ON THE VALIDITY OF THE RIEMANN HYPOTHESIS

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Abstract

In this paper, we have used the partial Euler product to examine the validity of the Riemann Hypothesis. The Dirichlet series with the Mobius function $M(s) = \sum_{n=1}^{\infty} \mu(n)/n^s$ has been modified and represented in terms of the partial Euler product by progressively eliminating the numbers that first have a prime factor 2, then 3, then 5, ..up to the prime number p_r to obtain the series $M(s, p_r)$. It is shown that the series $M(s)$ and the new series $M(s, p_r)$ have the same region of convergence for every p_r. Unlike the partial sum of $M(s)$ that has irregular behavior, the partial sum of the new series exhibits regular behavior as p_r approaches infinity. This has allowed the use of integration methods to compute the partial sum of the new series to determine its region of convergence and to provide an answer for the validity of the Riemann Hypothesis.

Keywords: Riemann zeta function, Mobius function, Riemann hypothesis, conditional convergence, Euler product.

Classification: Number Theory, 11M26

1 Introduction

The Riemann zeta function $\zeta(s)$ satisfies the following functional equation over the complex plain [1]

$$\zeta(1-s) = 2(2\pi)^2 \cos(0.5s\pi)\Gamma(s)\zeta(s), \tag{1}$$

where, $s = \sigma + it$ is a complex variable and $s \neq 1$.

For $\sigma > 1$ (or $\Re(s) > 1$), $\zeta(s)$ can be expressed by the following series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{2}$$

or by the following product over the primes p_i's

$$\frac{1}{\zeta(s)} = \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^s}\right). \tag{3}$$

where, $p_1 = 2$, $\prod_{i=1}^{\infty}(1 - 1/{p_i}^s)$ is the Euler product and $\prod_{i=1}^{r}(1 - 1/{p_i}^s)$ is the partial Euler product. The above series and product representations of $\zeta(s)$ are absolutely convergent for $\sigma > 1$.

The region of the convergence for the sum in Equation (2) can be extended to $\Re(s) > 0$ by using the alternating series $\eta(s)$ where

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \tag{4}$$

and

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \eta(s). \tag{5}$$

One may notice that the term $1 - 2^{1-s}$ is zero at $s = 1$. This zero cancels the simple pole that $\zeta(s)$ has at $s = 1$ enabling the extension (or analog continuation) of the zeta function series representation over the critical strip where $0 < \Re(s) < 1$.

It is well known that all of the non-trivial zeros of $\zeta(s)$ are located in the critical strip. Riemann stated that all non-trivial zeros were very probably located on the critical line $\Re(s) = 0.5$ [2]. There are many equivalent statements for the Riemann Hypothesis (RH) and one of them involves the Dirichlet series with the Mobius function.

The Mobius function $\mu(n)$ is defined as follows
$\mu(n) = 1$, if $n = 1$.
$\mu(n) = (-1)^k$, if $n = \prod_{i=1}^{k} p_i$, p_i's are distinct primes.
$\mu(n) = 0$, if $p^2|n$ for some prime number p.

The Dirichlet series $M(s)$ with the Mobius function is defined as

$$M(s) = \sum_{n=1}^{\infty} \frac{\mu(s)}{n^s}. \tag{6}$$

This series is absolutely convergent to $1/\zeta(s)$ for $\Re(s) > 1$ and conditionally convergent to $1/\zeta(s)$ for $\Re(s) = 1$. The Riemann hypothesis is equivalent to the statement that $M(s)$ is conditionally convergent to $1/\zeta(s)$ for $\Re(s) > 0.5$.

Gonek, Hughes and Keating [3] have done an extensive research into establishing a relationship between $\zeta(s)$ and its partial Euler product for $\Re(s) < 1$. Gonek stated "Analytic number theorists believe that an eventual proof of the Riemann Hypothesis must use both the Euler product and functional equation of the zeta-function. For there are functions with similar functional equations but no Euler product, and functions with an Euler product but no functional equation." In section 4, we will present a functional equation for $\zeta(s)$ using its partial Euler product. The method is based on writing the Euler product formula as follows

$$1/\zeta(s) = \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^s}\right) = \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right) \prod_{r+1}^{\infty} \left(1 - \frac{1}{p_i^s}\right).$$

The above equation is valid for $\sigma > 1$. To be able to represent $\zeta(s)$ in term of its partial Euler product for $\sigma \leq 1$, we have to replace the term $\prod_r^{\infty} (1 - 1/p_i^s)$ with an equivalent one that allows the analytic continuation for the representation of $\zeta(s)$ for $\sigma \leq 1$. Thus, the new term, that we need to introduce to replace $\prod_r^{\infty} (1 - 1/p_i^s)$, must have a zero that cancels the pole that $\zeta(s)$ has at $s = 1$. In the section 4, we will use the complex analysis to compute this new term and then represent $\zeta(s)$ in terms of its partial Euler product. In sections (2) and (5), we have introduced an alternative method to compute $\zeta(s)$ in terms of its partial Euler product. This alternative method is based on modifying the Dirichlet series with the Mobius function. In sections (6) and (7) we analyzed the results of these two methods to examine the validity of the Riemann Hypothesis

In this paper, we claim that the the Riemann Hypothesis is invalid. We support our claim by proving that the series $M(\sigma)$ is divergent for $\sigma < 1$. We have achieved this result by introducing a method to represent the Dirichlet series $M(s)$ (defined by Equation (6)) in terms of the partial Euler product. This task is achieved by first eliminating the numbers that have the prime factor 2 to generate the series $M(s, 2)$. For the series $M(s, 2)$, we then eliminate the numbers with the prime factor 3 to generate the series $M(s, 3)$, and so on, up to the prime number p_r. In essence, in sections 2, we have applied the sieving technique to modify the series $M(s)$ to include only the numbers with prime factors greater than p_r. In the literature, numbers with prime factors less than y are called y-smooth while numbers with prime factors greater than y are called y-rough. In essence, our approach is to compute the Dirichlet series over p_r-rough numbers. In section 3, we have shown that the series $M(s)$ and the new series $M(s, p_r)$ have the same region of convergence (Theorem 1).

So far, the efforts to use the series $M(\sigma)$ to examine the validity of the Riemann Hypothesis have failed due to the irregular behavior of the partial sum of the series $M(\sigma)$. In sections 5, 6 and 7, we have shown that the partial sum of the new series $M(\sigma, p_r)$ exhibits regular behavior as p_r approaches infinity. This has allowed the use of integration methods to compute the partial sum of the new series and consequently determine its region of convergence.

In section 4, we have used the complex analysis to compute a functional representation for $\zeta(s)$ in terms of its partial Euler product (Theorem 2). We have then used this theorem to represent the series $M(s, p_r)$ in terms of the prime counting function $\pi(x)$ (Theorem 3, where $\pi(x) = \mathrm{Li}(x) + J(x)$, $\mathrm{Li}(x)$ is the Logarithmic Integral of x and on RH, $J(x)$ is given by $O(\sqrt{x}\log x)$). In section 5, we have used integration methods to compute $M(1, p_r)$. We have then shown that $M(1, p_r)$ can be decomposed into two terms (Theorem 4). The first term, that we have called the regular component, is generated by the $\mathrm{Li}(x)$ component of the prime counting function. The second term is the irregular component. In sections 6 and 7, we have used the Fourier analysis methods to compute the irregular component for any value of s. These methods were also used to compute the irregular component in terms of $J(x)$ (Theorems 6 and 7). We have then exploited Theorem 6 to show that non-trivial zeros can be found arbitrary close to the line $\Re(s) = 1$.

2 Applying the Sieving Method to the Dirichlet Series $M(s)$.

The Dirichlet series $M(s)$ with the Mobius function is defined as

$$M(s) = \sum_{n=1}^{\infty} \frac{\mu(s)}{n^s},$$

where $\mu(n)$ is the Mobius function. Thus,

$$M(s) = 1 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{0}{4^s} - \frac{1}{5^s} + \frac{1}{6^s}....$$

It should be pointed out that our definition of $M(s)$ is different from Mertins function $M(x)$ defined in the literature as $M(x) = \sum_{1 \leq n \leq x} \mu(n)$.

Next, we introduce the series $M(s,2)$ by eliminating all the numbers that have a prime factor 2. Thus, $M(s,2)$ can be written as

$$M(s,2) = 1 - \frac{1}{3^s} - \frac{1}{5^s} - \frac{1}{7^s} + \frac{0}{9^s} - \frac{1}{11^s} - \frac{1}{13^s} + \frac{1}{15^s}.....$$

Our analysis to test the conditional convergence of these series ($M(s)$ and $M(s,2)$ for $\sigma \leq 1$) is based on comparing correspondent terms of these two series. Therefore, rearrangement and permutation of the terms may have a significant impact on analyzing the region of convergence of both series. Thus, it essential to have the same index for both series $M(s)$ and $M(s,2)$ refer to the same term. Hence, we will represent $M(s,2)$ as follows

$$M(s,2) = 1 + \frac{0}{2^s} - \frac{1}{3^s} + \frac{0}{4^s} - \frac{1}{5^s} + \frac{0}{6^s} - \frac{1}{7^s} - \frac{0}{8^s}....,$$

or

$$M(s,2) = \sum_{n=1}^{\infty} \frac{\mu(n,2)}{n^s}, \tag{7}$$

where

$\mu(n,2) = \mu(n)$, if n is an odd number,
$\mu(n,2) = 0$, if n is an even number.

The above series $M(s,2)$ can be further modified by eliminating all the numbers that have a prime factor 3 to get the series $M(s,3)$ where

$$M(s,3) = 1 - \frac{1}{5^s} - \frac{1}{7^s} - \frac{1}{11^s} - \frac{1}{13^s} - \frac{1}{17^s} - \frac{1}{19^s} - \frac{1}{23^s} + \frac{0}{25^s}....,$$

or more conveniently

$$M(s,3) = 1 + \frac{0}{2^s} - \frac{0}{3^s} + \frac{0}{4^s} - \frac{1}{5^s} + \frac{0}{6^s} - \frac{1}{7^s} - \frac{0}{8^s}....,$$

and so on.

Let $I(p_r)$ represent, in ascending order, the integers with distinct prime factors that belong to the set $\{p_i : p_i > p_r\}$. Let $\{1, I(p_r)\}$ be the set of 1 and $I(p_r)$ (for example, $\{1, I(2)\}$ is the set of square-free odd numbers), then we define the series $M(s, p_r)$ as

$$M(s,p_r) = \sum_{n=1}^{\infty} \frac{\mu(n,p_r)}{n^s}, \tag{8}$$

where

$\mu(n,p_r) = \mu(n)$, if $n \in \{1, I(p_r)\}$,
otherwise, $\mu(n,p_r) = 0$.

It can be easily shown that, for every prime number p_r, the series $M(s,p_r)$ converges absolutely for $\Re(s) > 1$. Furthermore, it can be shown that, for $\Re(s) > 1$, $M(s,p_r)$ satisfies the following equation

$$M(s) = M(s, p_r) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right). \tag{9}$$

Since

$$M(s) = \frac{1}{\zeta(s)} = \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^s}\right),$$

therefore we conclude that, for $\Re(s) > 1$, $M(s, p_r)$ approaches 1 as p_r approaches infinity.

3 The region of convergence for the series $M(s)$ and $M(s, p_r)$.

In this section, we will deal with the question of the relationship between the conditional convergence of the two series $M(s, p_r)$ and $M(s)$ over the strip $0.5 < \Re(s) \leq 1$. Theorem 1 establishes this relationship.

Theorem 1. *For $s = \sigma + it$, where $0.5 < \sigma \leq 1$ and for every prime number p_r, the series $M(s)$ converges conditionally if and only if the series $M(s, p_r)$ converges conditionally. Furthermore, $M(s)$ and $M(s, p_r)$ are related as follows*

$$M(s) = M(s, p_r) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right). \tag{10}$$

The proof of this theorem can be achieved either by applying the Cauchy convergence criteria or more conveniently by applying the complex analysis where we take advantage of the fact that both functions $\zeta(s)$ and $\zeta(s) \prod_{i=1}^{r}(1 - 1/p_i^s)$ have the same zeros (and a simple pole at $s = 1$) to the right of the line $\Re(s) = 1/2$.

In the following, we will use the complex analysis to prove Theorem 1 by using a method similar to the one outlined by Littlewood Theorem that shows that the Riemann Hypothesis is valid if and only if the sum $\sum_{n=1}^{\infty} \mu(n)/n^s$ is convergent to $1/\zeta(s)$ for every s with $\sigma > 0.5$. The prove of this theorem can be found in [7, Theorem 14.12] and it depends mainly on Lemma 3.12 of the same reference [7]. This Lemma states: Let $f(s) = \sum_{n=1}^{\infty} a_n/n^s$, where $\sigma > 1$, $a_n = O(\psi(n))$ being non-decreasing and $\sum_{n=1}^{\infty} |a_n|/n^{\sigma} = O(1/(\sigma - 1)^{\alpha})$ as $\sigma \to 1$. Then, if $c > 0$, $\sigma + c > 1$, x is not an integer and N is the integer nearest to x, we have

$$\sum_{n<x} \frac{a_n}{n^s} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s+w) \frac{x^w}{w} dw + O\left(\frac{x^c}{T(\sigma + c - 1)^{\alpha}}\right) + O\left(\frac{\psi(2x) x^{1-\sigma} \log x}{T}\right) + O\left(\frac{\psi(N) x^{1-\sigma}}{T|x - N|}\right)$$

To prove the first part of Theorem 1 (i.e. for $s = \sigma + it$ and $0.5 < \sigma \leq 1$, the series $M(s, p_r)$ converges conditionally if $M(s)$ converges conditionally), we note that for $\sigma > 1$,

$$M(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)},$$

and

$$M(s, p_r) = \sum_{n=1}^{\infty} \frac{\mu(n, p_r)}{n^s} = \frac{1}{\zeta(s) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right)}.$$

If we assume that $M(s)$ is convergent for $\sigma > h > 0.5$, then $\zeta(s)$ has no zeros in the complex plane to the right of the line $\Re(s) = h$ [7, Theorem 14.12]. Consequently, the function $\zeta(s)\prod_{i=1}^{r}(1 - 1/p_i^s)$ has no zeros in the complex plane to the right of the line $\Re(s) = h$. Thus, we may apply Lemma 3.12 [7] with $a_n = \mu(n, p_r)$, $f(s) = 1/(\zeta(s)\prod_{i=1}^{r}(1 - 1/p_i^s))$, $c = 2$ and x half an odd integer to obtain (refer to [7, Theorem 14.12])

$$\sum_{n<x} \frac{\mu(n, p_r)}{n^s} = \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{1}{\zeta(s+w)\prod_{i=1}^{r}\left(1 - \frac{1}{p_i^{s+w}}\right)} \frac{x^w}{w} dw + O\left(\frac{x^2}{T}\right)$$

However, by the calculus of residues we have

$$\frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{1}{\zeta(s+w)\prod_{i=1}^{r}\left(1 - \frac{1}{p_i^{s+w}}\right)} \frac{x^w}{w} dw = \frac{1}{\zeta(s)\prod_{i=1}^{r}\left(1 - \frac{1}{p_i^{s}}\right)} +$$

$$\frac{1}{2\pi i}\left(\int_{2-iT}^{h-\sigma+\gamma-iT} + \int_{h-\sigma+\gamma-iT}^{h-\sigma+\gamma+iT} + \int_{h-\sigma+\gamma+iT}^{2+iT}\right) \frac{1}{\zeta(s+w)\prod_{i=1}^{r}\left(1 - \frac{1}{p_i^{s+w}}\right)} \frac{x^w}{w} dw$$

where, $0 < \gamma < \sigma - h$. Since, along the line of integration and for an arbitrary small ϵ, we have $1/\zeta(\sigma + iT) = O(T^\epsilon)$ [7], therefore the first and third integrals on right side of the above equation are given by $O(T^{-1+\epsilon}x^2)$ while the second integral is given by $O(x^{h-\sigma+\gamma}T^\epsilon)$. Hence

$$\sum_{n<x} \frac{\mu(n, p_r)}{n^s} = \frac{1}{\zeta(s)\prod_{i=1}^{r}\left(1 - \frac{1}{p_i^{s}}\right)} + O(T^{-1+\epsilon}x^2) + O(T^\epsilon x^{h-\sigma+\gamma})$$

Taking $T = x^3$, the $O-$terms tend to zero as x approaches infinity. Consequently, the partial sum $M(s, p_r; 1, x)$ is convergent as x approaches infinity and it is given by

$$M(s, p_r) = \sum_{n=1}^{\infty} \frac{\mu(n, p_r)}{n^s} = \frac{1}{\zeta(s)\prod_{i=1}^{r}\left(1 - \frac{1}{p_i^{s}}\right)}.$$

or

$$M(s) = M(s, p_r) \prod_{i=1}^{r}\left(1 - \frac{1}{{p_i}^s}\right).$$

The second part of the theorem can be also proved by first defining $M(s, p_r; N_1, N_2)$ as the partial sum

$$M(s, p_r; N_1, N_2) = \sum_{n=N_1}^{N_2} \frac{\mu(n, p_r)}{n^s}, \tag{11}$$

where $N_2 \geq N_1 \geq p_r$. Then, we have

$$M(s, p_{r-1}; 1, Np_r) = M(s, p_r; 1, Np_r) - \frac{1}{p_r^s} M(s, p_r; 1, N). \tag{12}$$

Since the series $M(s, p_r)$ is conditionally convergent, then the partial sums $M(s, p_r; 1, Np_r)$ and $M(s, p_r; 1, N)$ are both convergent to $M(s, p_r)$ as N approaches infinity. Hence, as N approaches infinity, we obtain

$$M(s, p_{r-1}) = \lim_{N\to\infty} M(s, p_{r-1}; 1, Np_r) = M(s, p_r)\left(1 - \frac{1}{p_r^s}\right).$$

By repeating this process $r-1$ times, we then obtain

$$M(s) = M(s, p_r)\prod_{i=1}^{r}\left(1 - \frac{1}{p_i^s}\right).$$

4 Functional representation of $\zeta(s)$ using its partial Euler product.

In this section, we will use the prime counting function to derive a functional representation for $\zeta(s)$ using its partial Euler product. We will start this task by first writing $\zeta(s)$ for $\sigma > 1$ as follows

$$1/\zeta(s) = \prod_{i=1}^{\infty}\left(1 - \frac{1}{p_i^s}\right) = \prod_{i=1}^{r}\left(1 - \frac{1}{p_i^s}\right)\prod_{r+1}^{\infty}\left(1 - \frac{1}{p_i^s}\right). \tag{13}$$

For $\sigma > 0.5$, we have

$$\log\prod_{i=r1}^{r2}\left(1 - \frac{1}{p_i^s}\right) = \sum_{i=r1}^{r2}\log\left(1 - \frac{1}{{p_i}^s}\right),$$

or

$$\log\prod_{i=r1}^{r2}\left(1 - \frac{1}{p_i^s}\right) = \sum_{i=r1}^{r2}\left(-\frac{1}{{p_i}^s} - \frac{1}{2{p_i}^{2s}} - \frac{1}{3{p_i}^{3s}} - ...\right).$$

Let δ_p be defined as the sum

$$\delta_p = \sum_{i=r1}^{r2}\left(-\frac{1}{2{p_i}^{2s}} - \frac{1}{3{p_i}^{3s}} - \frac{1}{4{p_i}^{4s}}...\right). \tag{14}$$

Thus,

$$\log\prod_{i=r1}^{r2}\left(1 - \frac{1}{p_i^s}\right) = -\sum_{i=r1}^{r2}\frac{1}{{p_i}^s} + \delta_p. \tag{15}$$

Since $|\delta_p| < \sum_{n=p_{r1}}^{\infty}\left(\frac{1}{2n^{2\sigma}} + \frac{1}{3n^{3s}} + \frac{1}{4n^{4s}}...\right)$, thus $\delta_p = O({p_{r1}^{1-2\sigma}}/(2\sigma-1))$. Furthermore, if $2\sigma - 1$ is a fixed positive number, then $\delta_p = O({p_{r1}^{1-2\sigma}})$.

Using the Prime Number Theorem (PNT) with a suitable constant $a > 0$, the number of primes less than x is given by [4, page 43]

$$\pi(x) = \mathrm{Li}(x) + J(x), \tag{16}$$

where $\mathrm{Li}(x)$ is the Logarithmic Integral of x and

$$J(x) = O\left(xe^{-a\sqrt{\log x}}\right), \tag{17}$$

or

$$J(x) = O\left(x/(\log x)^k\right), \tag{18}$$

where k is a number greater than zero.

Using Stieltjes integral [5], we may write the sum $\sum_{i=r1}^{r2} \frac{1}{p_i{}^\sigma}$ for $\sigma > 1$ as follows

$$\sum_{i=r1}^{r2} \frac{1}{p_i{}^\sigma} = \int_{x=p_{r1}}^{p_{r2}} \frac{d\pi(x)}{x^\sigma}. \tag{19}$$

Using Equation (18) for the representation of $\pi(x)$, we may then write the integral in Equation (19) as [5, Theorem 2, page 57]

$$\sum_{i=r1}^{r2} \frac{1}{p_i^\sigma} = \int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma} \frac{1}{\log x} dx + O\left(\frac{1}{(\log p_{r1})^k}\right), \tag{20}$$

where k is a number greater than zero. Therefore,

$$\sum_{i=r1}^{r2} \frac{1}{p_i^\sigma} = \int_{p_{r1}}^{\infty} \frac{1}{x^\sigma} \frac{1}{\log x} dx - \int_{p_{r2}}^{\infty} \frac{1}{x^\sigma} \frac{1}{\log x} dx + O\left(\frac{1}{(\log p_{r1})^k}\right). \tag{21}$$

Recalling that the Exponential Integral $E_1(r)$ is given by

$$E_1(r) = \int_r^\infty \frac{e^{-u}}{u} du,$$

and using the substitutions $u = (\sigma - 1)\log x$, $du = (\sigma - 1)dx/x$ and $x^\sigma/x = e^u$, then for $\sigma > 1$, we may write Equation (21) as

$$\sum_{i=r1}^{r2} \frac{1}{p_i^\sigma} = E_1\left((\sigma - 1)\log p_{r1}\right) - E_1\left((\sigma - 1)\log p_{r2}\right) + O\left(\frac{1}{(\log p_{r1})^k}\right). \tag{22}$$

Combining Equations (15) and ((22)) and noting that, for $\sigma > 1$, $E_1\left((\sigma - 1)\log p_{r2}\right)$ approaches zero as p_{r2} approaches infinity, we may write Equation (13) for $s = \sigma$ and $\sigma > 1$ as

$$-\log\zeta(\sigma) = \sum_{i=1}^{r} \log\left(1 - \frac{1}{p_i{}^\sigma}\right) - \sum_{i=r+1}^{\infty} \frac{1}{p_i{}^\sigma} + \delta_p,$$

or

$$\log\zeta(\sigma) + \sum_{i=1}^{r} \log\left(1 - \frac{1}{p_i{}^\sigma}\right) - E_1\left((\sigma - 1)\log p_{r+1}\right) = \epsilon,$$

where $\epsilon = O(1/(\log p_{r1})^k)$ is an arbitrarily small number attained by setting p_r sufficiently large. Therefore,

$$\zeta(\sigma)\prod_{i=1}^{r}\left(1 - \frac{1}{p_i^\sigma}\right)\exp\left(-E_1((\sigma - 1)\log p_{r+1})\right) = 1 + \epsilon + O(\epsilon^2). \tag{23}$$

As p_r approaches infinity, ϵ approaches zero. Hence, the right side of the above equation approaches 1 as p_r approaches infinity.

Similarly, for $\Re(s) > 1$, we can use the following expression for $E_1(s)$

$$E_1(s) = \int_1^\infty \frac{e^{-xs}}{x} dx,$$

to show that

$$\lim_{r\to\infty}\left\{\zeta(s)\prod_{i=1}^{r}\left(1-\frac{1}{p_i^s}\right)\exp\left(-E_1((s-1)\log p_{r+1})\right)\right\}=1. \tag{24}$$

Let the function $G(s,p_r)$ be defined as

$$G(s,p_r)=\zeta(s)\prod_{i=1}^{r}\left(1-\frac{1}{p_i^s}\right)\exp\left(-E_1((s-1)\log p_{r+1})\right) \tag{25}$$

where, $G(s,p_r)$ is a regular function for $\Re(s)>1$. Referring to Equation (24), the function $G(s,p_r)$ approaches 1 as p_r approaches infinity. It should be noted that, for every p_r, the function $\exp\left(-E_1((s-1)\log p_{r+1})\right)$ is an entire function, the function $\zeta(s)$ is analytic everywhere except at $s=1$ and the function $\prod_{i=1}^{r}(1-1/p_i^s)$ is analytic for $\Re(s)>0$. Thus, for any $\sigma>1$, the function $G(s,p_r)$ can be considered as a sequence of analytic functions. Furthermore, as p_r (or r) approaches infinity, this sequence is uniformly convergent over the half plane with $\sigma>1+\epsilon$ (where, ϵ is an arbitrary small number). Therefore, by the virtue of the Weiestrass theorem, the limit is also analytic function [6] (Weiestrass theorem states that if the function sequence f_n is analytic over the region Ω and f_n is uniformly convergent to a function f, then f is also analytic on Ω and ${f_n}'$ converges uniformly to f' on Ω). If we define this limit as $G(s)$, where

$$G(s)=\lim_{r\to\infty}G(s,p_r) \tag{26}$$

then, $G(s)$ is analytic over the half plane $\Re(s)>1$ and it is equal to 1 by the virtue of Equation (24).

Next, we will extend the above results to the line $s=1+it$. We will then show that if RH is valid, then for the strip $s=\sigma+it$ where $0.5<\sigma<1$, the above results will also be valid with the limit of $G(s,p_r)$ is 1 as p_r approaches infinity.

We will start this task by showing that although both $\zeta(s)$ and $E_1((s-1)\log p_{r+1})$ have a singularity at $s=1$, the product $G(s,p_r)$ has a removable singularity at $s=1$ for every p_r. This can be shown by first expanding $\zeta(s)$ as a Laurent series about its singularity at $s=1$

$$\zeta(s)=\frac{1}{s-1}+\gamma-\gamma_1(s-1)+\gamma_2\frac{(s-1)^2}{2!}-\gamma_3\frac{(s-1)^3}{3!}+...,\tag{27}$$

where γ is the Euler-Mascheroni constant and γ_i's are the Stieltjes constants. For $s=1+\epsilon$, where $\epsilon=\epsilon_1+i\epsilon_2$, ϵ_1 and ϵ_2 are arbitrary small numbers, the above equation can be written as

$$\zeta(s)=\frac{1}{\epsilon}+\gamma-\gamma_1\epsilon+\gamma_2\frac{\epsilon^2}{2!}-\gamma_3\frac{\epsilon^3}{3!}+... \tag{28}$$

Furthermore, for $\sigma>1$, using the definition of the Exponential Integral, we may write $E_1(s)$ as

$$E_1(s)=-\gamma-\log s+s-\frac{s^2}{2\,2!}+\frac{s^3}{3\,3!}-\frac{s^4}{4\,4!}+.... \tag{29}$$

Thus, for $s=1+\epsilon$, we have

$$\exp\left(-E_1((s-1)\log p_r)\right)=e^{\gamma}\epsilon\,\log p_r\,\exp\left(-\epsilon\log p_r+\frac{(\epsilon\log p_r)^2}{2\,2!}-\frac{(\epsilon\log p_r)^3}{3\,3!}+....\right). \tag{30}$$

By taking the product $\zeta(s)\exp\left(-E_1((s-1)\log p_r)\right)$ and allowing ϵ to approach zero, we then have

$$\lim_{s\to 1}\left\{\zeta(s)\ \exp\left(-E_1((s-1)\log p_r)\right)\right\} = e^{\gamma}\ \log p_r. \tag{31}$$

However, it is well known that the partial Euler product at $s=1$ can be written as [8]

$$\prod_{i=1}^{r}\left(1-\frac{1}{p_i}\right) = \frac{e^{-\gamma}}{\log p_r} + O\left(\frac{1}{(\log p_r)^2}\right). \tag{32}$$

Multiplying Equations (31) and (32), we then conclude that at $s=1$, $G(s,p_r)$ approaches 1 as p_r approaches infinity. Furthermore, for $s=1+it$ and $t\neq 0$, the value of $\exp(-E_1(it\log p_r))$ approaches 1 as p_r approaches infinity and since

$$\lim_{r\to\infty}\left\{\zeta(s)\prod_{i=1}^{r}\left(1-\frac{1}{{p_i}^s}\right)\right\} = 1,$$

therefore, for $s=1+it$, we have the following

$$\lim_{r\to\infty} G(s,p_r) = \lim_{r\to\infty}\left\{\zeta(s)\prod_{i=1}^{r}\left(1-\frac{1}{p_i^s}\right)\exp\left(-E_1((s-1)\log p_{r+1})\right)\right\} = 1.$$

So far, we have shown that the function $G(s,p_r)$ is uniformly convergent to 1 when $\Re(s)>1$. We have also shown that $G(s,p_r)$ is convergent to 1 for $\Re(s)=1$. In the following, we will show that, assuming the validity of the Riemann Hypothesis, the function $G(s,p_r)$ is uniformly convergent to 1 for every value of s with $\Re(s)>0.5+\epsilon$, where ϵ is an arbitrary small number. Toward this end, we will first show that the function $G(s,p_r)$ is convergent for any value of s on the real axis with $\sigma>0.5$. This can be achieved by first writing the expressions for $G(\sigma,p_{r1})$ and $G(\sigma,p_{r2})$ (where $r2$ is an arbitrary large number greater than $r1$)

$$G(\sigma,p_{r1}) = \zeta(\sigma)\exp\left(-E_1((\sigma-1)\log p_{r1+1})\right)\prod_{i=1}^{r1}\left(1-\frac{1}{p_i^{\sigma}}\right), \tag{33}$$

$$G(\sigma,p_{r2}) = \zeta(\sigma)\exp\left(-E_1((\sigma-1)\log p_{r2+1})\right)\prod_{i=1}^{r2}\left(1-\frac{1}{p_i^{\sigma}}\right). \tag{34}$$

Since the function $G(s,p_r)$ is analytic that is not equal to 0 for $\sigma>0.5$, hence we can divide Equation (34) by Equation (33) and then take the logarithm to obtain

$$\log\left(\frac{G(\sigma,p_{r2})}{G(\sigma,p_{r1})}\right) = E_1\left((\sigma-1)\log p_{r1+1}\right) - E_1\left((\sigma-1)\log p_{r2+1}\right) + \log\left(\prod_{i=r1+1}^{r2}\left(1-\frac{1}{{p_i}^{\sigma}}\right)\right). \tag{35}$$

To compute the logarithm of the partial Euler product in Equation (35), we recall Equation (15)

$$\log\prod_{r1+1}^{r2}\left(1-\frac{1}{p_i^s}\right) = -\sum_{i=r1+1}^{r2}\frac{1}{{p_i}^s} + \delta_p,$$

where $\delta_p = O(p_{r1}^{1-2\sigma}/(2\sigma-1))$. Furthermore, we have

$$\pi(x) = \mathrm{Li}(x) + J(x), \tag{36}$$

where $\mathrm{Li}(x)$ is the Logarithmic Integral of x and on RH, we have

$$J(x) = O\left(\sqrt{x}\,\log x\right). \tag{37}$$

Using the above equation for the representation of the prime counting function, we may then obtain (Appendix 1)

$$\sum_{i=r1+1}^{r2} \frac{1}{{p_i}^{\sigma}} = E_1((\sigma-1)\log p_{r1+1}) - E_1((\sigma-1)\log p_{r2}) + \varepsilon_p,$$

where $\varepsilon_p = \int_{p_{r1}}^{p_{r2}} dJ(x)/x^{\sigma} = O\left({p_{r1}}^{0.5-\sigma}\log p_{r1}/(\sigma-0.5)^2\right)$. Hence, Equation (35) can be written as

$$\log\left(\frac{G(\sigma,p_{r2})}{G(\sigma,p_{r1})}\right) = -\varepsilon_p + \delta_p + E_1((\sigma-1)\log p_{r2}) - E_1((\sigma-1)\log p_{r2+1}).$$

Since $E_1((\sigma-1)\log p_{r2}) - E_1((\sigma-1)\log p_{r2+1})$ approaches zero as p_{r2} approaches infinity (this follows from Cramer's theorem on the gap between primes that states that on RH, the gap between a prime number p_r and p_{r+1} is less than $k\sqrt{p_r}\log p_r$ for some constant k), thus

$$\lim_{p_{r2}\to\infty} \log\left(\frac{G(\sigma,p_{r2})}{G(\sigma,p_{r1})}\right) = -\varepsilon_p + \delta_p.$$

For the above equation, it should be pointed that we have kept p_{r1} fixed while we allowed p_{r2} to approach infinity. Hence, $G(\sigma,p_r)$ for any arbitrary large p_r is bounded. Furthermore, for $\sigma > 0.5+\epsilon$, the term $-\varepsilon_p+\delta_p$ can be made arbitrary small by choosing p_{r1} arbitrary large, thus the limit of $G(\sigma,p_r)$ as p_r approaches infinity exists and it is given by

$$G(\sigma) = \lim_{r\to\infty} G(\sigma,p_r) \tag{38}$$

This proves that, on RH, $G(\sigma,p_r)$ is convergent as p_r approaches infinity and thus $G(\sigma)$ exists for $\sigma > 0.5$. In Appendix 2, we have shown that, on RH and for $\Re(s) > 0.5$, we have

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^{s}} = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}) + \varepsilon_p, \tag{39}$$

where $\varepsilon_p = \int_{p_{r1}}^{p_{r2}} dJ(x)/x^{s} = O\left(\frac{|s|}{(\sigma-0.5)^2}\,{p_{r1}}^{0.5-\sigma}\log p_{r1}\right)$. Thus, we can follow the same steps and show that $G(s,p_r)$ is convergent as p_r approaches infinity and thus $G(s)$ exists for $\Re(s) > 0.5$ (it is worth mentioning here that the term ε_p in Equation (39) can be determined in terms of the non-trivial zero if von Mangoldt function is used in deriving Equation (39) instead of using the prime counting function).

It should be noted that, while the function sequence $G(s,p_r)$ is not uniformly convergent when the region of convergence is extended all the way to the line $\sigma = 0.5$, it is however uniformly convergent for any rectangle extending from $-iT$ to iT (for any arbitrary large T) and with $\sigma > 0.5+\epsilon$ (for any arbitrary small ϵ). This follows from the fact that, on RH, ε_p (or, the O term) is bounded for any $\sigma > 0.5+\epsilon$. Since $G(s,p_r)$ is analytic for $\Re(s) > 0$ and it is uniformly convergent for $\Re(s) > 0.5+\epsilon$, thus $G(s)$ is analytic for the half right complex plain with $\Re(s) > 0.5+\epsilon$ (Weiestrass theorem [6]). Since we have shown that $G(s) = 1$ for $\Re(s) \geq 1$, thus on RH, $G(s) = 1$ for $\Re(s) > 0.5+\epsilon$. Hence, we have the following theorem

Theorem 2. *For $s = \sigma + it$ and $\sigma > 0.5$, the following holds if RH is valid*

$$\lim_{r\to\infty} \left\{ \zeta(s) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right) \exp\left(-E_1((s-1)\log p_{r+1})\right) \right\} = 1. \tag{40}$$

$$\lim_{r\to\infty} \left\{ M(s, p_r) \exp\left(E_1((s-1)\log p_{r+1})\right) \right\} = 1. \tag{41}$$

It should be also pointed out that Theorem 2 can be generalized to the case where there are no non-trivial zeros for values of s with $\Re(s) > h$ (where $h > 0.5$). For this case, Equation (40) is valid for every s with $\Re(s) > h$ and ε_p in Appendix 2 is given by $O\left(\frac{|s|}{(\sigma-h)^2} {p_{r1}}^{h-\sigma} \log p_{r1}\right)$.

Equation (40) of Theorem 2 can be written as follows

$$\log \zeta(s) + \log \prod_{i=1}^{r2} \left(1 - \frac{1}{p_i^s}\right) - E_1\left((s-1)\log p_{r2+1}\right) = 0,$$

where the equality of both sides is attained as r_2 (or p_{r2}) approaches infinity (or more appropriately, for sufficiently large p_{r2}, the right side can be made arbitrary close to zero). It should be noted that while both functions $\log \zeta(s)$ and $E_1((s-1)\log p_{r2+1})$ have a branch cut along the real axis where $0.5 < \sigma < 1$, the difference (i.e. $\log \zeta(s) - E_1((s-1)\log p_{r2+1})$) does not have a branch cut. For $r < r2$, the above equation can be then written as

$$\log \zeta(s) = E_1\left((s-1)\log p_{r2+1}\right) - \sum_{i=1}^{r} \log\left(1 - \frac{1}{p_i^s}\right) - \sum_{i=r+1}^{r2} \log\left(1 - \frac{1}{p_i^s}\right).$$

Since for the region of convergence of the series $M(s, p_r)$, we have (refer to Appendix 2)

$$-\sum_{i=r+1}^{r2} \log\left(1 - \frac{1}{p_i^s}\right) = \sum_{i=r+1}^{r2} \frac{1}{{p_i}^s} - \delta_p = E_1\left((s-1)\log p_{r+1}\right) - E_1\left((s-1)\log p_{r2}\right) + \varepsilon_p - \delta_p$$

where $\varepsilon_p = \int_{p_r}^{\infty} dJ(x)/x^s$ (on RH, $\varepsilon_p = O((|s|\, {p_r}^{0.5-\sigma} \log p_r)/(\sigma - 0.5)^2)$). Therefore, as p_{r2} approaches infinity, we have

$$\log \zeta(s) = -\sum_{i=1}^{r} \log\left(1 - \frac{1}{p_i^s}\right) + E_1\left((s-1)\log p_{r+1}\right) + \varepsilon_p - \delta_p. \tag{42}$$

where for sufficiently large p_r, δ_p is negligible compared to ε_p (in fact, δ_p is of the same order of magnitude as ${\varepsilon_p}^2$). Taking the exponential of both side, we then obtain the following theorem

Theorem 3. *For the region of convergence of the series $M(s, p_r) = \sum_1^{\infty} \mu(n, p_r)/n^s$, we have*

$$M(s, p_r) = e^{-E_1((s-1)\log p_{r+1}) - \varepsilon_p + \delta_p}, \tag{43}$$

where $\varepsilon_p = \int_{p_r}^{\infty} dJ(x)/x^s$, $J(x) = \pi(x) - \mathrm{Li}(x)$ and $\delta_p = \sum_{i=r}^{\infty} \left(-\frac{1}{2{p_i}^{2s}} - \frac{1}{3{p_i}^{3s}} - \frac{1}{4{p_i}^{4s}} \ldots\right)$. Furthermore, on RH and for sufficiently large p_r, we have for $\Re(s) > 0.5$

$$M(s, p_r) = e^{-E_1((s-1)\log p_r)} \left(1 - \varepsilon_p + O({\varepsilon_p}^2)\right). \tag{44}$$

While we have used in this section the complex analysis to compute $M(s,p_r)$, in the next section, we will employ integration methods to compute the partial sum $M(s,p_r;1,p_r^a)$. The results obtained in this section and the following section will be then combined (using the Fourier analysis methods) in sections (6) and (7) and then used to examine the validity of the Riemann Hypothesis.

5 The series $M(\sigma,p_r)$ at $\sigma=1$.

In this section, we will provide an estimate for the partial sum $M(1,p_r;1,{p_r}^a)$ as a approaches infinity. This estimate will be computed using integration methods and noting that $M(1,p_r)$ equals zero for every p_r. Therefore, for every p_r, $M(1,p_r;1,{p_r}^a)$ approaches zero as a approaches infinity.

Before we present the details of our method, it is important to mention that the partial sum $M(1,p_r;1,{p_r}^a)$ can be also generated using y-smooth numbers. The y-smooth numbers are the numbers that have only prime factors less than or equal to y. These numbers have been extensively analyzed in the literature [10][12]. In [10], a clever method was presented to generate the partial sum $M(1,p_r;1,{p_r}^a)$. With this method and using the inclusion-exclusion principle [10, page 248], one can then provide an estimate for the partial sum $M(1,p_r;1,{p_r}^a)$. In this section, we will provide a more general approach to compute $M(1,p_r;1,{p_r}^a)$. The main advantage of our approach is the ability to extend it to compute the partial sum for values of s other than 1. We will present our method in the following two steps.

- In the first step of our approach, we will show that, for every a and as p_r approaches infinity, the partial sum $M(1,p_r;1,{p_r}^a)$ approaches a function that is dependent on only a (independent of p_r).

Toward this end, we define the function $f(a,p_r)$ as

$$f(a,p_r)=M(1,p_r;1,{p_r}^a)=\sum_{n=1}^{{p_r}^a}\frac{\mu(n,p_r)}{n}.$$

We will then show that, for every a and as p_r approaches infinity, the function $f(a,p_r)$ approaches a deterministic function $\rho(a)$. In other words; if we plot $M(1,p_r;1,N)$ (where $N={p_r}^a$) as a function of $a=\log N/\log p_r$, then for each value of a and as p_r approaches infinity, $f(a,p_r)$ approaches a unique value $\rho(a)$. This is equivalent to the statement

$$\rho(a)=\lim_{p_r\to\infty}f(a,p_r)=\lim_{p_r\to\infty}M(1,p_r;1,{p_r}^a).$$

This result can be achieved by first noting that the partial sum $M(1,p_r;1,{p_r}^a)$ for $1<a<2$ is given by

$$M(1,p_r;1,{p_r}^a)=1-\sum_{p_r\le p_i<{p_r}^a}\frac{1}{p_i}.$$

If we define $M_1(1,p_r;1,{p_r}^a)$ as

$$M_1(1,p_r;1,{p_r}^a)=\sum_{p_r\le p_i<{p_r}^a}\frac{1}{p_i},$$

then, using Stieltjes integral, we obtain

$$M(1,p_r;1,{p_r}^a) = 1 - M_1(1,p_r;1,{p_r}^a) = 1 - \int_{x=p_r}^{{p_r}^a} \frac{d\pi(x)}{x} = 1 - \int_{y=1}^{a} \frac{d\pi({p_r}^y)}{{p_r}^y}.$$

Since

$$d\pi({p_r}^y) = d\mathrm{Li}({p_r}^y) + dJ(p_r^y),$$

therefore

$$d\pi({p_r}^y) = \frac{1}{\log({p_r}^y)} d{p_r}^y + dJ({p_r}^y) = \frac{p_r^y}{y} dy + dJ({p_r}^y),$$

where on RH, $J(p_r^y) = O(\sqrt{{p_r}^y}\log({p_r}^y))$. Hence, for $1 < a < 2$, we have

$$M(1,p_r;1,{p_r}^a) = 1 - \int_1^a \frac{dy}{y} - \int_1^a \frac{dJ({p_r}^y)}{{p_r}^y} = 1 - \log(a) + g_1(p_r,a), \tag{45}$$

where

$$g_1(p_r,a) = -\int_1^a \frac{dJ({p_r}^y)}{{p_r}^y}. \tag{46}$$

As p_r approaches infinity, $g_1(p_r,a)$ approaches zero. Consequently,

$$\lim_{p_r\to\infty} M(1,p_r;1,{p_r}^a) = 1 - \log a.$$

The terms of the partial sum $M(1,p_r;1,{p_r}^a)$ for a in the range $1 < a < 3$ are either a reciprocal of a prime or a reciprocal of the product of two primes. Therefore, for $1 < a < 3$, we have

$$M(1,p_r;1,{p_r}^a) = 1 - \sum_{p_r \le p_i < {p_r}^a} \frac{1}{p_i} + \sum_{p_r \le p_{i1} < p_{i2} < p_{i1}p_{i2} < {p_r}^a} \frac{1}{p_{i1}p_{i2}},$$

where p_{i1} and p_{i2} are two distinct primes that are greater than or equal to p_r. Let $M_2(1,p_r;1,{p_r}^a)$ be defined as

$$M_2(1,p_r;1,{p_r}^a) = \sum_{p_r \le p_{i1} < p_{i2} < p_{i1}p_{i2} < {p_r}^a} \frac{1}{p_{i1}p_{i2}} = \frac{1}{2} \sum_{p_r \le p_i < {p_r}^{a-1}} \frac{1}{p_i} M_1(1,p_r;1,p_r^a/p_i) + r_2.$$

Note that, for the second sum (i.e. $\sum_{p_r \le p_i < {p_r}^{a-1}} \frac{1}{p_i} M_1(1,p_r;1,p_r^a/p_i)$), the factor of half was added since each term of the form $1/(p_{i1}p_{i2})$ is repeated twice. Furthermore, this sum includes non square-free terms (notice that, there is no repetition in any of the non square-free terms). The term r_2 was added to offset the contribution by these non square-free terms. We will show later that the contribution by these terms (or r_2) approaches zero as p_r approaches infinity. Using Stieltjes integral, we then have

$$M_2(1,p_r;1,{p_r}^a) = \frac{1}{2} \int_1^{a-1} \frac{d\pi({p_r}^y)}{{p_r}^y} \left(\log(a-y) + g_1(p_r,a-y)\right) + r_2.$$

Hence

$$M(1,p_r;1,{p_r}^a) = 1-\log(a)+g_1(p_r,a)+\frac{1}{2}\int_1^{a-1}\frac{\log(a-y)}{y}dy+g_2(p_r,a),$$

where

$$g_2(p_r,a) = \frac{1}{2}\int_1^{a-1}\frac{g_1(p_r,a-y)}{y}dy+\frac{1}{2}\int_1^{a-1}\log(a-y)\frac{dJ({p_r}^y)}{p_r^y}+$$
$$\frac{1}{2}\int_1^{a-1}g_1(p_r,a-y)\frac{dJ({p_r}^y)}{p_r^y}+r_2.$$

It can be easily shown that, for any fixed value of a, the three integrals on the right side of the above equation approach zero as p_r approaches infinity. We will also show later that r_2 approaches zero as p_r approaches infinity. Thus, for $1\le a<3$, we have

$$\lim_{p_r\to\infty} M(1,p_r;1,{p_r}^a) = 1-\log a+\int_1^{a-1}\frac{\log(a-y)}{y}dy$$

Therefore, as p_r approaches infinity, $M(1,p_r;1,{p_r}^a)$ approaches a function that is dependent on only a.

Repeating the previous process $\lfloor a\rfloor$ times (where $\lfloor x\rfloor$ is the integer value of x) and by using the induction method, we can show that, as p_r approaches infinity, the partial sum $M(1,p_r;1,{p_r}^a)$ approaches a function that is dependent on only a. Specifically, we first write the partial sum $M(1,p_r;1,{p_r}^a)$ as follows

$$M(1,p_r;1,{p_r}^a) = 1-M_1(1,p_r;1,{p_r}^a)+M_2(1,p_r;1,{p_r}^a)-...+(-1)^jM_j(1,p_r;1,{p_r}^a)+...+$$
$$(-1)^{\lfloor a\rfloor-1}M_{\lfloor a\rfloor-1}(1,p_r;1,{p_r}^a)+(-1)^{\lfloor a\rfloor}M_{\lfloor a\rfloor}(1,p_r;1,{p_r}^a),$$

where

$$M_j(1,p_r;1,{p_r}^a) = \sum_{p_r\le p_{i1}<p_{i2}<..<p_{ij}<p_{i1}p_{i2}..p_{ij}<{p_r}^a}\frac{1}{p_{i1}p_{i2}...p_{ij}}.$$

and $p_{i1},p_{i2},...,p_{ij}$ are j distinct prime numbers greater than or equal to p_r. If we assume that $M_{j-1}(1,p_r;1,{p_r}^a)$ is given by

$$M_{j-1}(1,p_r;1,{p_r}^a) = h_{j-1}(a)+g_{j-1}(p_r,a)$$

where $h_{j-1}(a)$ is a function of a and $g_{j-1}(p_r,a)$ approaches zero as p_r approaches infinity, then

$$M_j(1,p_r;1,{p_r}^a) = \frac{1}{j}\sum_{p_r\le p_i<{p_r}^{a-1}}\frac{1}{p_i}M_{j-1}(1,p_r;p_r,p_r^a/p_i)+r_j,$$

where the factor of $1/j$ was added since each term of the form $1/(p_{i1}p_{i2}...p_{ij})$ is repeated j times. It should be also noted that the sum of the above equation includes non square-free terms. The term r_j was added to offset the contribution by these non square-free terms. We will show later that the contribution by these terms (or r_j) approaches zero as p_r approaches infinity. Using Stieltjes integral, we then have

$$M_j(1, p_r; 1, {p_r}^a) = \frac{1}{j}\int_1^{a-1} \frac{d\pi({p_r}^y)}{{p_r}^y}\left(h_{j-1}(a-y) + g_{j-1}(p_r, a-y)\right) + r_j.$$

Hence

$$M_j(1, p_r; 1, {p_r}^a) = \frac{1}{j}\int_1^{a-1} \frac{h_{j-1}(a-y)}{y}dy + g_j(p_r, a),$$

where the first term is a definite integral with only one variable y integrated over the range $1 \le y \le a-1$. Thus, the definite integral is a function of only a. We define this function as $h_j(a)$. The second term is given by

$$g_j(p_r, a) = \frac{1}{j}\int_1^{a-1} \frac{g_{j-1}(p_r, a-y)}{y}dy + \frac{1}{j}\int_1^{a-1} h_{j-1}(a-y)\frac{dJ({p_r}^y)}{p_r^y} +$$
$$\frac{1}{j}\int_1^{a-1} g_{j-1}(p_r, a-y)\frac{dJ({p_r}^y)}{p_r^y} + r_j.$$

It can be easily shown that, for a fixed value of a, the three integrals on the right side of the above equation approach zero as p_r approaches infinity. We will also show later that r_j approaches zero as p_r approaches infinity. Hence, as p_r approaches infinity, we have

$$\lim_{p_r \to \infty} M_j(1, p_r; 1, {p_r}^a) = \frac{1}{j}\int_1^{a-1} \frac{h_{j-1}(a-y)}{y}dy = h_j(a)$$

where $h_1(a) = \log(a)$. Hence, for every a and as p_r approaches infinity, we have

$$\lim_{p_r \to \infty} M(1, p_r; 1, {p_r}^a) = 1 - h_1(a) + h_2(a) - h_3(a) + ... + (-1)^{\lfloor a \rfloor} h_{\lfloor a \rfloor}(a) = \rho(a). \quad (47)$$

It should be pointed out that the above equation implies that the partial sums $M(1, p_r; 1, {p_r}^a)$ and $M(1, p_r^y; 1, {p_r}^{ay})$ (where, p_r^y is a prime number) have the same limit as p_r approaches infinity. Hence,

$$\lim_{p_r \to \infty} M(1, p_r; 1, {p_r}^a) = \lim_{p_r \to \infty} M(1, p_r^y; 1, {p_r}^{ay}) = \rho(a). \quad (48)$$

Equation (48) will be used in the next step to estimate the asymptotic behavior of the function $\rho(a)$ as a approaches infinity.

As mentioned earlier, the partial sum $M(1, p_r; 1, {p_r}^a)$ constructed by this process included non square-free terms (i.e r_i's). In the following, we will show that, for every a and as p_r approaches infinity, the total contribution by these non square-free terms approaches zero as well. Toward this end, let S_0 be the sum of the terms with the factor $1/p_r^2$. Therefore, S_0 can be expressed as K_0/p_r^2. Let S_1 be the sum of the remaining terms with the factor $1/(p_{r+1})^2$. Therefore, S_1 can be expressed as $K_1/(p_{r+1})^2$. Let S_2 be the sum of the remaining terms with the factor $1/(p_{r+2})^2$ where S_2 can be expressed as $K_2/(p_{r+2})^2$, and so on. Let S be sum of all the terms associated with non square-free terms. Thus, S is given by

$$S = \frac{1}{{p_r}^2}K_0 + \frac{1}{{p_{r+1}}^2}K_1 + ... + \frac{1}{{p_{r+L}}^2}K_L,$$

where p_{r+L} is the largest prime that satisfies the condition ${p_{r+L}}^2 \leq {p_r}^a$. However,

$$|K_0|, |K_1|, ..., |K_L| < 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{{p_r}^a}.$$

Thus,

$$|K_0|, |K_1|, ..., |K_L| = O(a \log p_r).$$

Therefore,

$$S = \left(\frac{1}{{p_r}^2} + \frac{1}{{p_{r+1}}^2} + ... + \frac{1}{{p_{r+L}}^2} \right) O(a \log p_r).$$

Hence, the contribution by the non square-free terms S is given by,

$$S = O(a \log p_r / p_r).$$

Consequently, for every a and as p_r approaches infinity, S (or the contribution by the non square-free terms) approaches zero.

- In the second step, we write the partial sum $M(1, p_r; 1, {p_r}^a)$ as the sum of two components. The first one is the deterministic or regular component and it is given by $\rho(a)$. The second one is the irregular component $R(1, p_r; 1, {p_r}^a)$ given by $M(1, p_r; 1, {p_r}^a) - \rho(a)$. We will then show that the function $\rho(a)$ is the Dickman function that has been extensively used to analyze the properties of y-smooth numbers.

Toward this end, we write the partial sum $M(1, p_r; 1, {p_r}^a)$ as the following sum

$$M(1, p_r; 1, {p_r}^a) = 1 - \sum_{p_r \leq p_i < {p_r}^{a/2}} \frac{1}{p_i} M(1, p_i; 1, {p_r}^a / p_i) - \sum_{{p_r}^{a/2} \leq p_i < {p_r}^a} \frac{1}{p_i}. \tag{49}$$

The second sum was added since the first sum is void of the terms $1/p_i$'s for ${p_i}^{a/2} \leq p_i \leq p_r^a$. It can be easily shown that every term on the right side of Equation (49) is a term on the left side of the equation and vice versa. Furthermore, there is no repetition of any term on the right side of Equation (49). Using Stieltjes integral, we can write the above equation as follows

$$M(1, p_r; 1, {p_r}^a) = 1 - \int_1^{a/2} \frac{d\pi({p_r}^y)}{p_r^y} M(1, {p_r}^y; 1, p_r^a / p_r^y) - \int_{a/2}^a \frac{d\pi({p_r}^y)}{p_r^y}, \tag{50}$$

where $d\pi({p_r}^y) = d\mathrm{Li}({p_r}^y) + dJ({p_r}^y)$. It should pointed out that while Equations (49) and (50) provide the value of the partial sum $M(s, p_r; 1, p_r^a)$ at $s = 1$, they can be easily modified to compute the partial sum for any value of s to the right of the line $\Re(s) = 1$ (and on RH, to the right of the line $\Re(s) = 0.5$). This task will be achieved in the next section and it will be a key step to examine the validity of the Riemann Hypothesis

For any fixed a, as p_r approaches infinity, $M(1, {p_r}^y; 1, p_r^{a-y})$ approaches $\rho(a/y - 1)$ (refer to Equation (48)). Therefore, as p_r approaches infinity, we have

$$\rho(a) = 1 - \int_1^{a/2} \frac{\rho\left(\frac{a}{y} - 1\right)}{y} dy - \int_{a/2}^a \frac{dy}{y}. \tag{51}$$

In the following, we will show that $\rho(a)$ is the Dickman function that has been extensively used in the analysis of the y-smooth numbers. This task will be achieved by using Equation (51) to compute the difference $\rho(a+\Delta a)-\rho(a)$ (where, Δa is an arbitrary small number) to obtain

$$\rho(a+\Delta a)-\rho(a) = -\int_1^{(a+\Delta a)/2} \frac{\rho\left(\frac{a+\Delta a}{y}-1\right)}{y}dy + \int_1^{a/2} \frac{\rho\left(\frac{a}{y}-1\right)}{y}dy - \int_{(a+\Delta a)/2}^{(a+\Delta a)} \frac{dy}{y} + \int_{a/2}^{a} \frac{dy}{y}.$$

Since the third integral of the above equation is equal to the fourth integral, therefore

$$\rho(a+\Delta a)-\rho(a) = -\int_1^{(a+\Delta a)/2} \frac{\rho\left(\frac{a+\Delta a}{y}-1\right)}{y}dy + \int_1^{a/2} \frac{\rho\left(\frac{a}{y}-1\right)}{y}dy.$$

If we define $z = y/(1+\Delta a/a)$, then we have

$$\rho(a+\Delta a)-\rho(a) = -\int_{1/(1+\Delta a/a)}^{((a+\Delta a)/2)/(1+\Delta a/a)} \frac{\rho\left(\frac{a}{z}-1\right)}{z}dz + \int_1^{a/2} \frac{\rho\left(\frac{a}{y}-1\right)}{y}dy.$$

Thus,

$$\rho(a+\Delta a)-\rho(a) = -\int_{1/(1+\Delta a/a)}^{1} \frac{\rho\left(\frac{a}{z}-1\right)}{z}dz.$$

Dividing both sides of the above equation by Δa and letting Δa approach zero, we then obtain

$$\frac{d\rho(a)}{da} = -\frac{\rho(a-1)}{a}, \tag{52}$$

where $\rho(a) = 1-\log(a)$ for $1 \leq a \leq 2$. Equation (52) is a first order delay differential equation that has been extensively analyzed in the literature [10][12]. The function $\rho(a)$ is known as the Dickman function. As a approaches infinity, $\rho(a)$ can be given by the following estimate [10]

$$\rho(a) = \left(\frac{e+o(1)}{a\log a}\right)^a. \tag{53}$$

For sufficiently large values of a, we have $\rho(a) < a^{-a}$.

To compute the irregular component of $M(1,p_r;1,p_r^a)$, we notice that $R(1,p_r;1,{p_r}^a)$ is given by

$$R(1,p_r;1,{p_r}^a) = M(1,p_r;1,p_r^a) - \rho(a).$$

Thus, $R(1,p_r;1,{p_r}^a)$ can be computed by subtracting Equation (51) from Equation (50) to obtain the following theorem

Theorem 4. *The partial sum* $M(1,p_r;1,p_r^a) = \sum_{n=1}^{\lfloor p_r^a \rfloor} u(n,p_r)/n$ *can be expressed as*

$$M(1,p_r;1,p_r^a) = \rho(a) + R(1,p_r;1,{p_r}^a) \tag{54}$$

where $\rho(a)$ is Dickman function and the regular component of $M(1,p_r;1,p_r^a)$ is given by

$$\rho(a)=\lim_{p_r\to\infty}M(1,p_r;1,p_r^a) \tag{55}$$

while $R(1,p_r;1,{p_r}^a)$ or the irregular component of $M(1,p_r;1,p_r^a)$ is given by

$$R(1,p_r;1,{p_r}^a)=-\int_1^{a/2}\rho\left(a/y-1\right)\frac{dJ({p_r}^y)}{p_r^y}-\int_{a/2}^{a}\frac{dJ({p_r}^y)}{p_r^y}-\int_1^{a/2}R(1,{p_r}^y;1,p_r^{a-y})\frac{d\pi(p_r^y)}{p_r^y}. \tag{56}$$

The term $J(x)$ on the right side of Equation (56) is given by $\Omega(x^{0.5-\epsilon})$ (where ϵ is an arbitrary small number). This follows directly from the Riemann explicit formula where $J(x)=\pi(x)-\mathrm{Li}(x)$ is given by the sum of terms of the form $\mathrm{Li}(x^\rho)$ (where ρ's are the non-trivial zeros) and many of these terms grow at least as fast as $\sqrt{x}/\log x$ [1]. Thus, we have unconditionally $J(x)=\Omega(x^{0.5-\epsilon})$ (In fact, in 1914, Littlewood have shown that $J(x)=\Omega_\pm(x^{1/2}\log\log\log x/\log x)$)

In the following, we will show that, for sufficiently large a, $R(1,p_r;1,x)$ is given by $\Omega(x^{-0.5})$ and on RH, it is given by $O(x^{-0.5+\epsilon})$. Toward this end, we first recall that on RH, $M(0;1,x)$ (or the Mertens function) is given by [7, Theorems 14.25-C and 14.26-B]

$$M(0;1,x)=\sum_{n=1}^{x}\mu(n)=O(x^{0.5+\epsilon}),$$

and

$$M(0;1,x)=\Omega(x^{0.5}),$$

where ϵ is an arbitrary small number. Using the method of partial summation, we then have

$$M(1;1,x)=\sum_{n=1}^{x}\frac{\mu(n)}{n}=O(x^{-0.5+\epsilon}),$$

and

$$M(1;1,x)=\Omega(x^{-0.5}).$$

Similarly, for sufficiently large x, we can show that

$$M(0,p_r;1,x)=\sum_{n=1}^{x}\mu(n,p_r)=O(x^{0.5+\epsilon}),$$

and

$$M(0,p_r;1,x)=\Omega(x^{0.5}).$$

Using the method of partial summation, we then have

$$M(1,p_r;1,x)=\sum_{n=1}^{x}\frac{\mu(n,p_r)}{n}=O(x^{-0.5+\epsilon}),$$

and

$$M(1,p_r;1,x)=\Omega(x^{-0.5}).$$

Since $M(1, p_r; 1, p_r^a)$ is given by

$$M(1, p_r; 1, {p_r}^a) = \rho(a) + R(1, p_r; 1, {p_r}^a)$$

and since $\rho(a)$ decays to zero faster than e^{-ca} for any arbitrary large c, therefore on RH, we have the following theorem

Theorem 5. *On RH and as a approaches infinity, we have*

$$R(1, p_r; 1, {p_r}^a) = O({p_r}^{-a/2+a\epsilon}),$$

and

$$R(1, p_r; 1, {p_r}^a) = \Omega({p_r}^{-a/2}).$$

where ϵ is arbitrary small number.

Our examination for the validity of the Riemann Hypothesis is based on establishing a relationship between $R(1, p_r; 1, {p_r}^a)$ and $J(p_r^a)/p_r^a$ (such as Equation (56)). With this relationship, we may then examine the validity of the Riemann Hypothesis by analyzing the decay of functions $R(1, p_r; 1, {p_r}^a)$ and $J(p_r^a)/p_r^a$ as a approaches infinity. For Equation (56), the presence of the term $\int_{y=1}^{a/2} R(1, {p_r}^y; 1, p_r^{a-y}) d\pi(p_r^y)/p_r^y$ hinders our attempts to achieve this task. In the rest of this section and sections (6) to (7), we will use the Fourier analysis to provide simpler expressions for $R(1, p_r; 1, {p_r}^a)$ in terms of $J(p_r^a)/p_r^a$.

So far, we have shown that the regular component of $M(1, p_r; 1, p_r^a)$ is given by $\rho(a)$. Since $\rho(a) = 1$ for $0 \leq a \leq 1$, therefore the regular component of $M(1, p_r; 1, p_r^a)$ can be written as

$$\rho(a) = 1 + \int_1^a d\rho(x) = 1 + \int_1^a \rho'(x)dx.$$

Note that, since $\rho'(x) = 0$ for $0 < a < 1$, the integral $\int_1^a \rho'(x)dx$ in the above equation can be replaced by the integral $\int_0^a \rho'(x)dx$.

Similarly, for values of $s \neq 1$, we can consider that $M(s, p_r; 1, p_r^a)$ is comprised of two components. The first component is the regular component defined as $F(\alpha, a)$ (where $\alpha = (s-1)\log p_r$) and is given by

$$F(\alpha, a) = 1 + \int_1^a \frac{{p_r}^x}{{p_r}^{sx}} d\rho(x) = 1 + \int_1^a {p_r}^{(1-s)x} \rho'(x)dx,$$

or,

$$F(\alpha, a) = 1 + \int_1^a e^{-\alpha x} \rho'(x)dx, \tag{57}$$

while the irregular component is given by $R(s, p_r; 1, p_r^a) = M(s, p_r; 1, p_r^a) - F(\alpha, a)$. Notice that for $s = 1$, we have $\alpha = 0$ and $F(0, a) = \rho(a)$. We now define $F(\alpha)$ as

$$F(\alpha) = \lim_{a \to \infty} F(\alpha, a) = 1 + \int_1^\infty e^{-\alpha x} \rho'(x)dx. \tag{58}$$

Thus, for $\Re(s) \geq 1$, α is a complex variable in the complex plane to the right of the line

$\Re(s) = 1$. Hence, the integral $\int_1^\infty e^{-\alpha x}\rho'(x)dx$ is the Laplace transform of the function $\rho^{'}(x)$ and is given by $F(\alpha) - 1$ (where $F(\alpha)$ is the regular component of the series $M(s, p_r)$, i.e. $M(s, p_r; 1, \infty)$). Since the Laplace transform of $\rho(x)$ is given by $e^{-E_1(s)}/s$ [11, page 569][12], therefore the Laplace transform of $\rho^{'}(x)$ is then given by $s\mathcal{L}(\rho(x)) - \rho(0)$. Hence

$$F(\alpha) = e^{-E_1(\alpha)}.$$

Remarkably, these results agree with what we have obtained in Theorem 2. In Theorem 2, we have shown that

$$\lim_{r\to\infty} \{M(s, p_r) \exp\left(E_1((s-1)\log p_{r+1})\right)\} = 1,$$

or referring to Equation (44), we have

$$M(s, p_r) = e^{-E_1(\alpha)} \left(1 - \varepsilon_p(p_r, s) + O((\varepsilon_p(p_r, s))^2)\right), \tag{59}$$

where $\varepsilon_p(p_r, s) = \int_{p_r}^\infty dJ(x)/x^s$ and $J(x) = \pi(x) - \mathrm{Li}(x)$. Consequently, for $\Re(s) \geq 1$, we then obtain

$$M(s, p_r) = F(\alpha) \left(1 - \varepsilon_p(p_r, s) + O((\varepsilon_p(p_r, s))^2\right). \tag{60}$$

where $F(\alpha)$ is the regular component of the series $M(s, p_r)$ and $-F(\alpha)\varepsilon_p(p_r, s)(1+O(\varepsilon_p(p_r, s))$ is the irregular component of the series $M(s, p_r)$. It should be emphasized here that the regular component $F(\alpha)$ is the value of $M(s, p_r)$ due to the $\mathrm{Li}(x)$ component of the prime counting function $\pi(x)$. It is also important to note that the irregular component is not the same as the difference between the partial sum $M(s, p_r; 1, {p_r}^a)$ and the series $M(s, p_r)$.Therefore, except for $s = 1$ (where the irregular component $F(0)\varepsilon_p(p_r, 1)(1 + O(\varepsilon_p(p_r, s))$ is zero for every p_r), $F(\alpha)\varepsilon_p(p_r, s)(1 + O(\varepsilon_p(p_r, s))$ may have values different from zero although it approaches zero as p_r approaches infinity

Notice that on RH, the previous analysis should also hold for $\Re(s) > 0.5$. This analysis and its application to examine the validity of the Riemann Hypothesis will be presented in the following two sections.

6 The regular component of $M(s, p_r; 1, p_r^a)$ for $\Re(s) < 1$.

In the previous section, Equation (49) was used to compute $M(1, p_r; 1, {p_r}^a)$. In this section, we will modify this equation to compute $M(s, p_r; 1, {p_r}^a)$ for $s \neq 1$ as follows

$$M(s, p_r; 1, {p_r}^a) = 1 - \sum_{p_r \leq p_i < {p_r}^{a/2}} \frac{1}{p_i^s} M(s, p_i; 1, {p_r}^a/p_i) - \sum_{{p_r}^{a/2} \leq p_i < {p_r}^a} \frac{1}{p_i^s}. \tag{61}$$

Using Stieltjes integral, we can write the above equation as

$$M(s, p_r; 1, {p_r}^a) = 1 - \int_1^{a/2} \frac{d\pi({p_r}^y)}{p_r^{sy}} M(s, {p_r}^y; 1, p_r^a/p_r^y) - \int_{a/2}^a \frac{d\pi({p_r}^y)}{p_r^{sy}}. \tag{62}$$

On the real axis (i.e. $s = \sigma$), we then have

$$M(\sigma, p_r; 1, {p_r}^a) = 1 - \int_1^{a/2} \frac{d\pi({p_r}^y)}{p_r^{\sigma y}} M(\sigma, {p_r}^y; 1, p_r^{a-y}) - \int_{a/2}^a \frac{d\pi({p_r}^y)}{p_r^{\sigma y}}. \tag{63}$$

Using Theorem 2, on RH and for $\sigma > 0.5$, the partial sum $M(\sigma, p_r; 1, {p_r}^a)$ is convergent as a approaches infinity and its value is given by

$$\lim_{a\to\infty} M(\sigma, p_r; 1, {p_r}^a) = M(\sigma, p_r) = e^{-E_1(-\beta)} \left(1 - \varepsilon_p(p_r, s) + O((\varepsilon_p(p_r, s))^2\right), \tag{64}$$

where $\beta = -\alpha = (1-\sigma)\log p_r$ (note that $\beta > 0$ for $\sigma < 1$). Therefore, as a approaches infinity, the left side of Equation (63) can be split into the regular component $e^{-E_1(-\beta)}$ and the irregular component $-e^{-E_1(-\beta)}\varepsilon_p(p_r, s)(1 + O(\varepsilon_p(p_r, s))$. Similarly, referring to the previous section, we can split the right side of Equation (63) can into regular and irregular components. Toward this end, we write the first integral in Equation (63) as follows

$$\int_1^{a/2} M(\sigma, {p_r}^y; 1, p_r^{a-y}) \frac{d\pi({p_r}^y)}{p_r^{\sigma y}} = \int_1^{a/2} F((\sigma-1)\log p_r^y\,,\, a/y - 1) \frac{d\pi({p_r}^y)}{p_r^{\sigma y}} + \int_1^{a/2} R(\sigma, {p_r}^y; 1, p_r^{a-y}) \frac{d\pi({p_r}^y)}{p_r^{\sigma y}}. \tag{65}$$

The first integral on the right side of Equation (65) can be then written as

$$\int_1^{a/2} F((\sigma-1)\log p_r^y\,,\, a/y - 1) \frac{d\pi({p_r}^y)}{p_r^{\sigma y}} = \int_1^{a/2} F((\sigma-1)\log p_r^y\,,\, a/y - 1) \frac{d\mathrm{Li}({p_r}^y)}{p_r^{\sigma y}} + \int_1^{a/2} F((\sigma-1)\log p_r^y\,,\, a/y - 1) \frac{dJ({p_r}^y)}{p_r^{\sigma y}}.$$

where $J(x) = \pi(x) - \mathrm{Li}(x)$ and

$$F((\sigma-1)\log p_r\,,\, a) = 1 + \int_1^a \rho'(x) e^{x(1-\sigma)\log p_r} dx = 1 + \int_1^a \rho'(x) e^{\beta x} dx,$$

and

$$F((\sigma-1)\log p_r^y\,,\, a/y - 1) = 1 + \int_1^{a/y-1} \rho'(x) e^{x(1-\sigma)\log p_r^y} dx = 1 + \int_1^{a/y-1} \rho'(x) e^{\beta y x} dx.$$

Hence, the first integral on the right side of Equation (65) can be then written as

$$\int_1^{a/2} F((\sigma-1)\log p_r^y\,,\, a/y - 1) \frac{d\pi({p_r}^y)}{p_r^{\sigma y}} = \int_1^{a/2} \frac{d\mathrm{Li}({p_r}^y)}{p_r^{\sigma y}} + \int_1^{a/2} \frac{d\mathrm{Li}({p_r}^y)}{p_r^{\sigma y}} \int_1^{a/y-1} \rho'(x) e^{\beta y x} dx + \int_1^{a/2} F((\sigma-1)\log p_r^y\,,\, a/y - 1) \frac{dJ({p_r}^y)}{p_r^{\sigma y}}. \tag{66}$$

Therefore, Equation (63) can be written as

$$M(\sigma, p_r; 1, {p_r}^a) = 1 - \int_1^{a/2} \frac{d\mathrm{Li}({p_r}^y)}{p_r^{\sigma y}} \int_1^{a/y-1} \rho'(x) e^{\beta y x} dx - \int_1^{a} \frac{d\mathrm{Li}({p_r}^y)}{p_r^{\sigma y}} - \int_1^{a/2} F((\sigma-1)\log p_r^y\,,\, a/y - 1) \frac{dJ({p_r}^y)}{p_r^{\sigma y}} - \int_{a/2}^{a} \frac{dJ({p_r}^y)}{p_r^{\sigma y}} - \int_1^{a/2} R(\sigma, {p_r}^y; 1, p_r^{a-y}) \frac{d\pi(p_r^y)}{p_r^{\sigma y}}.$$

Consequently, the regular component of $M(\sigma, p_r; 1, p_r^a)$ is given by

$$F(\alpha, a) = 1 - \int_1^{a/2} \frac{\mathrm{dLi}(p_r{}^y)}{p_r^{\sigma y}} \int_1^{a/y-1} \rho'(x) e^{\beta yx} dx - \int_1^a \frac{\mathrm{dLi}(p_r{}^y)}{p_r^{\sigma y}}, \tag{67}$$

and

$$e^{-E_1(-\beta)} = \lim_{a\to\infty} \left(1 - \int_1^{a/2} \frac{\mathrm{dLi}(p_r{}^y)}{p_r^{\sigma y}} \int_1^{a/y-1} \rho'(x) e^{\beta yx} dx - \int_1^a \frac{\mathrm{dLi}(p_r{}^y)}{p_r^{\sigma y}} \right), \tag{68}$$

while the irregular component of $M(\sigma, p_r; 1, p_r^a)$ is given by

$$R(\sigma, p_r; 1, p_r{}^a) = -\int_1^{a/2} F((\sigma-1)\log p_r^y\,,\ a/y-1) \frac{dJ(p_r{}^y)}{p_r^{\sigma y}} - \int_{a/2}^a \frac{dJ(p_r{}^y)}{p_r^{\sigma y}} -$$

$$\int_1^{a/2} R(\sigma, p_r{}^y; 1, p_r^{a-y}) \frac{d\pi(p_r^y)}{p_r^{\sigma y}}, \tag{69}$$

and

$$\lim_{a\to\infty} R(\sigma, p_r; 1, p_r{}^a) = -e^{-E_1(-\beta)} \varepsilon_p(p_r, \sigma) \left(1 + O(\varepsilon_p(p_r, \sigma))\right). \tag{70}$$

For the Riemann hypothesis to be valid, Equations (68), (69) and (70) have to be satisfied for $\sigma > 0.5$ as a approaches infinity. For the remaining of this section, we will analyze the convergence of the right side of Equation (68) as a approaches infinity. In the next section, we will analyze the convergence of Equations (69) and (70) and examine their implication on the validity of the Riemann hypothesis.

Since the regular component is void of the function $J(x)$, one may expect that Equation (68) is not only valid for $\sigma > 0.5$ but it is also valid for $\sigma > 0$. This requires the convergence of the right side of Equation (68) as a approaches infinity for values of $\sigma > 0$. A necessary condition for the convergence of the right side of Equation (68) is that its derivative with respect to a should approach zero as a approaches infinity. In other words;

$$\lim_{a\to\infty} \left(\frac{d}{da} \int_1^a \frac{\mathrm{dLi}(p_r{}^y)}{p_r^{\sigma y}} + \frac{d}{da} \int_1^{a/2} \frac{\mathrm{dLi}(p_r{}^y)}{p_r^{\sigma y}} \int_0^{a/y-1} \rho'(x) e^{\beta yx} dx \right) = 0.$$

To show that the above equation is valid for $\sigma > 0$, we first write the derivative of the first integral as follows

$$\frac{d}{da} \int_1^a \frac{\mathrm{dLi}(p_r{}^y)}{p_r^{\sigma y}} = \frac{d}{da} \int_1^a \frac{1}{p_r{}^{\sigma y}} \frac{p_r{}^y dy}{y} = \frac{d}{da} \int_1^a \frac{e^{\beta y}}{y} dy = \frac{e^{\beta a}}{a}.$$

The derivative of the second integral can be computed as follows

$$\frac{d}{da} \int_1^{a/2} \frac{\mathrm{dLi}(p_r{}^y)}{p_r^{\sigma y}} \int_1^{a/y-1} \rho'(x) e^{\beta yx} dx =$$

$$\lim_{\Delta a\to 0} \frac{1}{\Delta a} \left(\int_1^{(a+\Delta a)/2} \frac{e^{\beta y}}{y} \left(\int_1^{(a+\Delta a)/y-1} \rho'(x) e^{\beta yx} dx \right) dy - \int_1^{a/2} \frac{e^{\beta y}}{y} \left(\int_1^{a/y-1} \rho'(x) e^{\beta yx} dx \right) dy \right).$$

Simplifying the above equation and noting that $\rho'(x) = 0$ for $0 < x < 1$, we then have

$$\frac{d}{da}\int_1^{a/2}\frac{e^{\beta y}}{y}\left(\int_1^{a/y-1}\rho'(x)e^{\beta yx}dx\right)dy = \lim_{\Delta a\to 0}\frac{1}{\Delta a}\left(\int_1^{a/2}\frac{e^{\beta y}}{y}\left(\int_{a/y-1}^{(a+\Delta a)/y-1}\rho'(x)e^{\beta yx}dx\right)dy\right),$$

or

$$\frac{d}{da}\int_1^{a/2}\frac{e^{\beta y}}{y}\left(\int_1^{a/y-1}\rho'(x)e^{\beta yx}dx\right)dy = \lim_{\Delta a\to 0}\frac{1}{\Delta a}\left(\int_1^{a/2}\frac{e^{\beta y}}{y}\rho'(a/y-1)e^{\beta y(a/y-1)}\frac{\Delta a}{y}dy\right).$$

Therefore,

$$\frac{d}{da}\int_1^{a/2}\frac{e^{\beta y}}{y}\left(\int_1^{a/y-1}\rho'(x)e^{\beta yx}dx\right)dy = e^{a\beta}\int_1^{a/2}\frac{\rho'(a/y-1)}{y^2}dy.$$

The integral on the right side of the above equation can be simplified by substituting u for $a/y-1$ to obtain

$$\int_1^{a/2}\frac{\rho'(a/y-1)}{y^2}dy = \int_{a-1}^{1}\frac{\rho'(u)(u+1)^2}{a^2}\frac{-adu}{(u+1)^2} = \frac{1}{a}\int_1^{a-1}\rho'(u)du = \frac{\rho(a-1)-1}{a}.$$

Therefore,

$$\frac{d}{da}\left(\int_1^{a/2}\frac{d\mathrm{Li}({p_r}^y)}{p_r^{\sigma y}}F((\sigma-1)\log p_r^y\,,\,a/y-1)+\int_{a/2}^{a}\frac{d\mathrm{Li}({p_r}^y)}{p_r^{\sigma y}}\right) = \frac{e^{\beta a}}{a}\rho(a-1).$$

It is clear that, as a approaches infinity, the above derivative with respect to a approaches zero for any value of β. Furthermore, the integral $\int_a^\infty(e^{\beta x}\rho(x-1)/x)dx$ is finite for $a>1$. Since $\rho(a)$ decays to zero faster than $e^{-a\log a}$, therefore the integral $\int_a^\infty(e^{\beta x}\rho(x-1)/x)dx$ approaches zero as a approaches infinity. Thus, as expected, the regular component of $M(\sigma,p_r;1,p_r^a)$ is convergent as a approaches infinity for any value of $\beta>0$ (or for any value of $\sigma>0$). In the next section, we will analyze the convergence of Equations (69) and (70) and then examine their implication on the validity of the Riemann hypothesis.

7 The irregular component of $M(s,p_r;1,p_r^a)$ and the Riemann Hypothesis.

The irregular component of $M(1,p_r;1,p_r^a)$ for values of $a>1$ is given by Equation (56) of Theorem 4

$$R(1,p_r;1,{p_r}^a) = -\int_{y=1}^{a/2}\rho\left(a/y-1\right)\frac{dJ(p_r^y)}{p_r^y} - \int_{y=a/2}^{a}\frac{dJ(p_r^y)}{p_r^y} -$$
$$\int_{y=1}^{a/2}R(1,{p_r}^y;1,p_r^{a-y})\frac{d\pi(p_r^y)}{p_r^y} \qquad .$$

Using the Fourier analysis methods, $M(s,p_r;1,p_r^a)$ was then computed in the previous section for any value of s in the region of convergence of the series $M(s,p_r)$

$$R(s,p_r;1,{p_r}^a) = -\int_{y=1}^{a/2} F((s-1)\log p_r^y\,,\ a/y-1)\frac{dJ({p_r}^y)}{p_r^{\sigma y}} - \int_{y=a/2}^{a}\frac{dJ({p_r}^y)}{p_r^{sy}} -$$

$$\int_{y=1}^{a/2} R(s,{p_r}^y;1,p_r^{a-y})\frac{d\pi(p_r^y)}{p_r^{sy}},$$

As mentioned earlier, the presence of the term $\int_{y=1}^{a/2} R(1,{p_r}^y;1,p_r^{a-y})d\pi(p_r^y)/p_r^y$ hinders our attempt to exploit the above equations to examine the validity of the Riemann Hypothesis. Theorem 3 provides a simpler expression for the irregular component of $M(s,p_r;1,\infty)$. With the aid of this theorem, we have shown that (refer to Equation (70))

$$R(s,p_r;1,\infty) = -e^{-E_1(-\beta)}\varepsilon_p(p_r,s)\left(1+O(\varepsilon_p(p_r,s))\right),$$

where $\varepsilon_p(p_r,s) = \int_{y=1}^{\infty} e^{\beta y}\frac{dJ(p_r^y)}{p_r^y}$ and for sufficiently large p_r, the term $(1+O(\varepsilon_p(p_r,s)))$ can be made arbitrary close to one. Thus

$$R(s,p_r;1,\infty) = -\left(1+O(\varepsilon_p(p_r,s))\right)e^{-E_1(-\beta)}\int_{y=1}^{\infty} e^{\beta y}\frac{dJ(p_r^y)}{p_r^y}.$$

where $\beta = 1-s$. Using Stieltjes integral, we also have

$$R(s,p_r;1,{p_r}^a) = \int_{y=1}^{a} e^{\beta y} dR(1,p_r;1,{p_r}^y).$$

Hence

$$\int_{y=1}^{\infty} e^{\beta y} dR(1,p_r;1,{p_r}^y) = -\left(1+O(\varepsilon_p(p_r,s))\right)e^{-E_1(-\beta)}\int_{y=1}^{\infty} e^{\beta y}\frac{dJ(p_r^y)}{p_r^y}. \tag{71}$$

Equation (71) establishes the relationship between the Laplace transforms of the functions $dR(1,p_r;1,{p_r}^a)/dy$ and $p^{-y}dJ(p_r^a)/dy$. With this relationship, we will establish a much simplified relationship between $R(1,p_r;1,{p_r}^a)$ and $J(p_r^a)$ than that given by Equation (56). First, we note that for sufficiently small β, Equation (71) can be written as follows

$$\int_{y=1}^{\infty} e^{\beta y} dR(1,p_r;1,{p_r}^y) = -\left(1+O(\varepsilon_p(p_r,s))\right)e^{\gamma}(-\beta+O(\beta^2))\int_{y=1}^{\infty} e^{\beta y}\frac{dJ(p_r^y)}{p_r^y}.$$

By differentiating the above equation with respect to β and allowing β to approach zero, we then have

$$\int_{y=1}^{\infty} y\, dR(1,p_r;1,{p_r}^y) = \left(1+O(\varepsilon_p(p_r,1))\right)e^{\gamma}\int_{y=1}^{\infty}\frac{dJ(p_r^y)}{p_r^y}. \tag{72}$$

The integral $\int_{y=1}^{\infty} y\, dR(1,p_r;1,{p_r}^y)$ is the first moment of the function $dR(1,p_r;1,{p_r}^y)/dy$. The computation of the second and third moments of the function $dR(1,p_r;1,{p_r}^y)/dy$ is outlined in Appendix 3. These moments are given by the following theorem.

Theorem 6. *For sufficiently large N and for every $p_r > N$, the first, second and third moments of the function $dR(1,p_r;1,p_r^a)/dy$ where $R(1,p_r;1,p_r^a)$ is the irregular component of the partial sum $M(1,p_r;1,p_r^a)$ are given by*

$$\int_{y=1}^{\infty} y\,dR(1,p_r;1,{p_r}^y) = e^{\gamma}\left(1+O(\varepsilon_p(p_r,1))\right)\int_{y=1}^{\infty}\frac{dJ(p_r^y)}{p_r^y}, \tag{73}$$

$$\int_{y=1}^{\infty} y^2\,dR(1,p_r;1,{p_r}^y) = 2e^{\gamma}\left(1+O(\varepsilon_p(p_r,1))\right)\int_{y=1}^{\infty}\frac{dJ(p_r^y)}{p_r^y}+ e^{\gamma}\left(1+O(\varepsilon_p(p_r,1))\right)\int_{y=1}^{\infty} y\frac{dJ(p_r^y)}{p_r^y}, \tag{74}$$

$$\int_{y=1}^{\infty} y^3\,dR(1,p_r;1,{p_r}^y) = \frac{9}{2}e^{\gamma}\left(1+O(\varepsilon_p(p_r,1))\right)\int_{y=1}^{\infty}\frac{dJ(p_r^y)}{p_r^y}+ 2e^{\gamma}\left(1+O(\varepsilon_p(p_r,1))\right)\int_{y=1}^{\infty} y\frac{dJ(p_r^y)}{p_r^y}+ e^{\gamma}\left(1+O(\varepsilon_p(p_r,1))\right)\int_{y=1}^{\infty} y^2\frac{dJ(p_r^y)}{p_r^y}. \tag{75}$$

The term $e^{\gamma} = \int_0^{\infty}\rho(y)dy$ signifies the importance of the Dickman function in establishing the relationship between $R(1,p_r;1,{p_r}^a)$ and $J(p_r^a)/p_r^a$.

Another interesting relationship between $R(1,p_r;1,{p_r}^a)$ and $J(p_r^a)$ can be established by substituting $-\alpha$ for $\beta = 1-s$ in Equation (71) to obtain for sufficiently large p_r

$$\int_{y=1}^{\infty} e^{-\alpha y} dR(1,p_r;1,{p_r}^y) = -e^{-E_1(\alpha)}\int_{y=1}^{\infty} e^{-\alpha y}\frac{dJ(p_r^y)}{p_r^y}. \tag{76}$$

where the term $(1+O(\varepsilon_p(p_r,s)))$ was ignored because of its negligible contribution to the following analysis. Let $f_1(y)$ and $f_2(y)$ be defined as

$$f_1(y) = \frac{dR(1,p_r;1,{p_r}^y)}{dy},$$

and

$$f_2(y) = \frac{dJ(p_r^y)/p_r^y}{dy}.$$

Thus, Equation (76) can be written as

$$\mathcal{L}f_1(y) = -e^{-E_1(\alpha)}\mathcal{L}f_2(y).$$

Since $\mathcal{L}^{-1}e^{-E_1(\alpha)} = \rho'(y)+\delta(y)$, therefore

$$f_1(y) = -\left((\rho'+\delta)*f_2\right)(y)$$

Since $f_1(y)$, $f_2(y)$ and $\rho'(y)$ are zero for $y<1$, hence

$$f_1(y) = -\int_1^{y-1}\rho'(y-x)f_2(x)dx - f_2(y)$$

Consequently,

$$\int_{y=1}^{a} f_1(y)dy = -\int_{y=2}^{a} dy \int_{x=1}^{y-1} \rho'(y-x) f_2(x)dx - \int_{y=1}^{a} f_2(y)dy$$

Thus,

$$\int_{y=1}^{a} dR(1, p_r; 1, {p_r}^y) = -\int_{y=2}^{a} dy \int_{x=1}^{y-1} \rho'(y-x) \frac{dJ(p_r^x)}{p_r^x} - \int_{x=1}^{a} \frac{dJ(p_r^x)}{p_r^x}. \tag{77}$$

The right side of the above equation can be written as the following sums,

$$\int_{y=2}^{a} dy \int_{x=1}^{y-1} \rho'(y-x) \frac{dJ(p_r^x)}{p_r^x} + \int_{x=1}^{a} \frac{dJ(p_r^x)}{p_r^x} =$$

$$\lim_{N\to\infty} \lim_{M\to\infty} \left(\sum_{j=\lfloor 2M/a \rfloor}^{M} \Delta y \sum_{i=\lfloor N/a \rfloor}^{j-\lfloor M/a \rfloor} \rho^{'}(y_j - x_i) \frac{J(p_r^{x_{i+1}}) - J(p_r^{x_i})}{p_r^{x_i}} + \sum_{i=\lfloor N/a \rfloor}^{M} \frac{J(p_r^{x_{i+1}}) - J(p_r^{x_i})}{p_r^{x_i}} \right)$$

where $\Delta y = 1/M$, $y_j = ja/M$, $\Delta x = 1/N$ and $x_i = ia/N$. From the above sum, we notice that, for every x_i, the term $\left(J(p_r^{x_{i+1}}) - J(p_r^{x_i})\right)/p_r^{x_i}$ is multiplied by $\rho^{'}(y_j - x_i)$'s for values of y_j's in the range $x_i \le y_j \le a$. Since $\rho(z) = 1 + \int_0^z \rho'(x)dx$, thus $\rho(a-x) = 1 + \int_x^a \rho'(y-x)dy$. Consequently, we have the following theorem

Theorem 7. *For sufficiently large N and for every $p_r > N$, the relationship between the irregular component $R(1, p_r; 1, p_r^a)$ of the partial sum $M(1, p_r; 1, p_r^a)$ and $J(x)$ is given by*

$$R(1, p_r; 1, {p_r}^a) = -\int_{x=1}^{a} \rho(a-x) \frac{dJ(p_r^x)}{p_r^x}. \tag{78}$$

where $R(1, p_r; 1, p_r^a) = M(1, p_r; 1, p_r^a) - \rho(a)$ and $J(x) = \pi(x) - \mathrm{Li}(x)$.

In the following, we will examine the validity of the Riemann Hypothesis by analyzing Equations (73), (74) and (75) with sufficiently large values for p_r so that the integral $\int_{y=1}^{\infty} dJ(p_r^y)/p_r^y$ is determined by the values of y in the vicinity of one. More specifically, referring to Appendix 1, on RH, we have

$$\int_{y=1}^{\infty} \frac{dJ(p_r^y)}{p_r^y} = O\left({p_r}^{-1/2} \log p_r\right).$$

Furthermore, on RH and due to the presence of non-trivial zeros on the line $\Re(s) = 1/2$, we also have

$$\int_{y=1}^{\infty} \frac{dJ(p_r^y)}{p_r^y} = \Omega\left({p_r}^{-1/2}\right).$$

Note that if the above equation does not hold, then $\int_{y=1}^{\infty} dJ(p_r^y)/{p_r}^{(\sigma+it)y} = O({p_r}^{1/2-\sigma})$. Hence, in the right vicinity and at the zeros on the line $\Re(s) = 0.5$, the term $\varepsilon(p_r, s)$ in Equation (44) is bounded leading to a contradiction by giving bounded values for $M(s, p_r)$ in the right vicinity and at these zeros. Therefore, for sufficiently large N and for some constant k, there are an infinite number of p_r's (that are greater than N) such that

$$\left| \int_{y=1}^{\infty} \frac{dJ(p_r^y)}{p_r^y} \right| > k{p_r}^{-1/2} > 0.$$

Furthermore, for any positive number h, we also have

$$\int_{y=1+h}^{\infty} \frac{dJ(p_r^y)}{p_r^y} = O\left((1+h){p_r}^{-h}{p_r}^{-1/2}\log p_r\right).$$

Thus

$$\int_{y=1}^{\infty} \frac{dJ(p_r^y)}{p_r^y} = \int_{y=1}^{1+h} \frac{dJ(p_r^y)}{p_r^y} + O\left((1+h){p_r}^{-h}{p_r}^{-1/2}\log p_r\right).$$

Therefore, on RH and for sufficiently small h, we can always choose sufficiently large p_r so that the integral $\int_{y=1}^{\infty} dJ(p_r^y)/p_r^y$ is determined by values of y in the vicinity of one. In other words; we have

$$\int_{y=1}^{\infty} \frac{dJ(p_r^y)}{p_r^y} = \int_{y=1}^{1+h} \frac{dJ(p_r^y)}{p_r^y} + \int_{y=1+h}^{\infty} \frac{dJ(p_r^y)}{p_r^y}.$$

where,

$$\left|\int_{y=1}^{\infty} \frac{dJ(p_r^y)}{p_r^y}\right| > k{p_r}^{-1/2} > 0,$$

and

$$\left|\int_{y=1+h}^{\infty} \frac{dJ(p_r^y)}{p_r^y}\right| < k_1(1+h){p_r}^{-h}{p_r}^{-1/2}\log p_r,$$

for some constant k_1. Therefore, for any h, we can always find sufficiently large p_r such

$$\int_{y=1}^{\infty} \frac{dJ(p_r^y)}{p_r^y} = (1+\delta_1)\int_{y=1}^{1+h} \frac{dJ(p_r^y)}{p_r^y},$$

where δ_1 can be made arbitrary close to zero by choosing p_r sufficiently large.

For the integral $\int_{y=1}^{\infty} y^m\, dR(1,p_r;1,{p_r}^y)$, we note that by the virtue of Theorem 7, on RH and for sufficiently large p_r, the integral $\int_{y=2}^{\infty} y^m\, dR(1,p_r;1,{p_r}^y)$ becomes negligible compared to the integral $\int_{y=1}^{2} y^m\, dR(1,p_r;1,{p_r}^y)$. Therefore, as is the case with the integral $\int_{y=1}^{\infty} dJ(p_r^y)/p_r^y$, on RH and for sufficiently large p_r, we have for any $h < 2$

$$\int_{y=1}^{\infty} y^m\, dR(1,p_r;1,{p_r}^y) = (1+\delta_2)\int_{y=1}^{1+h} y^m\, dR(1,p_r;1,{p_r}^y).$$

where δ_2 can be made arbitrary close to zero by choosing p_r sufficiently large. However, referring to Equation (46), for $1 \le h < 2$, we also have

$$R(1,p_r;1,{p_r}^{1+h}) = \int_{y=1}^{1+h} dR(1,p_r;1,{p_r}^y) = -\int_{y=1}^{1+h} \frac{dJ(p_r^y)}{p_r^y}. \tag{79}$$

In other words, for $1 \le y \le 2$, we have

$$dR(1,p_r;1,{p_r}^y) = -\frac{dJ(p_r^y)}{p_r^y}. \tag{80}$$

Therefore, referring to Equation (73), we have

$$\int_{y=1}^{1+h} dR(1,p_r;1,{p_r}^y) = e^{\gamma}(1+\delta_3)\int_{y=1}^{1+h}\frac{dJ(p_r^y)}{p_r^y}. \tag{81}$$

where h can be made arbitrary close to one and and δ_3 can be made arbitrary close to zero by choosing p_r sufficiently large. Subtracting Equation (79) from Equation (81), we then obtain when assuming the validity of the Riemann Hypothesis

$$\int_{y=1}^{1+h} (y-1)dR(1,p_r;1,{p_r}^y) = (e^{\gamma}-1+\delta_3 e^{\gamma})\int_{y=1}^{1+h}\frac{dJ(p_r^y)}{p_r^y},$$

or

$$\int_{y=1}^{1+h} (y-1)\frac{dJ(p_r^y)}{p_r^y} = -(e^{\gamma}-1+\delta_3 e^{\gamma})\int_{y=1}^{1+h}\frac{dJ(p_r^y)}{p_r^y}. \tag{82}$$

where the variable $(y-1)$ (for $1 \le y \le h$) can be made arbitrary close to zero by choosing p_r sufficiently large. Furthermore, using Equations (74), (79) and (82), we can also show that

$$\int_{y=1}^{1+h} (y-1)^2 dR(1,p_r;1,{p_r}^y) = (e^{2\gamma}-3e^{\gamma}+3+\delta_4)\int_{y=1}^{1+h}\frac{dJ(p_r^y)}{p_r^y},$$

or (for $h < 2$)

$$\int_{y=1}^{1+h} (y-1)^2\frac{dJ(p_r^y)}{p_r^y} = -(e^{2\gamma}-3e^{\gamma}+3+\delta_4)\int_{y=1}^{1+h}\frac{dJ(p_r^y)}{p_r^y}. \tag{83}$$

Similarly, on RH, using Equations (75), (79), (82) and (83), we have

$$\int_{y=1}^{1+h} (y-1)^3\frac{dJ(p_r^y)}{p_r^y} = -(e^{3\gamma}-4e^{2\gamma}+10.5e^{\gamma}-6+\delta_5)\int_{y=1}^{1+h}\frac{dJ(p_r^y)}{p_r^y}. \tag{84}$$

As mentioned earlier, the integral $\int_{y=1}^{\infty} dJ(p_r^y)/p_r^y$ is given by $\Omega\left({p_r}^{-1/2}\right)$. Therefore, for sufficiently large N and for some constant k, there are an infinite number of p_r's (that are greater than N) such that

$$\left|\int_{y=1}^{\infty}\frac{dJ(p_r^y)}{p_r^y}\right| = (1+\delta_1)\left|\int_{y=1}^{1+h}\frac{dJ(p_r^y)}{p_r^y}\right| > k{p_r}^{-1/2} > 0.$$

Thus, for some constant k_2 the absolute value of the right side of Equation (84) is given by

$$\left|-(e^{3\gamma}-4e^{2\gamma}+10.5e^{\gamma}-6+\delta_5)\int_{y=1}^{1+h}\frac{dJ(p_r^y)}{p_r^y}\right| > k_2{p_r}^{-1/2}$$

Furthermore, on RH, the integral $\int_{y=1}^{\infty} dJ(p_r^y)/p_r^y$ is given by $O\left({p_r}^{-1/2}\log p_r\right)$. Thus, by setting $h = k_3/\log p_r$ (where the constant k_3 is chosen such that p_r^{-h} is sufficiently small), we then have

$$\left|\int_{y=1}^{1+h} (y-1)^3\frac{dJ(p_r^y)}{p_r^y}\right| = h^2\, O\left({p_r}^{-1/2}\log p_r\right) < k_4 h{p_r}^{-1/2}.$$

Consequently, we have

$$k_2{p_r}^{-1/2} < k_4 h{p_r}^{-1/2}.$$

or

$$k_2 < k_4 h.$$

However, this contradicts our assertion that h can be chosen arbitrary small (be choosing p_r sufficiently large).

Similar contradiction is also attained if we assume that there are no zeros to the right of the line $\Re(s) = c$ for any $c < 1$. This follows from the fact if there are no zeros to right of the line $\Re(s) = c$ for any $c > 1$, then $J(x)$ is given by $O(x^{1-c} \log x)$ and this will lead to similar contradiction. Hence, we conclude that non-trivial zeros can be found arbitrary close to the line $\Re(s) = 1$.

Appendix 1

Assuming RH is valid and for $\sigma > 0.5$, to show that

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^{\sigma}} = E_1((\sigma - 1)\log p_{r1}) - E_1((\sigma - 1)\log p_{r2}) + \varepsilon_p$$

where, $\varepsilon_p = \int_{p_{r1}}^{p_{r2}} dJ(x)/x^{\sigma} = O\left(\frac{1}{(\sigma-0.5)^2} {p_{r1}}^{1/2-\sigma} \log p_{r1}\right)$ and $J(x) = \pi(x) - \mathrm{Li}(x)$, we first recall that

$$\sum_{i=r1}^{r2} \frac{1}{p_i^{\sigma}} = \int_{p_{r1}}^{p_{r2}} \frac{d\pi(x)}{x^{\sigma}} = \int_{p_{r1}}^{p_{r2}} \frac{d\mathrm{Li}(x)}{x^{\sigma}} + \int_{p_{r1}}^{p_{r2}} \frac{dJ(x)}{x^{\sigma}},$$

or

$$\sum_{i=r1}^{r2} \frac{1}{p_i^{\sigma}} = \int_{p_{r1}}^{p_{r2}} \frac{d\pi(x)}{x^{\sigma}} = \int_{p_{r1}}^{p_{r2}} \frac{1}{x^{\sigma}\log x} dx + \int_{p_{r1}}^{p_{r2}} \frac{1}{x^{\sigma}} dO\left(\sqrt{x}\log x\right).$$

We will first compute the integral with the O notation. This can be done by integration by parts to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^{\sigma}} dO\left(\sqrt{x}\log x\right) = \frac{O\left(\sqrt{p_{r2}}\log p_{r2}\right)}{{p_{r2}}^{\sigma}} - \frac{O\left(\sqrt{p_{r1}}\log p_{r1}\right)}{{p_{r1}}^{\sigma}} - \int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) d\left(\frac{1}{x^{\sigma}}\right)$$

Since $x > 0$, thus

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^{\sigma}} dO\left(\sqrt{x}\log x\right) = \frac{O\left(\sqrt{p_{r2}}\log p_{r2}\right)}{{p_{r2}}^{\sigma}} - \frac{O\left(\sqrt{p_{r1}}\log p_{r1}\right)}{{p_{r1}}^{\sigma}} - O\left(\int_{p_{r1}}^{p_{r2}} \sqrt{x}\log x \, d\left(\frac{1}{x^{\sigma}}\right)\right)$$

With the substitution of variables $y = \log x$, we then obtain

$$\int_{p_{r1}}^{p_{r2}} \sqrt{x}\log x \, d\left(\frac{1}{x^{\sigma}}\right) = -\int_{p_{r1}}^{p_{r2}} \sigma y e^{(\frac{1}{2}-\sigma)y} dy.$$

Since

$$\int x e^{ax} dx = \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax},$$

therefore

$$\int_{p_{r1}}^{p_{r2}} \sqrt{x}\log x \, d\left(\frac{1}{x^{\sigma}}\right) = -\sigma\left(\frac{\log p_{r2}}{0.5-\sigma} - \frac{1}{(0.5-\sigma)^2}\right) {p_{r2}}^{0.5-\sigma} + \sigma\left(\frac{\log p_{r1}}{0.5-\sigma} - \frac{1}{(0.5-\sigma)^2}\right) {p_{r1}}^{0.5-\sigma}.$$

Hence, for $\sigma > 0.5$, we have

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma} dO\left(\sqrt{x}\log x\right) = O\left(\frac{{p_{r1}}^{0.5-\sigma}\log p_{r1}}{(\sigma-0.5)^2}\right) \tag{85}$$

For $\sigma \geq 1$, the integral $\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x}dx$ can be computed directly from the definition of the Exponential Integral $E_1(r) = \int_r^\infty \frac{e^{-u}}{u}du$ (where $r \geq 0$) to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x}dx = E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2})$$

It should be pointed out that although the functions $E_1((\sigma-1)\log p_{r1})$ and $E_1((\sigma-1)\log p_{r2})$ have a singularity at $\sigma = 1$, the difference has a removable singularity at $\sigma = 1$. This follows from the fact that as σ approaches 1, the difference can be written as

$$E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2}) = -\log\left((1-\sigma)\log p_{r1}\right) - \gamma + \log\left((1-\sigma)\log p_{r2}\right) + \gamma$$

or,

$$\lim_{\sigma\to 1}\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x}dx = \lim_{\sigma\to 1}\{E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2})\} = -\log\log p_{r1} + \log\log p_{r2}$$

To compute the integral $\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x}dx$ for $\sigma < 0$, we first use the substantiation $y = \log x$ to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x}dx = \int_{\log p_{r1}}^{\log p_{r2}} \frac{e^{(1-\sigma)y}}{y}dy = \int_{\epsilon}^{\log p_{r2}} \frac{e^{(1-\sigma)y}}{y}dy - \int_{\epsilon}^{\log p_{r1}} \frac{e^{(1-\sigma)y}}{y}dy$$

where, ϵ is an arbitrary small positive number. With the variable substantiations $z_1 = y/\log p_{r1}$ and $z_2 = y/\log p_{r2}$, we then obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x}dx = \int_{\epsilon/\log p_{r2}}^{1} \frac{e^{(1-\sigma)(\log p_{r2})z_2}}{z_2}dz_2 - \int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)(\log p_{r1})z_1}}{z_1}dz_1.$$

With the variable substantiations $w_1 = (1-\sigma)(\log p_{r1})z_1$ and $w_2 = (1-\sigma)(\log p_{r2})z_1$ and by adding and subtracting the terms $-\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} + \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{dw_1}{w_1}$, we then have

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x}dx = \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{e^{w_2}-1}{w_2}dw_2 - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{e^{w_1}-1}{w_1}dw_1 + \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{dw_1}{w_1}.$$

Using the following identity [9, page 230]

$$\int_0^a \frac{e^t-1}{t}dt = -E_1(-a) - \log(a) - \gamma$$

where $a > 0$, we then obtain for $\sigma < 1$,

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x}dx = E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2})$$

Hence, for $\sigma > 0.5$, we have

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^{\sigma}} = E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2}) + \varepsilon_p$$

In general, if there are no non-trivial zeros for values of s with $\Re(s) > a$, then by following the same steps, we can also show that for $\sigma > a$, we have

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^{\sigma}} = E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2}) + \varepsilon_p$$

where, $\varepsilon_p = \int_{p_{r1}}^{p_{r2}} dJ(x)/x^{\sigma} = O\left({p_{r1}}^{a-\sigma}\log p_{r1}/(\sigma-a)^2\right)$.

Appendix 2

Assuming RH is valid and for $\sigma > 0.5$, to show that

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^{s}} = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}) + \varepsilon_p$$

where, $\varepsilon_p = O\left(\frac{|s|+1}{(\sigma-0.5)^2}\,{p_{r1}}^{1/2-\sigma}\log p_{r1}\right)$, we first recall that

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^{s}} = \int_{p_{r1}}^{p_{r2}} \frac{d\pi(x)}{x^s} = \int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx + \int_{p_{r1}}^{p_{r2}} \frac{1}{x^s} dO\left(\sqrt{x}\log x\right).$$

We will first compute the integral with the O notation. This can be done by integration by parts to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s} dO\left(\sqrt{x}\log x\right) = \frac{O\left(\sqrt{p_{r2}}\log p_{r2}\right)}{{p_{r2}}^s} - \frac{O\left(\sqrt{p_{r1}}\log p_{r1}\right)}{{p_{r1}}^s} - \int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) d\left(\frac{1}{x^s}\right)$$

The integral on the right side of the above equation can be then written as

$$\int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) d\left(\frac{1}{x^s}\right) = -s\int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) x^{-s-1} dx.$$

Hence,

$$\left|\int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) d\left(\frac{1}{x^s}\right)\right| \le |s| \int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) |x^{-s-1}| dx.$$

Consequently,

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s} dO\left(\sqrt{x}\log x\right) = O\left(|s|\;\frac{{p_{r1}}^{0.5-\sigma}\log p_{r1}}{(\sigma-0.5)^2}\right).$$

For $\Re(s) \geq 1$, the integral $\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx$ can be computed directly from the definition of the Exponential Integral $E_1(z) = \int_1^\infty \frac{e^{-tz}}{t} dt$ (where $\Re(z) \geq 0$) to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2})$$

To compute the integral $\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx$ for $\Re(z) < 1$, we first write the integral as follows

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx = \int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx - i \int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \sin(t \log x)}{\log x} dx.$$

The first integral on the right side $\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx$ can be computed by using the substitution $y = \log x$ to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \int_{p_{r1}}^{p_{r2}} \frac{e^{(1-\sigma)y} \cos(ty)}{y} dy,$$

or

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \int_{p_{r1}}^{p_{r2}} \frac{e^{(1-\sigma)y} \cos(ty)}{y} dy + \int_{p_{r1}}^{p_{r2}} \frac{e^{(1-\sigma)y}}{y} dy - \int_{p_{r1}}^{p_{r2}} \frac{e^{(1-\sigma)y}}{y} dy.$$

Hence,

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \int_{\epsilon}^{p_{r1}} \frac{e^{(1-\sigma)y}(1-\cos(ty))}{y} dy - \int_{\epsilon}^{p_{r2}} \frac{e^{(1-\sigma)y}(1-\cos(ty))}{y} dy -$$

$$\int_{\epsilon}^{p_{r1}} \frac{e^{(1-\sigma)y}}{y} dy + \int_{\epsilon}^{p_{r2}} \frac{e^{(1-\sigma)y}}{y} dy$$

where, ϵ is an arbitrary small positive number. With the variable substantiations $z_1 = y/\log p_{r1}$ and $z_2 = y/\log p_{r2}$, we then obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)(\log p_{r1}) z_1}(1-\cos(t(\log p_{r1}) z_1))}{z_1} dz_1 -$$

$$\int_{\epsilon/\log p_{r2}}^{1} \frac{e^{(1-\sigma)(\log p_{r2}) z_2}(1-\cos(t(\log p_{r2}) z_2))}{z_2} dz_2 -$$

$$\int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)(\log p_{r1}) z_1}}{z_1} dz_1 + \int_{\epsilon/\log p_{r2}}^{1} \frac{e^{(1-\sigma)(\log p_{r2}) z_2}}{z_2} dz_2$$

By the virtue of the following identity ([9], page 230)

$$\int_0^1 \frac{e^{at}(1-\cos(bt))}{t} dt = \frac{1}{2} \log(1 + b^2/a^2) + \mathrm{Li}(a) + \Re[E_1(-a+ib)],$$

where $a > 0$, we then obtain the following

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \Re[E_1((s-1)\log p_{r1})] + \mathrm{Li}((1-\sigma)\log p_{r1}) -$$

$$\Re[E_1((s-1)\log p_{r2})] - \text{Li}((1-\sigma)\log p_{r2}) -$$

$$\int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)(\log p_{r1})z_1}}{z_1} dz_1 + \int_{\epsilon/\log p_{r2}}^{1} \frac{e^{(1-\sigma)(\log p_{r2})z_2}}{z_2} dz_2$$

With the variable substantiations $w_1 = (1-\sigma)(\log p_{r1})z_1$ and $w_1 = (1-\sigma)(\log p_{r1})z_1$ and by adding and subtracting the terms $-\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} + \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{dw_1}{w_1}$, we then have

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \Re[E_1((s-1)\log p_{r1})] + \text{Li}((1-\sigma)\log p_{r1}) -$$

$$\Re[E_1((s-1)\log p_{r2})] - \text{Li}((1-\sigma)\log p_{r2}) +$$

$$\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{e^{w_2}-1}{w_2} dw_2 - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{e^{w_1}-1}{w_1} dw_1 +$$

$$\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{dw_1}{w_1}.$$

Using the following identity [9, page 230]

$$\int_0^a \frac{e^t - 1}{t} dt = \text{Ei}(a) - \log(a) - \gamma$$

where $a > 0$, we then obtain for $\sigma < 1$,

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \Re[E_1((s-1)\log p_{r1})] - \Re[E_1((s-1)\log p_{r2})]$$

Similarly, using the identity [9, page 230]

$$\int_{p_0}^{1} \frac{e^{at} \sin(bt)}{t} dt = \pi - \arctan(b/a) + \Im[E_1(-a+ib)],$$

where $a > 0$, we can show that for $\sigma < 1$, we have

$$-\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \sin(t \log x)}{\log x} dx = \Im[E_1((s-1)\log p_{r1})] - \Im[E_1((s-1)\log p_{r2})].$$

Therefore, for $\Re(s) > 0.5$, we have

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^s} = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}) + \varepsilon_p$$

where, on RH, $\varepsilon_p = O\left(\frac{|s|}{(\sigma-0.5)^2} {p_{r1}}^{1/2-\sigma} \log p_{r1}\right)$ and if we write $\pi(x) = \text{Li} + J(x)$, then ε_p is also given by

$$\varepsilon_p = \int_{p_{r1}}^{p_{r2}} \frac{dJ(x)}{x^s}$$

Appendix 3

To compute the m-th derivative of the function $f(\beta) = e^{-E_1(-\beta)-\epsilon_p(\beta,p_r)} - e^{-E_1(-\beta)}$ (where $f(\beta) = R(\sigma, p_r; 1, p_r{}^{\infty}) = \int_{y=1}^{\infty} e^{\beta y} dR(1, p_r; 1, p_r{}^{y})$, $\beta = (1-\sigma)\log p_r$, $\epsilon_p(\beta, p_r) = \varepsilon_p(p_r, \sigma) - \delta_p$ and $\varepsilon_p(p_r, \sigma) = \int_{y=1}^{\infty} e^{\beta y} dJ(p_r^y)/p_r^y$), we write this function as the product of the following two functions

$$f(\beta) = f_1(\beta) f_2(\beta),$$

where

$$f_1(\beta) = e^{-E_1(-\beta)},$$

and

$$f_2(\beta) = e^{-\epsilon_p(\beta,p_r)} - 1$$

The m-th derivative of the function f is then given by

$$\frac{d^m f(\beta)}{d\beta^m} = \frac{d^m f_1}{d\beta^m} f_2 + m \frac{d^{m-1} f_1}{d\beta^{m-1}} \frac{df_2}{d\beta} + ... + \frac{m!}{k!\,(m-k)!} \frac{d^{m-k} f_1}{d\beta^{m-k}} \frac{d^k f_2}{d\beta^k} + ... + f_1 \frac{d^m f_2}{d\beta^m}$$

For the function $f_2(\beta)$, it can be easily shown (by noting that $e^{-\epsilon_p(\beta,p_r)} = 1 - \varepsilon(p_r, \sigma) + O(\varepsilon^2(p_r, \sigma))$, $\delta_p = O(\varepsilon^2(p_r, \sigma))$ and for $m \geq 1$, $d^m \delta_p / d\beta^m = O(\varepsilon^2(p_r, \sigma))$) that

$$f_2(0) = -\varepsilon(p_r, 1) + O(\varepsilon^2(p_r, 1)),$$

$$\frac{df_2(\beta)}{d\beta}|_{\beta=0} = -\left(1 + O(\varepsilon(p_r, 1)\right) \int_{y=1}^{\infty} y \frac{dJ(p_r^y)}{p_r^y},$$

and for $m \geq 2$

$$\frac{d{f_2}^m(\beta)}{d\beta^m}|_{\beta=0} = -\left(1 + O(\varepsilon(p_r, 1)\right) \int_{y=1}^{\infty} y^{m-1} \frac{dJ(p_r^y)}{p_r^y}.$$

For the function $f_1(\beta)$, we write f_1 as follows (refer to Equations (29))

$$f_1(\beta) = e^{-E_1(-\beta)} = -e^{\gamma} \beta e^{A}$$

where

$$A = \beta + \frac{\beta^2}{2\,2!} + \frac{\beta^3}{3\,3!} + \frac{\beta^4}{4\,4!} + ...$$

Thus,

$$f_1(0) = 0,$$

and the first derivative of f_1 is given by

$$\frac{df_1(\beta)}{d\beta} = -e^{\gamma} e^{A} \left(1 + \beta \frac{dA}{d\beta}\right),$$

At $\beta = 0$, $A = 0$, $e^A = 1$ and $dA/d\beta = 1$, thus

$$\frac{df_1(\beta)}{d\beta}|_{\beta=0} = -e^{\gamma}.$$

The second derivative of f_1 is given by

$$\frac{d^2 f_1(\beta)}{d\beta^2} = -e^{\gamma} e^{A} \left(2 \frac{dA}{d\beta} + \beta \left(\frac{d^2 A}{d\beta^2} + \left(\frac{dA}{d\beta}\right)^2\right)\right),$$

and

$$\frac{d^2 f_1(\beta)}{d\beta^2}|_{\beta=0} = -2e^{\gamma}.$$

The third derivative of f_1 is given by

$$\frac{d^3 f_1(\beta)}{d\beta^3} = -e^{\gamma}e^{A}\left(\left(3\frac{d^2A}{d\beta^2}+3\left(\frac{dA}{d\beta}\right)^2\right)+\beta\left(\frac{d^3A}{d\beta^3}+3\frac{dA}{d\beta}\frac{d^2A}{d\beta^2}+\left(\frac{dA}{d\beta}\right)^3\right)\right),$$

At $\beta = 0$, $e^A = 1$, $dA/d\beta = 1$ and $d^2A/d\beta^2 = 0.5$, thus

$$\frac{d^3 f_1(\beta)}{d\beta^3}|_{\beta=0} = -\frac{9}{2}e^{\gamma}.$$

Similarly, the fourth derivative of f_1 at $\beta = 0$ is given by

$$\frac{d^4 f_1(\beta)}{d\beta^4}|_{\beta=0} = -\frac{34}{3}e^{\gamma}.$$

In general, using the method of induction, we can show that the m-th derivative of f_1 is given by

$$\frac{d^m f_1(\beta)}{d\beta^m}|_{\beta=0} = -e^{\gamma}e^{A}\left(\frac{m^2}{2}+k\right),$$

where $k \geq 0$. This can be achieved by writing the m-th derivative as

$$\frac{d^m f_1(\beta)}{d\beta^m} = -e^{\gamma}e^{A}\left(m\left(\frac{dA}{d\beta}\right)^{m-1}+\beta\frac{d^mA}{d\beta^m}+\beta\left(\frac{dA}{d\beta}\right)^m+k_1\right),$$

where $k_1 \geq 0$ (note that the second derivative of f_1 satisfies the above equation). Then, the $(m+1)$-th derivative is given by

$$\frac{d^{m+1} f_1(\beta)}{d\beta^{m+1}} = -e^{\gamma}e^{A}\left((m+1)\left(\frac{dA}{d\beta}\right)^m+m(m-1)\left(\frac{dA}{d\beta}\right)^{m-2}\frac{d^2A}{d\beta^2}+\beta\frac{d^{m+1}A}{d\beta^{m+1}}+k_2\right).$$

where $k_2 \geq 0$. At $\beta = 0$, $e^A = 1$, $dA/d\beta = 1$ and $d^2A/d\beta^2 = 0.5$. Thus

$$\frac{d^m f_1(\beta)}{d\beta^m}|_{\beta=0} = -e^{\gamma}e^{A}\left(\frac{m^2}{2}+k\right).$$

Referring the the binomial expression for the m-th derivative of the function $f(\beta) = e^{-E_1(-\beta)-\epsilon_p(\beta,p_r)}$, we then have

$$\frac{df(\beta)}{d\beta}|_{\beta=0} = e^{\gamma}\left(1+O(\varepsilon_p(p_r,1))\right)\varepsilon_p(p_r,1)) = e^{\gamma}\left(1+O(\varepsilon_p(p_r,1))\right)\int_{y=1}^{\infty}\frac{dJ(p_r^y)}{p_r^y},$$

$$\frac{d^2 f(\beta)}{d\beta^2}|_{\beta=0} = 2e^{\gamma}\left(1+O(\varepsilon_p(p_r,1))\right)\int_{y=1}^{\infty}\frac{dJ(p_r^y)}{p_r^y}+e^{\gamma}\left(1+O(\varepsilon_p(p_r,1))\right)\int_{y=1}^{\infty}y\frac{dJ(p_r^y)}{p_r^y},$$

and

$$\frac{d^3 f(\beta)}{d\beta^3}|_{\beta=0} = \frac{9}{2}e^{\gamma}\left(1+O(\varepsilon_p(p_r,1))\right)\int_{y=1}^{\infty}\frac{dJ(p_r^y)}{p_r^y}+2e^{\gamma}\left(1+O(\varepsilon_p(p_r,1))\right)\int_{y=1}^{\infty}y\frac{dJ(p_r^y)}{p_r^y}+$$
$$e^{\gamma}\left(1+O(\varepsilon_p(p_r,1))\right)\int_{y=1}^{\infty}y^2\frac{dJ(p_r^y)}{p_r^y}$$

References

[1] Edwards H. M., *Riemann Zeta Function.*, Academic Press, 1974.
[2] Riemann R., *On the Number of Prime Numbers less than a Given Quantity.*, Nov. 1856. Translated by Wilkins, D. R., Dec. 1998.
[3] Gonek S. M., Hughes C.P., and Keating J.P, *A Hybrid Euler-Hadamard Product for the Riemann Zeta Function*, Duke Mathematical Journal, Vol. 136, No. 3,c 2007
[4] Gerald Tenenbaum and Michel Mendes France, Translated by Philip G Spain, *The prime numbers and their distribution.*, Student Medical Library, Volume 6, AMS, USA, 2000.
[5] Daniel H. Green and Donald E. Knuth, *Mathematics for the analysis of Algorithms.*, Third Edition, Birkhauser. 1990.
[6] Eberhard Freitag and Rolf Busam, *Complex Analysis*, Second Edition, Springer, 2005.
[7] Titchmarsh E. C., *The Theory of the Riemann Zeta-function.*, Second Revised Edition (Heath-Brown), Oxford Science Publications, 1986, Reprinted 2007.
[8] William J. LeVeque, *Fundamentals of Number Theory.*, LeVeque, W. L., Dover Duplications, 1996.
[9] Abramowitz and Stegun , *Handbook of Mathematical Functions.*, Tenth Edition, United States Printing Office, Washington DC, 1972.
[10] Granville A., *Smooth numbers: Computational number theory and beyond.*, Algorithmic Number Theory, Vol. 44, 267-323, 2008.
[11] Lagarias J. C., *Euler constant: Euler work and modern developments.*, Bulletin (New Series) American Mathematical Society, Vol. 50, No. 4, 527-628, Oct. 2013.
[12] Hilderrand A. and Tenenbaum G., *Integers without large prime factors*, Journal de Theorie des Numbers de Bordeaux, vol. 5, no. 2, 411-484, 1993.