Series: **PHYSICS and ASTRONOMY**

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Principle extremum of full action

Annotation

A new variational principle extremum of full action is proposed, which extends the Lagrange formalism on dissipative systems. It is shown that this principle is applicable in electrical engineering, mechanics, taking into account the friction forces. Its applicability to electrodynamics and hydrodynamics is also indicated. The proposed variational principle may be considered as a new formalism used as an universal method of physical equations derivation, and also as a method for solving these equations. The formalism consists in building a functional with a sole saddle line; the equation that describes it presents the equation with dynamic variables for a certain domain of physics. The solution method consists in a search for global saddle line for given conditions of a physical problem.

Contents

Introduction

Here we formulate principle extremum of full action, allowing to construct a functional for various physical systems and, which is most important, for dissipative systems.

The first step in building such functional is to write for a certain physical system an equation of energy conservation or an equation of powers balance. There we take into account as the energy losses (for example, for friction or heating), as also the energy flow into the system and from it.

Here we shall describe this principle applied to electric engineering and mechanics.

1. The Principle Formulation

The Lagrange formalism is widely known – it is an universal method of deriving physical equations from the principle of least action. The action here is determined as a definite integral - functional

$$S(q)=\int_{t_1}^{t_2}\left(K(q)-P(q)\right)dt \qquad (1)$$

from the difference of kinetic energy $K(q)$ and potential energy $P(q)$, which is called Lagrangian

$$\Lambda(q)=K(q)-P(q). \qquad (2)$$

Here the integral is taken on a definite time interval $t_1 \le t \le t_2$, and q is a vector of generalized coordinates, dynamic variables, which, in their turn, are depending on time. The principle of least action states that the extremals of this functional (i.e. the equations for which it assumes the minimal value), on which it reaches its minimum, are equations of real dynamic variables (i.e. existing in reality).

For example, if the energy of system depends only on functions q and their derivatives with respect to time q', then the extremal is determined by the Euler formula [1]

$$\frac{\partial(K-P)}{\partial q}-\frac{d}{dt}\left(\frac{\partial(K-P)}{\partial q'}\right)=0, \qquad (3)$$

As a result we get the Lagrange equations.

The Lagrange formalism is applicable to those systems where the full energy (the sum of kinetic and potential energies) is kept constant. The principle does not reflect the fact that in real systems the full energy (the sum of kinetic and potential energies) decreases during motion, turning into other types of energy, for example, into thermal energy Q, i. e. there occurs energy dissipation. The fact, that for dissipative systems (i.e., for system with energy dissipation) there is no formalism similar to Lagrange formalism, seems to be strange: so the physical world is found to be divided to a harmonious (with the principle of least action) part, and a chaotic ("unprincipled") part.

The author puts forward the **principle extremum of full action**, applicable to dissipative systems. We propose calling full action a definite integral – the functional

$$\Phi(q)=\int_{t_1}^{t_2}\Re(q)dt \tag{4}$$

from the value

$$\Re(q)=\left(K(q)-P(q)-Q(q)\right), \tag{5}$$

which we shall call energian (by analogy with Lagrangian). In it $Q(q)$ is the thermal energy. Further we shall consider a full action quasiextremal, having the form:

$$\frac{\partial(K-P)}{\partial q}-\frac{d}{dt}\left(\frac{\partial(K-P)}{\partial q'}\right)-\frac{\partial Q}{\partial q}=0. \tag{6}$$

Functional (4) reaches its extremal value (*defined further*) on quasiextremals. The principle extremum of full action states that the quasiextremals of this functional are equations of real dynamic processes.

Right away we must note that the extremals of functional (4) coincide with extremals of functional (1) - the component corresponding to $Q(q)$, disappears

Let us determine the extremal value of functional (5). For this purpose we shall "split" (i.e. replace) the function $q(t)$ into two independent functions $x(t)$ and $y(t)$, and the functional (4) will be associated with functional

$$\Phi_2(x,y)=\int_{t_1}^{t_2}\Re_2(x,y)dt, \tag{7}$$

which we shall call "split" full action. The function $\Re_2(x,y)$ will be called "split" energian. This functional is minimized along function $x(t)$ with a fixed function $y(t)$ and is maximized along function $y(t)$ with a fixed function $x(t)$. The minimum and the maximum are sole ones. Thus, the extremum of functional (7) is a saddle line, where one group of functions x_o minimizes the functional, and another - y_o, maximizes it. The sum of the pair of optimal values of the split functions gives us the sought function $q = x_o + y_o$, satisfying the quasiextremal equation (6). In other words, the quasiextremal of the functional (4) is a sum of extremals x_o, y_o of functional (7), determining the saddle point of this functional. It is important to note that this point is the sole extremal point – there is no other saddle points and no other minimum or maximum points. Therein lies the essence of the expression "extremal value on quasiextremals". Our **statement 1** is as follows:

> In every area of physics we may find correspondence between full action and split full action, and by this we may prove that full action takes global extremal value on quasiextremals.

Let us consider the relevance of statement 1 for several fields of physics.

2. Energian in Electrical Engineering

Full action in electrical engineering takes the form (1.4, 1.5), where

$$K(q) = \frac{Lq'^2}{2}, \;\; P(q) = \left(\frac{Sq^2}{2} - Eq \right), \;\; Q(q) = Rq'q. \tag{1}$$

Here stroke means derivative , q - vector of functions-charges with respect to time, E - vector of functions-voltages with respect to time, L - matrix of inductivities and mutual inductivities, R - matrix of resistances, S - matrix of inverse capacities, and functions $K(q),\ P(q),\ Q(q)$ present magnetic, electric and thermal energies correspondingly. Here and further vectors and matrices are considered in the sense of vector algebra, and the operation with them are written in short form. Thus, a product of vectors is a product of column-vector by row-vector, and a quadratic form, as, for example, $Rq'q$ is a product of row-vector q' by quadratic matrix R and by column-vector q.

The equation of quasiextremal (1.6) in this case takes the form:

$$Sq + Lq'' + Rq' - E = 0. \tag{2}$$

Substituting (1) to (1.5), we shall write the Energian (1.5) in expanded form:

$$\Re(q)=\left(\frac{Lq'^2}{2}-\frac{Sq^2}{2}+Eq-Rq'q\right). \tag{3}$$

Let us present the split energian in the form

$$\Re_2(x,y)=\begin{bmatrix}\left(Ly'^2-Sy^2+Ey-Rx'y\right)- \\ \left(Lx'^2-Sx^2+Ex-Rxy'\right)\end{bmatrix}. \tag{4}$$

Here the extremals of integral (1.7) by functions $x(t)$ and $y(t)$, found by Euler equation, will assume accordingly the form:

$$2Sx+2Lx''+2Ry'-E=0, \tag{5}$$

$$2Sy+2Ly''+2Rx'-E=0. \tag{6}$$

By symmetry of equations (5, 6) it follows that optimal functions x_0 and y_0, satisfying these equations, satisfy also the condition

$$x_0=y_0. \tag{7}$$

Adding the equations (5) and (6), we get equation (2), where

$$q=x_o+y_o. \tag{8}$$

Consequently, conditions (5, 6) are necessary for the existence of a sole saddle line. In [2, 3] showed that sufficient condition for this is that the matrix L has a fixed sign, which is true for any electric circuit.

Thus, the statement 1 for electrical engineering is proved. From it follows also **statement 2**:

> Any physical process described by an equation of the form (2), satisfies the principle extremum of full action.

Note that equation (2) is an equation of the circuit without knots. However, in [2, 3] has shown that to a similar form can be transformed into an equation of any electrical circuit (with any accuracy).

3. Energian in mechanics

Here we shall discuss only one example - line motion of a body with mass m under the influence of a force f and drag force kq', where k - known coefficient, q - body's coordinate. It is well known that

$$f=mq''+kq'. \tag{1}$$

In this case the kinetic, potential and thermal energies are accordingly:

$$K(q)=mq'^2/2,\ \ P(q)=-fq,\ \ Q(q)=kqq'. \tag{2}$$

Let us write the energian (1.5) for this case:

$$\Re(q)=mq'^2/2+fq-kqq'. \tag{3}$$

The equation for energian in this case is (1)/
Let us present the split energian as:

$$\Re_2(x,y)=\begin{bmatrix}\left(my'^2+fy-kx'y\right)-\\ \left(mx'^2+fx-kxy'\right)\end{bmatrix}. \tag{4}$$

It is easy to notice an analogy between energians for electrical engineering and for this case, whence it follows that Statement 1 for this case is proved. However, it also follows directly from Statement 2.

4. Mathematical Excursus

Let us introduce the following notations:

$$y'=dy/dt,\ \ \hat{y}=\int_0^t ydt. \tag{1}$$

There is a known Euler's formula for the variation of a functional of function $f(y,y',y'',...)$ [1]. By analogy we shall now write a similar formula for function $f(...,\hat{y},y,y',y'',...)$:

$$f(...,\hat{y},y,y',y'',...): \tag{2}$$

$$\text{var}=...-\int_0^t f'_{\hat{y}}dt+f'_y-\frac{d}{dt}f'_{y'}+\frac{d^2}{dt^2}f'_{y''}-... \tag{3}$$

In particular, if $f()=xy'$, then $\text{var}=-x'$; if $f()=x\hat{y}$, then $\text{var}=-\hat{x}$. The equality to zero of the variation (1) is a necessary condition of the extremum of functional from function (2).

5. Action for Powers

Further we shall refer to the power of energy (kinetic, potential, thermal) as to the variation of this energy in a time unit. We shall consider these powers as the functions of integral generalized coordinates $\hat{i}=q$ - integrals i from generalized coordinates q. We shall denote these powers as $\hat{K}(i),\ \hat{P}(i),\ \hat{Q}(i)$. It is important to note the following. The energy functions contain as an argument the generalized coordinates q and their derivatives q', q''. The energy functions contain as their arguments the integral generalized coordinates i, their derivatives i' and their integrals $\hat{i}$.

Let us consider action-2 for powers and define it as a definite integral - functional

$$\hat{S}(i) = \int_{t_1}^{t_2} \left(\hat{K}(i) + \hat{P}(i) \right) dt \tag{1}$$

from the sum of kinetic and potential powers

$$\hat{\Lambda}(i) = \hat{K}(i) + \hat{P}(i). \tag{2}$$

and we shall call this sum Lagrangian-2.

The principle of minimal action may be extended also on action-2, i.e. assert that the extremals of functional (1) are equations of real physical processes over the same integral generalized coordinates as quasiexstremals. But the extremals should be calculated by the formulas (4.3).

Example 1. Let us consider the example from Section 3, for which the equation (3.1) is applicable, or, if the thermal losses are absent,

$$f = m \cdot i'. \tag{3}$$

In this case the kinetic and potential powers are accordingly:

$$\hat{K}(i) = m \cdot i \cdot i', \quad \hat{P}(i) = -f \cdot i. \tag{4}$$

Let us write the Lagrangian-2 (2) for this case:

$$\hat{\Re}(i) = m \cdot i \cdot i' - f \cdot i. \tag{5}$$

The equation of extremal for functional (1) in this case coincides with equation (3).

Example 2. Let us consider the example from Section 2, for which the equation (2.2) is applicable, or, if the thermal losses are absent,

$$S\hat{i} + Li' - E = 0. \tag{6}$$

In this case the kinetic and potential powers are accordingly:

$$\hat{K}(i) = L \cdot i \cdot i', \quad \hat{P}(i) = S \cdot \hat{i} \cdot i - E \cdot i. \tag{7}$$

Let us write the Lagrangian-2 (2) for this case:

$$\hat{\Re}(i) = L \cdot i \cdot i' + S \cdot \hat{i} \cdot i - E \cdot i. \tag{8}$$

The equation of extremal for functional (1) in this case may be obtained by formula (4.3) and coincides with equation (6).

6. Full Action for Powers

In this case full action-2 is a definite integral - functional

$$\hat{\Phi}(i)=\int_{t_1}^{t_2}\hat{\Re}(i)dt \tag{1}$$

from the value

$$\hat{\Re}(i)=\left(\hat{K}(i)+\hat{P}(i)+\hat{Q}(i)\right), \tag{2}$$

which we shall call Energian-2. In this case we shall define full action quasiextremal-2 as

$$\frac{\partial\left(\frac{\hat{Q}}{2}+\hat{P}+\hat{K}\right)}{\partial i}=0. \tag{3}$$

Functional (1) assumes extremal value on these quasiextremals. **The principle extremal of full action-2** asserts that quasiextremals of this functional are equations of real dynamic processes over integral generalized coordinates i.

Let us now determine the extremal value of functional (1, 2). For this purpose we, as before, will "split" the function $i(t)$ to two independent functions $x(t)$ and $y(t)$, and put in accordance to functional (1) the functional

$$\hat{\Phi}_2(x,y)=\int_{t_1}^{t_2}\hat{\Re}_2(x,y)dt, \tag{4}$$

which we shall call "split full action-2. We shall call the function $\hat{\Re}_2(x,y)$ "split " Energian--2. This functional is being minimized by the function $x(t)$ with fixed function $y(t)$ and maximized by function $y(t)$ with fixed function $x(t)$. As before, the quasiextremal (3) of functional (1) is a sum $i = x_o + y_o$ of extremals x_o, y_o of the functional (4), determining the saddle point of this functional.

7. Energian-2 in mechanics

As in Section 3 we shall consider an example, for which the equation (3.1) is applicable, or

$$f=m\cdot i'+k\cdot i. \tag{1}$$

In this case the kinetic, potential and thermal powers are accordingly:

$$\hat{K}(i) = m \cdot i \cdot i', \quad \hat{P}(i) = -f \cdot i, \quad \hat{Q}(q) = k \cdot i^2. \qquad (2)$$

Let us write the energian-2 (6.2) for this case:

$$\hat{\Re}(i) = m \cdot i \cdot i' - f \cdot i + k \cdot i^2. \qquad (3)$$

Уравнение квазиэкстремали в этом случае принимает вид (1).

8. Energian-2 in Electrical Engineering

Let us consider an electrical circuit which equation has the form, (2.2) or

$$S \cdot \hat{i} + L \cdot i' + R \cdot i - E = 0. \qquad (1)$$

In this case the kinetic, potential and thermal powers are accordingly:

$$\hat{K}(i) = L \cdot i \cdot i', \quad \hat{P}(i) = S \cdot \hat{i} \cdot i - E \cdot i, \quad \hat{Q}(i) = R \cdot i^2. \qquad (2)$$

Let us write the energian-2 (6.2) for this case:

$$\hat{\Re}(i) = L \cdot i \cdot i' + S \cdot \hat{i} \cdot i - E \cdot i + R \cdot i^2. \qquad (3)$$

The equation of quasiextremal in this case assumes the form (1).

Let us now present the "split" Energian-2 as

$$\hat{\Re}_2(x, y) = \begin{bmatrix} S(x\hat{y} - \hat{x}y) + L(xy' - x'y) + \\ + R(x^2 - y^2) - E(x - y) \end{bmatrix}. \qquad (4)$$

The extremals of integral (6.4) by the functions $x(t)$ and $y(t)$, found according to equation (4.3), will assume accordingly the form:

$$2S\hat{y} + 2Ly' + 2Rx - E = 0, \qquad (5)$$

$$2S\hat{x} + 2Lx' + 2Ry - E = 0. \qquad (6)$$

From the symmetry of equations (5, 6) it follows that optimal functions x_0 and y_0, satisfying these equations, satisfy also the condition

$$x_0 = y_0. \qquad (7)$$

Adding together the equations (5) and (6), we get the equation (1), where

$$q = x_o + y_o. \qquad (8)$$

Therefore, the equation (1) is the necessary condition of the existence of saddle line. In [2, 3] it is shown that the sufficient condition for the existence of a sole saddle line is matrix L having fixed sign, which is true for every electrical circuit.

Conclusion

The functionals (1.7) and (6.4) have global saddle line and therefore the gradient descent to saddle point method may be used for calculating physical systems with such functional. As the global extremum exists, then the solution also always exists. Such method applied to electrical engineering and electro mechanics is described in [2, 3].

The author has applied the full time extremum principle for powers, and also the calculation method applicable for electrodynamics [2, 3] and hydrodynamics [4, 5].

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