-Let $A$ be any finite set and let $n$ be its cardinal. The set $B$ of the firsts $n + 1$ finite cardinals satisfies:

$$B = \{1, 2, \ldots, n, n + 1\} \subset \{1, 2, \ldots, n, n + 1, n + 2\} \subset \ldots$$

(38)

It is, therefore, a finite proper subset of $\mathbb{N}$. And evidently $|A| < |B|$. In consequence, and according to Definition 540, $\mathbb{N}$ is potentially infinite.

We will finally prove the following basic result:

**Theorem 543.**-If $X$ is a potentially infinite set and $A$ any of its finite subsets then the set $X - A$ is also potentially infinite.
Proof.-Evidently we will have:

\[ X = A \cup (X - A) \]  \hspace{1cm} (39)

So if \( X - A \) were finite then, according to the theorem 537 of the finite union, the set \( X \) would also be finite. Consequently \( X - A \) must be potentially infinite.
Appendix C
Platonism and biology

Living beings as extravagant objects

544 In 1973 Dobzhansky published a celebrated paper whose title summarizes modern biological thought [64]:

Nothing in Biology Makes Sense Except in the Light of Evolution.

I think it would have been more appropriate to write reproduction in the place of evolution. And not only because evolution is powered by reproduction. Mainly it is because only reproduction may account for the extravagances of living beings.

545 Living beings are, in fact, extravagant objects, i.e. objects with properties that cannot be derived exclusively from physical laws. To have red feathers, or yellow feathers, or to move by jumping, or to be devoured by the female in exchange for copulating with it, are examples (and the list would be interminable) of properties that cannot be derived exclusively from physical laws but from the peculiar competitive and reproductive history of each organism. Thus, living beings are subjected to a biological law that dominates over all physical laws, the Law of Reproduction: Reproduce as you might.

546 The informed nature of living beings [112] and the law of reproduction make it possible the fixation of arbitrary extravagances. The success in reproducing depends upon certain characteristics of living beings that frequently have nothing to do with the efficient accomplishment of physical laws but with arbitrary preferences such as singing, or dancing, or having brilliant colors. Although, on the other hand, to achieve reproduction is previously necessary to be alive, which in turn involves a lot of functional abilities related to the particular ecological niche each living being occupies. But this is in fact secondary: adapted and efficient as an organism may be, if it does not reproduce, all its physical excellence will be immediately removed from the biosphere. The Law of Reproduction opens the door to innovations in living beings, and then almost anything can be expected. Even writing this.

1 Of course, evolution is a natural process and denying it is so stupid as denying photosynthesis or glycolysis. Other thing is its theoretical explanation. As any scientific theory, the theory of organic evolution remains unfinished and currently opened to numerous discussions. See for instance [179], [25], [184], [160], [167], [125], [70], [159], [48], [86], [166], [47] etc.
Biology and abstract knowledge

Living beings are topically viewed as systems efficiently adapted to their environment. No attention is usually payed to their extravagant nature, although being extravagant is a very remarkable feature. We, living beings, are the only (known) extravagant objects in the Universe. By the way, those extravagances could only be the result of a capricious evolution rather than of an intelligent design. Capricious evolution restricted by the physical laws governing the world. One of the latest extravagances appeared in the biosphere is the consciousness exhibited by most of the human beings. Surely, that sensation of individual subjectivity is responsible for some peculiar ways of interpreting the world, as platonic essentialism, the belief that ideas do exist independently of the mind that elaborate them.

Animals do have the ability to compose abstract representations of their environment, particularly of all those objects and processes involved in their survival and reproduction. A leopard, for instance, has in its brain the (abstract) idea of gazelle, it knows what to do with a gazelle (as is well known by gazelles), whatsoever be the particular gazelle it encounters with. The abstract idea of gazelle, and of any other thing, is elaborated in the brain by means of different components (the so called atoms of knowledge) that not only serve to form the idea of gazelle but of many other abstract ideas. And not only ideas, sensorial perceptions are also elaborated, by similar processes, in atomic and abstract terms, which surely also serves to filter the irrelevant details of the highly variable and useless information coming from the physical world, and thus to identify with sufficient security the (biologically) significant objects and process that form part of their ecological niches.

To have the ability of composing abstract representations of the world is indispensable for animals in order to survive and reproduce, And a mistake in this affair may cost them the higher of the prices. A ball rolling down towards a precipice will not stop to avoid falling down; but the dog running behind it, will do; dogs know gravity and its consequences. Animals interact with their surroundings and need to know its singularities, its peculiar ways of being and evolving, i.e. its physical logic, and even its mathematical logic.

Animals need abstract representations of the physical world, and that is not a minor detail (the maintenance and continuous functioning of this internal representation of the world consumes up to 80% of the total energy consumed by a human brain [157].) It must be an efficient and precise representation, if not animal life would be impossible. It is through their own actions and experiences, including imitation and innovation that they develop their Neu-

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2[207], [137]
3Primates and humans could dispose of neural networks to deal with numbers [61], [62], [97].
4[108], [80], [161], [201]
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Abstract knowledge is then indispensable for animal life.

Perception and cognition are constructive neuronal processes in which elementary units of abstract knowledge are involved. The processes take place in different brain areas, as we are now beginning to know with certain detail. This way of functioning seems incompatible with platonist essentialism. Accordingly, concepts and ideas seem to be brain elaborations rather than transcendent entities we have the ability to connect with. Through our personal cognitive actions and experiences (that, in addition, have a transpersonal cumulative potential through cultural heritage and cultural networks) we have end up by developing that great cognitive system we call science.

The consciousness of ideas and the ability of recursive thinking (perhaps an exclusive ability of humans) could have promoted the raising and persistence of platonist essentialism. But that way of thinking is simply incompatible with both evolutionary biology and neurobiology. It seems reasonable that Plato were platonist in Plato times, but it is certainly surprising the persistence of that old way of thinking in the community of contemporary mathematicians. Though, as could be expected, a certain level of disagreement on this affair also exists. It is remarkable the fact that many non platonist authors, such as Wittgenstein, were against both the actual infinity and self-reference, two capital concepts in the history of platonist mathematics.

The reader may come to his own conclusions on the consequences the above biological criticism of platonist essentialism could have on self-reference and the actual infinity. Although, evidently, he can also maintain that he does not know through neural networks and persists in his platonist habits. But for those of us that believe in the organic nature of our brains and in its abilities of perceiving and knowing modeled through more than 3600 millions years of organic evolution, platonism has no longer sense. The actual infinity and self-reference could lose all their meaning away from the platonist scenario.

In my opinion, the actual infinity hypothesis is not only useless in order to explain the physical world, it is also annoying in certain disciplines as quantum gravity and quantum electrodynamics (renormalization). Physics and even mathematics could go without it. Experimental sciences as chemistry, biology and geology have never been related to it. The potential infinity would suffice.

Even the number of distinguishable sites in the universe could be fi-

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5[162], [54], [178], [55], [109], [56], [172]
6[53], [97]
7[119], [113], [120], [14]
8[77], [105], [115], [204], [170], [182], [8].
9[175], [177]
10[141], [176]
11Except transfinite arithmetic and other related areas, most of contemporary mathematics are compatible with the potential infinity, including key concepts as those of limit or integral
12Some contemporary cosmological theories, as the theory of multiverse or the theory of
nite [102]. Matter, energy, and electric charge seem to be discrete entities with indivisible minima; space and time could also be of the same discrete nature, as is being suggested from some areas of contemporary physics.\textsuperscript{13}

Beyond Planck’s scale nature seems to lose all its physical sense. As the actual infinity and self-reference, the continuum spacetime could only be a useless rhetorical device. The reader can finally imagine the enormous simplification of mathematics and physics once liberated from the burden of the actual infinity and self-reference. Perhaps we should give Ockham razor a chance.
Bibliography


[31] _____, *The Purpose of Zeno’s Arguments on Motion*, Isis III (1920-21), 7–20.


[34] _____, *Grundlagen einer allgemeinen Mannichfaltigkeitslehre*, Mathemathischen Annalen 21 (1883), 545 – 591.


[38] _____, *Beiträge zur Begründung der transfiniten Mengenlehre*, Mathematishe Annalen XLIX (1897), 207 – 246.


[60] Richard Dedekind, *Qué son y para qué sirven los números (was sind Und was sollen die Zahlen(1888))*, Alianza, Madrid, 1998.


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