ON THE VALIDITY OF THE RIEMANN HYPOTHESIS

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Abstract

In this paper, we have used the partial Euler product to examine the validity of the Riemann Hypothesis. The Dirichlet series over the Mobius function $M(s) = \sum_{n=1}^{\infty} 1/n^s$ has been modified and represented in terms of the partial Euler product by progressively eliminating the numbers that first have a prime factor 2, then 3, then 5, ..up to the prime p_r. It is shown that the series $M(s)$ and the new series have the same region of convergence. Unlike the partial sum of $M(s)$ that has irregular behavior, the partial sum of the new series exhibits regular behavior as p_r approaches infinity. This has allowed the use of integration methods to compute the partial sum of the new series to determine its region of convergence and to provide an answer for the validity of the Riemann Hypothesis.

1 Introduction

The Riemann zeta function $\zeta(s)$ satisfies the following functional equation over the complex plain [1]

$$\zeta(1-s) = 2(2\pi)^2 \cos(0.5s\pi)\Gamma(s)\zeta(s), \tag{1}$$

where, $s = \sigma + it$ is a complex variable and $s \neq 0$.

For $\sigma > 1$ (or $\Re(s) > 1$), $\zeta(s)$ can be expressed by the following series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{2}$$

or by the following product over the primes p_i's

$$\frac{1}{\zeta(s)} = \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^s}\right). \tag{3}$$

where, $p_1 = 2$, $\prod_{i=1}^{\infty}(1 - 1/{p_i}^s)$ is the Euler product and $\prod_{i=1}^{r}(1 - 1/{p_i}^s)$ is the partial Euler product. The above series and product representations of $\zeta(s)$ are absolutely convergent for $\sigma > 1$.

The region of the convergence of the sum in Equation (2) can be extended to $\Re(s) > 0$ by using the alternating series $\eta(s)$ where

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \tag{4}$$

and

$$\zeta(s) = \frac{1}{1 - 2^{1-s}}\eta(s). \tag{5}$$

One may notice that the term $1 - 2^{1-s}$ is zero at $s = 1$. This zero cancels the simple pole that $\zeta(s)$ has at $s = 1$ enabling the extension (or analog continuation) of the zeta function series representation over the critical strip $0 < \Re(s) < 1$.

It is well known that all of the non-trivial zeros of $\zeta(s)$ are located in the critical strip $0 < \Re(s) < 1$. Riemann stated that all non-trivial zeros were very probably located on the critical line $\Re(s) = 0.5$ [2]. There are many equivalent statements for the Riemann Hypothesis (RH) and one of them involves the Dirichlet series with the Mobius function.

The Mobius function $\mu(n)$ is define as follows
$\mu(n) = 1$, if $n = 1$.
$\mu(n) = (-1)^k$, if $n = \prod_{i=1}^{k} p_i$, p_i's are distinct primes.
$\mu(n) = 0$, if $p^2 | n$ for some p.

The Dirichlet series $M(s)$ with the Mobius function is defined as

$$M(s) = \sum_{n=1}^{\infty} \frac{\mu(s)}{n^s}. \tag{6}$$

This series is absolutely convergent to $1/\zeta(s)$ for $\Re(s) > 1$ and conditionally convergent to $1/\zeta(s)$ for $\Re(s) = 1$. The Riemann hypothesis is equivalent to the statement that $M(s)$ is conditionally convergent to $1/\zeta(s)$ for $\Re(s) > 0.5$.

Gonek, Hughes and Keating [3] have done an extensive research into establishing a relationship between $\zeta(s)$ and its partial Euler product for $\Re(s) < 1$. Gonek stated "Analytic number theorists believe that an eventual proof of the Riemann Hypothesis must use both the Euler product and functional equation of the zeta-function. For there are functions with similar functional equations but no Euler product, and functions with an Euler product but no functional equation." In sections 4, we will present a functional equation for $\zeta(s)$ using its partial Euler product. The method is based on writing the Euler product formula as follows

$$1/\zeta(s) = \prod_{i=1}^{\infty}\left(1 - \frac{1}{p_i^s}\right) = \prod_{i=1}^{r}\left(1 - \frac{1}{p_i^s}\right)\prod_{r+1}^{\infty}\left(1 - \frac{1}{p_i^s}\right).$$

The above equation is valid for $\sigma > 1$. To be able to represent $\zeta(s)$ in term of its partial Euler product for $\sigma \leq 1$, we have to replace the term $\prod_{r}^{\infty}(1 - 1/p_i^s)$ with an equivalent one that allows the analytic continuation for the representation of $\zeta(s)$ for $\sigma \leq 1$. Thus, the new term, that we need to introduce to replace $\prod_{r}^{\infty}(1 - 1/p_i^s)$, must have a zero that cancels the pole that $\zeta(s)$ has at $s = 1$. In section 4, we will use the complex analysis to compute this new term. We then use the new representation to compute the sum $\sum_{i=1}^{r} {p_i}^{\sigma}$ for $\sigma < 1$. This sum is then used to examine the validity of the Riemann Hypothesis.

In this paper, we claim the the Riemann Hypothesis is invalid. We support our claim by proving that the series $M(\sigma)$ is divergent for $\sigma < 1$. We have achieved this result by introducing a method to represent the Dirichlet series $M(s)$ (defined by Equation (6)) in terms of the partial Euler product. This task is achieved (sections 2) by first eliminating the numbers that have the prime factor 2 to generate the series $M(s,2)$. For the series $M(s,2)$, we then eliminate the numbers with the prime factor 3 to generate the series $M(s,3)$, and so on, up to the prime number p_r. In essence, we have applied the sieving technique to modify the series $M(s)$ to include only the numbers with prime factors greater than p_r. In section 3, we have shown that the series $M(s)$ and the new series $M(s,p_r)$ have the same region of convergence.

So far, the efforts to use the series $M(\sigma)$ to examine the validity of the Riemann Hypothesis have failed due to the irregular behavior of the partial sum of the series $M(\sigma)$. In section 6, we have shown that the partial sum of the new series $M(\sigma,p_r)$ exhibits regular behavior as p_r approaches infinity. This has allowed the use of integration methods to compute the partial sum of the new series and consequently determine its region of convergence. With this analysis, we have shown that the series $M(\sigma,p_r)$ and $M(\sigma)$ are divergent for $\sigma < 1$. Thus, non-trivial zeros can be found arbitrary close to the line $s = 1$.

2 Applying the Sieving Method to the Dirichlet Series $M(s)$.

The Dirichlet series $M(s)$ with the Mobius function is defined as

$$M(s) = \sum_{n=1}^{\infty} \frac{\mu(s)}{n^s},$$

where $\mu(n)$ is the Mobius function. Thus,

$$M(s) = 1 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{0}{4^s} - \frac{1}{5^s} + \frac{1}{6^s}....$$

It should be pointed out that our definition of $M(s)$ is different from $M(x)$ that is commonly defined in the literature as $M(x) = \sum_{n\leq x} \mu(n)$.

Next, we introduce the series $M(s,2)$ by eliminating all the numbers that have a prime factor 2. Thus, $M(s,2)$ can be written as

$$M(s,2) = 1 - \frac{1}{3^s} - \frac{1}{5^s} - \frac{1}{7^s} + \frac{0}{9^s} - \frac{1}{11^s} - \frac{1}{13^s} + \frac{1}{15^s}.....$$

Since our analysis to test the conditional convergence of these series ($M(s)$ and $M(s,2)$ for $\sigma \leq 1$) is based on comparing correspondent terms of the two series. Therefor, rearrangement and permutation of the terms may have a significant impact on the region of convergence of both series. Therefore, it essential to have the same index for both series $M(s)$ and $M(s,2)$ refer to the same term. Hence, we will represent $M(s,2)$ as follows

$$M(s,2) = 1 + \frac{0}{2^s} - \frac{1}{3^s} + \frac{0}{4^s} - \frac{1}{5^s} + \frac{0}{6^s} - \frac{1}{7^s} - \frac{0}{8^s}....,$$

or

$$M(s,2) = \sum_{n=1}^{\infty} \frac{\mu(n,2)}{n^s}, \tag{7}$$

where
$\mu(n,2) = \mu(n)$, if n is an odd number,

$\mu(n,2)=0$, if n is an even number.

The above series $M(s,2)$ can be further modified by eliminating all the numbers that have a prime factor 3 to get the series $M(s,3)$ where

$$M(s,3)=1-\frac{1}{5^s}-\frac{1}{7^s}-\frac{1}{11^s}-\frac{1}{13^s}-\frac{1}{17^s}-\frac{1}{19^s}-\frac{1}{23^s}+\frac{0}{25^s}....,$$

or more conveniently

$$M(s,3)=1+\frac{0}{2^s}-\frac{0}{3^s}+\frac{0}{4^s}-\frac{1}{5^s}+\frac{0}{6^s}-\frac{1}{7^s}-\frac{0}{8^s}....,$$

and so on.

Let $I(p_r)$ represent, in ascending order, the integers with distinct prime factors that belong to the set $\{p_i : p_i > p_r\}$. Let $\{1, I(p_r)\}$ be the set of 1 and $I(p_r)$ (for example, $\{1, I(2)\}$ is the set of square free odd numbers), then we define the series $M(s,p_r)$ as

$$M(s,p_r)=\sum_{n=1}^{\infty}\frac{\mu(n,p_r)}{n^s}, \tag{8}$$

where
$\mu(n,p_r)=\mu(n)$, if $n\in\{1, I(p_r)\}$,
otherwise, $\mu(n,p_r)=0$.

It can be easily shown that $M(s,p_r)$ converges absolutely for $\Re(s)>1$ for every prime number p_r. Furthermore, it can be shown that, for $\Re(s)>1$, $M(s,p_r)$ satisfies the following equation

$$M(s)=M(s,p_r)\prod_{i=1}^{r}\left(1-\frac{1}{p_i^s}\right). \tag{9}$$

Since

$$M(s)=\prod_{i=1}^{\infty}\left(1-\frac{1}{p_i^s}\right),$$

then we conclude that, for $\Re(s)>1$, $M(s,p_r)$ approaches 1 as p_r approaches infinity.

3 The region of convergence for the series $M(s)$ and $M(s,p_r)$.

In this section, we will deal with the question of relationship between the conditional convergence of the series of the two series $M(s,p_r)$ and $M(s)$ over the strip $0.5<\Re(s)\leq 1$. This task can be achieved by examining the convergence of the series $M(s,p_r)$ and $M(s)$ along the real axis (or along the line $0.5<\sigma\leq 1$). Theorems 1 and 2 establishes the relationship between the conditional convergence of the two series $M(s)$ and $M(s,p_r)$ for $0.5<\sigma\leq 1$.

Theorem 1 *For $s=\sigma+i0$, where $0.5<\sigma\leq 1$ and for every prime number p_r, the series $M(\sigma)$ converges conditionally if and only if the series $M(\sigma,p_r)$ converges conditionally. Furthermore, $M(\sigma)$ and $M(\sigma,p_r)$ are related as follows*

$$M(\sigma)=M(\sigma,p_r)\prod_{i=1}^{r}\left(1-\frac{1}{p_i^\sigma}\right). \tag{10}$$

The proof of Theorem 1 is outlined in Appendix 1.

Theorem 2 *For $s = \sigma + it$, where $0.5 < \sigma \leq 1$ and for every prime number p_r, the series $M(s)$ converges conditionally if and only if the series $M(s, p_r)$ converges conditionally. Furthermore, $M(s)$ and $M(s, p_r)$ are related as follows*

$$M(s) = M(s, p_r) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right). \tag{11}$$

The proof of the first part of Theorem 2 follows from the fact that $M(s, p_r)$ is a Dirichlet series and consequently this series is conditionally convergent if and only if the series $M(\sigma, p_r)$ is conditionally convergent.

The second part of the theorem can be proved by first defining $M(s, p_r; N_1, N_2)$ as the partial sum

$$M(s, p_r; N_1, N_2) = \sum_{n=N_1}^{N_2} \frac{\mu(n, p_r)}{n^s}. \tag{12}$$

Then, we have

$$M(s, p_{r-1}; 1, Np_r) = M(s, p_r; 1, Np_r) - \frac{1}{p_r^s} M(s, p_r; 1, N). \tag{13}$$

Since the series $M(s, p_r)$ is conditionally convergent, then the partial sums $M(s, p_r; 1, Np_r)$ and $M(s, p_r; 1, N)$ are both convergent to $M(s, p_r)$ as N approaches infinity. Hence, as N approaches infinity, we obtain

$$M(s, p_{r-1}) = \lim_{N\to\infty} M(s, p_{r-1}; 1, Np_r) = M(s, p_r) \left(1 - \frac{1}{p_i^s}\right).$$

By repeating this process $r - 1$ times, we then obtain

$$M(s) = M(s, p_r) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right).$$

Note that if we multiply both sides of the above equation by $\prod_{i=1}^{r} \left(1 + {p_i}^{-s}\right)$

$$M(s, p_r) = \frac{1}{\zeta(s) \prod_{i=1}^{r} \left(1 - p_i^{-2s}\right)} \prod_{i=1}^{r} \left(1 + \frac{1}{p_i^s}\right).$$

As p_r approaches infinity, we then have

$$M(s, p_r) = \frac{\zeta(2s)}{\zeta(s)} \prod_{i=1}^{r} \left(1 + \frac{1}{p_i^s}\right).$$

It should be pointed out that the sieving method applied to the Dirichlet series with Mobious function can be also applied to the Dirichlet series with Lioville function. The Dirichlet series L(s) with Lioville Function $\lambda(n)$ is defined as

$$L(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}, \tag{14}$$

where
$\lambda(n) = 1$, if $n = 1$,
$\lambda(n) = 1$, if n has an even number of prime factors including multiplicities,
$\lambda(n) = -1$, if n has an odd number of prime factors including multiplicities.

Following the same process, we define the series $L(s, p_r)$ as

$$L(s, p_r) = \sum_{n=1}^{\infty} \frac{\lambda(n, p_r)}{n^s}, \tag{15}$$

where
$\lambda(n, p_r) = \lambda(n)$, if $n \in \{1, I(p_r)\}$,
otherwise, $\lambda(n, p_r) = 0$.

It can be easily shown that $L(s, p_r)$ converges absolutely for $\Re(s) > 1$ for every prime number p_r. Furthermore, it can be also shown that, for $\Re(s) > 1$, $L(s, p_r)$ satisfies the following equation

$$L(s, p_r) = L(s) \prod_{i=1}^{r} \left(1 + \frac{1}{p_i^s}\right).$$

It is well known in the literature that, on RH, we have

$$\sum_{n \leq x} \lambda(n) = O(x^{1/2+\epsilon}),$$

where ϵ is an arbitrary small number.

Using the above equation and following similar steps to those used for Theorems (1) and (2), we may obtain the following theorem.

Theorem 3 *For $s = \sigma + it$, where $0.5 < \sigma \leq 1$ and for every prime number p_r, the series $L(s)$ converges conditionally if and only if the series $L(s, p_r)$ converges conditionally. Furthermore, $L(s)$ and $L(s, p_r)$ are related as follows*

$$L(s, p_r) = L(s) \prod_{i=1}^{r} \left(1 + \frac{1}{p_i^s}\right). \tag{16}$$

4 Functional representation of $\zeta(s)$ using its partial Euler product.

Theorem 1 of the previous section provides a relationship between $\zeta(s) = 1/M(s)$ and the partial Euler product $\prod_{i=1}^{r}(1 - 1/p_i^s)$. In this section, we will use the prime counting function to derive a functional representation for $\zeta(s)$ using its partial Euler product. This functional representation is then used to compute the sum $\sum_{i=1}^{r} {p_i}^{\sigma}$ for $\sigma < 1$. In section 6, we will use this sum to show that the series $M(\sigma, p_r)$ diverges for $\sigma < 1$.

We will start this task by first writing $\zeta(s)$ for $\sigma > 1$ as follows

$$1/\zeta(s) = \prod_{i=1}^{\infty}\left(1-\frac{1}{p_i^s}\right) = \prod_{i=1}^{r}\left(1-\frac{1}{p_i^s}\right)\prod_{r+1}^{\infty}\left(1-\frac{1}{p_i^s}\right). \tag{17}$$

For $\sigma > 0.5$, we have

$$\log\prod_{i=r1}^{r2}\left(1-\frac{1}{p_i^s}\right) = \sum_{i=r1}^{r2}\log\left(1-\frac{1}{{p_i}^s}\right),$$

or

$$\log\prod_{i=r1}^{r2}\left(1-\frac{1}{p_i^s}\right) = \sum_{i=r1}^{r2}\left(-\frac{1}{{p_i}^s}-\frac{1}{2{p_i}^{2s}}-\frac{1}{3{p_i}^{3s}}-...\right).$$

Let δ be defined as the sum

$$\delta = \sum_{i=r1}^{r2}\left(-\frac{1}{2{p_i}^{2s}}-\frac{1}{3{p_i}^{3s}}-\frac{1}{4{p_i}^{4s}}...\right). \tag{18}$$

Thus,

$$\log\prod_{i=r1}^{r2}\left(1-\frac{1}{p_i^s}\right) = -\sum_{i=r1}^{r2}\frac{1}{{p_i}^s}+\delta. \tag{19}$$

Since $|\delta| < \sum_{n=p_{r1}}^{\infty}\left(\frac{1}{2n^{2\sigma}}+\frac{1}{3n^{3s}}+\frac{1}{4n^{4s}}...\right)$, thus $\delta = O(p_{r1}^{1-2\sigma}/(2\sigma-1))$. Furthermore, if $2\sigma-1$ is a fixed positive number, then $\delta = O(p_{r1}^{1-2\sigma})$. It should be pointed out that for $\sigma = 0.5$ and $t \neq 0$, δ is convergent to a finite number by the virtue of the Prime Number Theorem.

Using the Prime Number Theorem (PNT) with a suitable constant $a > 0$, the number of primes less than x is given by [4, page 43]

$$\pi(x) = \mathrm{Li}(x) + O\left(xe^{-a\sqrt{\log x}}\right), \tag{20}$$

or

$$\pi(x) = \mathrm{Li}(x) + O\left(x/(\log x)^k\right), \tag{21}$$

where $\mathrm{Li}(x)$ is the Logarithmic Integral of x and k is a number greater than zero.

Using Stieltjes integral [5], we may write the sum $\sum_{i=r1}^{r2}\frac{1}{{p_i}^\sigma}$ for $\sigma > 1$ as follows

$$\sum_{i=r1}^{r2}\frac{1}{{p_i}^\sigma} = \int_{x=p_{r1}}^{p_{r2}}\frac{d\pi(x)}{x^\sigma}. \tag{22}$$

Using Equation (21) for the representation of $\pi(x)$, we may then write the integral in Equation (22) as [5, Theorem 2, page 57]

$$\sum_{i=r1}^{r2}\frac{1}{p_i^\sigma} = \int_{p_{r1}}^{p_{r2}}\frac{1}{x^\sigma}\frac{1}{\log x}dx + O\left(\frac{1}{(\log p_{r1})^k}\right), \tag{23}$$

where k is a number greater than zero. Therefore,

$$\sum_{i=r1}^{r2}\frac{1}{p_i^\sigma} = \int_{p_{r1}}^{\infty}\frac{1}{x^\sigma}\frac{1}{\log x}dx - \int_{p_{r2}}^{\infty}\frac{1}{x^\sigma}\frac{1}{\log x}dx + O\left(\frac{1}{(\log p_{r1})^k}\right). \tag{24}$$

Recalling that the Exponential Integral $E_1(r)$ is given by

$$E_1(r) = \int_r^\infty \frac{e^{-u}}{u} du,$$

and using the substitutions $u = (\sigma - 1)\log p_r$, $du = (\sigma - 1)dx/x$ and $x^\sigma/x = e^u$, then for $\sigma > 1$, we may write Equation (24) as

$$\sum_{i=r1}^{r2} \frac{1}{p_i^\sigma} = E_1\left((\sigma - 1)\log p_{r1}\right) - E_1\left((\sigma - 1)\log p_{r2}\right) + O\left(\frac{1}{(\log p_{r1})^k}\right). \quad (25)$$

Combining Equations (19) and ((25)) and noting that, for $\sigma > 1$, $E_1\left((\sigma - 1)\log p_{r2}\right)$ approaches zero as p_{r2} approaches infinity, we may write Equation (17) for $\sigma > 1$ as

$$-\log\zeta(\sigma) = \sum_{i=1}^{r} \log\left(1 - \frac{1}{{p_i}^\sigma}\right) - \sum_{i=r+1}^{\infty} \frac{1}{{p_i}^\sigma} + \delta,$$

or

$$\log\zeta(\sigma) + \sum_{i=1}^{r} \log\left(1 - \frac{1}{{p_i}^\sigma}\right) - E_1\left((\sigma - 1)\log p_{r+1}\right) = \epsilon,$$

where $\epsilon = O(1/(\log p_{r1})^k)$ is an arbitrarily small number attained by setting p_r sufficiently large. Therefore,

$$\zeta(\sigma)\prod_{i=1}^{r}\left(1 - \frac{1}{p_i^\sigma}\right)\exp\left(-E_1((\sigma - 1)\log p_{r+1})\right) = 1 + \epsilon. \quad (26)$$

As p_r approaches infinity, ϵ approaches zero. Hence, the right side of the above equation approaches 1 as p_r approaches infinity.

Similarly, for $\Re(s) > 1$, we can use the following expression for $E_1(s)$

$$E_1(s) = \int_1^\infty \frac{e^{-xs}}{x} dx,$$

to show that

$$\lim_{r\to\infty}\left\{\zeta(s)\prod_{i=1}^{r}\left(1 - \frac{1}{p_i^s}\right)\exp\left(-E_1((s - 1)\log p_{r+1})\right)\right\} = 1. \quad (27)$$

Let the function $G(s, p_r)$ be defined as

$$G(s, p_r) = \zeta(s)\prod_{i=1}^{r}\left(1 - \frac{1}{p_i^s}\right)\exp\left(-E_1((s - 1)\log p_{r+1})\right) \quad (28)$$

where, $G(s, p_r)$ is a regular function for $\Re(s) > 1$. Referring to Equation (27), the function $G(s, p_r)$ approaches 1 as p_r approaches infinity. It should be noted that, for each p_r, the function $\exp\left(-E_1((s - 1)\log p_{r+1})\right)$ is an entire function, the function $\zeta(s)$ is analytic everywhere except at $s = 1$ and the function $\prod_{i=1}^{r}(1 - 1/p_i^s)$ is analytic for $\Re(s) > 0$. Thus, for any $\sigma > 1$, the function $G(s, p_r)$ can be considered as a sequence of analytic functions. Furthermore, as p_r (or r) approaches infinity, this sequence is uniformly convergent over the half plane with $\sigma > 1 + \epsilon$ (where, ϵ is an arbitrary small number). Therefore, by the virtue of the Weiestrass theorem, the limit is also analytic function [6] (Weiestrass theorem states that if the function sequence f_n is analytic over the region Ω and f_n is uniformly convergent to a function f, then f is also analytic on Ω and $f_n{}'$ converges uniformly to f' on Ω). If we define this limit as $G(s)$, where

$$G(s) = \lim_{r\to\infty} G(s, p_r) \tag{29}$$

then, $G(s)$ is analytic over the half plane $\Re(s) > 1$ and it is equal to 1 by the virtue of Equation (27).

The Prime Number Theorem (PNT) allows us to extend the above results to the line $s = 1 + it$. Moreover, we will show that if RH is valid, then for the strip $s = \sigma + it$ where, $0.5 < \sigma < 1$, the above results will also be valid with the limit of $G(s, p_r)$ is 1 as p_r approaches infinity.

We will start this task by showing that although both $\zeta(s)$ and $E_1((s-1)\log p_{r+1})$ have a singularity at $s = 1$, the product $G(s, p_r)$ has a removable singularity at $s = 1$ for every p_r. This can be shown by first expanding $\zeta(s)$ as a Laurent series about its singularity at $s = 1$

$$\zeta(s) = \frac{1}{s-1} + \gamma - \gamma_1(s-1) + \gamma_2\frac{(s-1)^2}{2!} - \gamma_3\frac{(s-1)^3}{3!} + ..., \tag{30}$$

where γ is the Euler-Mascheroni constant and γ_i's are the Stieltjes constants. For $s = 1 + \epsilon$, where $\epsilon = \epsilon_1 + i\epsilon_2$, ϵ_1 and ϵ_2 are arbitrary small numbers, the above equation can be written as

$$\zeta(s) = \frac{1}{\epsilon} + \gamma - \gamma_1\epsilon + \gamma_2\frac{\epsilon^2}{2!} - \gamma_3\frac{\epsilon^3}{3!} + ... \tag{31}$$

Furthermore, for $\sigma > 1$, using the definition of the Exponential Integral, we may write $E_1(s)$ as

$$E_1(s) = -\gamma - \log s + s - \frac{s^2}{2\,2!} + \frac{s^3}{3\,3!} - \frac{s^4}{4\,4!} + ... \tag{32}$$

Thus, for $s = 1 + \epsilon$, we have

$$\exp\left(-E_1((s-1)\log p_r)\right) = e^{\gamma}\epsilon\,\log p_r\,\exp\left(-\epsilon\log p_r + \frac{(\epsilon\log p_r)^2}{2\,2!} - \frac{(\epsilon\log p_r)^3}{3\,3!} + ...\right). \tag{33}$$

By taking the product $\zeta(s)\exp\left(-E_1((s-1)\log p_r)\right)$ and allowing ϵ to approach zero, we then obtain at $s = 1$ (in the same sense as computing $\sin x/x$ at $x = 0$)

$$\zeta(s)\,\exp\left(-E_1((s-1)\log p_r)\right) = e^{\gamma}\,\log p_r. \tag{34}$$

However, it is well known that the partial Euler product at $s = 1$ can be written as [8]

$$\prod_{i=1}^{r}\left(1 - \frac{1}{p_i}\right) = \frac{e^{-\gamma}}{\log p_r} + O\left(\frac{1}{(\log p_r)^2}\right). \tag{35}$$

Multiplying Equations (34) and (35), we may conclude that at $s = 1$, $G(s, p_r)$ approaches 1 as p_r approaches infinity. Furthermore, for $s = 1 + it$ and $t \neq 1$, the value of $\exp(-E_1(it\log p_r))$ approaches 1 as p_r approaches infinity and since

$$\lim_{r\to\infty}\left\{\zeta(s)\prod_{i=1}^{r}\left(1 - \frac{1}{p_i{}^s}\right)\right\} = 1,$$

therefore, for $s = 1 + it$, we have the following

$$\lim_{r\to\infty} G(s, p_r) = \lim_{r\to\infty} \left\{ \zeta(s) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right) \exp\left(-E_1((s-1)\log p_{r+1})\right) \right\} = 1.$$

So far, we have shown that the function $G(s, p_r)$ is uniformly convergent to 1 when $\Re(s) > 1$ and using PNT, $G(s, p_r)$ is convergent to 1 for $\Re(s) = 1$. In the following, we will show that, assuming the validity of the Riemann Hypothesis, the function $G(s, p_r)$ is uniformly convergent to 1 for every value of s with $\Re(s) > 0.5 + \epsilon$, where ϵ is an arbitrary small number. Toward this goal, we will first show that the function $G(s, p_r)$ is convergent for any value of s on the real axis with $\sigma > 0.5$. This can be achieved by first writing the expressions for $G(\sigma, p_{r1})$ and $G(\sigma, p_{r2})$ (where $r2$ is an arbitrary large number greater than $r1$)

$$G(\sigma, p_{r1}) = \zeta(\sigma) \exp\left(-E_1((\sigma-1)\log p_{r1+1})\right) \prod_{i=1}^{r1} \left(1 - \frac{1}{p_i^\sigma}\right), \tag{36}$$

$$G(\sigma, p_{r2}) = \zeta(\sigma) \exp\left(-E_1((\sigma-1)\log p_{r2+1})\right) \prod_{i=1}^{r2} \left(1 - \frac{1}{p_i^\sigma}\right). \tag{37}$$

Since the function $G(s, p_r)$ is analytic that is not equal to 0 for $\sigma > 0.5$, hence we can divide Equation (37) by Equation (36) and then take the logarithm to obtain

$$\log\left(\frac{G(\sigma, p_{r2})}{G(\sigma, p_{r1})}\right) = E_1\left((\sigma-1)\log p_{r1+1}\right) - E_1\left((\sigma-1)\log p_{r2+1}\right) + \log\left(\prod_{i=r1+1}^{r2} \left(1 - \frac{1}{{p_i}^\sigma}\right)\right). \tag{38}$$

To compute the logarithm of the partial Euler product in Equation (38), we recall Equation (19)

$$\log \prod_{r1+1}^{r2} \left(1 - \frac{1}{p_i^s}\right) = -\sum_{i=r1+1}^{r2} \frac{1}{{p_i}^s} + \delta,$$

where $\delta = O(p_{r1}^{1-2\sigma}/(2\sigma-1))$. Furthermore, on RH, we have

$$\pi(x) = \mathrm{Li}(x) + O\left(\sqrt{x}\,\log x\right), \tag{39}$$

where $\mathrm{Li}(x)$ is the Logarithmic Integral of x. Using Equation (39) for the representation of the prime counting function, we may then obtain (Appendix 2)

$$\sum_{i=r1+1}^{r2} \frac{1}{{p_i}^\sigma} = E_1((\sigma-1)\log p_{r1+1}) - E_1((\sigma-1)\log p_{r2}) + \varepsilon,$$

where $\varepsilon = O\left(\frac{t}{(\sigma-0.5)^2}\, {p_{r1}}^{0.5-\sigma} \log p_{r1}\right)$. Hence, Equation (38) can be written as

$$\log\left(\frac{G(\sigma, p_{r2})}{G(\sigma, p_{r1})}\right) = \varepsilon + \delta + E_1((\sigma-1)\log p_{r2}) - E_1((\sigma-1)\log p_{r2+1}).$$

Since, for $\sigma > 0.5+\epsilon$, $\varepsilon+\delta$ and $E_1((\sigma-1)\log p_{r2}) - E_1((\sigma-1)\log p_{r2+1})$ can be made arbitrary small by choosing p_{r1} arbitrary large, thus the limit of $G(\sigma, p_r)$ exists as p_r approaches infinity and it is given by

$$G(\sigma) = \lim_{r\to\infty} G(\sigma, p_r) \tag{40}$$

This proves that, on RH, $G(\sigma, p_r)$ is convergent as p_r approaches infinity and thus $G(\sigma)$ exists for $\sigma > 0.5$. In Appendix 3, we have shown that, on RH and for $\Re(s) > 0.5$, we have

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^s} = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}) + \varepsilon,$$

where $\varepsilon = O\left(\frac{t+1}{(\sigma-0.5)^2}\, {p_{r1}}^{0.5-\sigma} \log p_{r1}\right)$. Thus, we can follow the same steps and show that $G(s, p_r)$ is convergent as p_r approaches infinity and thus $G(s)$ exists for $\Re(s) > 0.5$.

It should be noted that, while the function sequence $G(s, p_r)$ is not uniformly convergent when the region of convergence is extended all the way to the line $\sigma = 0.5$, it is however uniformly convergence for any strip with $\sigma > 0.5 + \epsilon$, where ϵ is an arbitrary small number. This follows from the fact that ε (or, the O term) is bounded for any $\sigma > 0.5 + \epsilon$. Since $G(s, p_r)$ is analytic for $\Re(s) > 0$ and it is uniformly convergent for $\Re(s) > 0.5 + \epsilon$, thus $G(s)$ is analytic for the half right complex plain with $\Re(s) > 0.5 + \epsilon$ (Weiestrass theorem [6]). Since we have shown that $G(s) = 1$ for $\Re(s) \geq 1$, thus on RH, $G(s) = 1$ for $\Re(s) > 0.5 + \epsilon$. Hence, we have the following theorem

Theorem 4 *For s = $\sigma + it$ and $\sigma > 0.5$, the following holds if RH is valid*

$$\lim_{r\to\infty} \left\{ \zeta(s) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right) \exp\left(-E_1((s-1)\log p_{r+1})\right) \right\} = 1. \tag{41}$$

$$\lim_{r\to\infty} \left\{ M(s, p_r) \exp\left(E_1((s-1)\log p_{r+1})\right) \right\} = 1. \tag{42}$$

It should be pointed out that Theorem 4 can be generalized to the case where there are no non-trivial zeros for values of s with $\Re(s) > a$ (where, $a > 0.5$). For this case, Equation (41) is valid for every s with $\Re(s) > a$ and ε in Appendix 3 is given by $O\left(\frac{t+1}{(\sigma-a)^2}\, {p_{r1}}^{a-\sigma} \log p_{r1}\right)$.

Equation (41) of Theorem 4 can be written as follows

$$\log \zeta(s) + \log \prod_{i=1}^{r2} \left(1 - \frac{1}{p_i^s}\right) - E_1\left((s-1)\log p_{r2+1}\right) = 0,$$

where the equality of both sides is attained as r_2 (or p_{r2}) approaches infinity. It should be pointed out that both functions $\log \zeta(s)$ and $E_1((s-1)\log p_{r2+1})$ have a branch cut along the real axis where $0.5 \leq \sigma < 1$, while the difference (i.e. $\log \zeta(s) - E_1((s-1)\log p_{r2+1})$) does not have a branch cut. For $r < r2$, the above equation can be then written as

$$\log \zeta(s) = E_1\left((s-1)\log p_{r2+1}\right) - \sum_{i=1}^{r} \log\left(1 - \frac{1}{p_i^s}\right) - \sum_{i=r+1}^{r2} \log\left(1 - \frac{1}{p_i^s}\right).$$

Since, on RH and for $\Re(s) > 0.5$, (refer to Appendix 3)

$$-\sum_{i=r+1}^{r2} \log\left(1 - \frac{1}{p_i^s}\right) = \sum_{i=r+1}^{r2} \frac{1}{{p_i}^s} + \delta = E_1\left((s-1)\log p_{r+1}\right) - E_1\left((s-1)\log p_{r2}\right) + \varepsilon + \delta$$

where $\varepsilon = O\left(\frac{t+1}{(\sigma-0.5)^2}\, {p_r}^{0.5-\sigma}\log p_r\right)$ and $\delta = O(p_r^{1-2\sigma}/(1-2\sigma))$, therefore

$$\log\zeta(s) = -\sum_{i=1}^{r}\log\left(1-\frac{1}{p_i^s}\right) + E_1\left((s-1)\log p_{r+1}\right) + O\left(\frac{t+1}{(\sigma-0.5)^2}\, {p_{r1}}^{0.5-\sigma}\log p_r\right). \quad (43)$$

Equation (43) represents well the singularity of $\log\zeta(s)$ at $s=1$ and it allows analytic continuation for values of s with $\Re(s)<1$. This analytic continuation should extend all the way to the non-trivial zeros with the highest value of σ. Unfortunately, Equation (43) poorly represents $\zeta(s)$ in the vicinity of the non-trivial zeros as the O term grows much faster than the growth of $\log\zeta(s)$ in the vicinity of the simple non-trivial zeros. In the next section, we will use the von Mangoldt function to provide a better representation for $\log\zeta(s)$ in the vicinity of the no-trivial zeros.

5 Partial Euler product functional representation of $\zeta(s)$ using von Mangoldt function.

The derivation of Equation (43) was based on computing the sum $\sum_{i=r1}^{r2} 1/p_i^s$ (Appendix 3) as follows

$$\sum_{i=r1}^{r2}\frac{1}{p_i^s} = \int_{p_{r1}}^{p_{r2}}\frac{d\pi(x)}{x^s} = \int_{p_{r1}}^{p_{r2}}\frac{1}{x^s\log x}dx + \int_{p_{r1}}^{p_{r2}}\frac{1}{x^s}dO\left(\sqrt{x}\log x\right)dx.$$

The above sum can be also computed using the von Mangoldt function $\Lambda(n)$ (where $\Lambda(n) = \log p$, if $n=p^k$ for some prime p and integer $k\geq 1$, otherwise, $\Lambda(n)=0$) to obtain

$$\sum_{i=r1}^{r2}\frac{1}{p_i^s} = \sum_{n=r1}^{r2}\frac{1}{n^s\log n}\Lambda(n) + \Delta, \quad (44)$$

where Δ is added to eliminate the contribution by the terms of the form m^{-s}, where $m=p^k$ and $2\leq k<\lfloor\log_2 p_{r2}\rfloor+1$. In other words, Δ is given by

$$\Delta = \sum_{p_i=\lfloor\sqrt{p_{r1}}\rfloor}^{\lfloor\sqrt{p_{r2}}\rfloor}\frac{1}{2{p_i}^{2s}} + \sum_{p_i=\lfloor\sqrt[3]{p_{r1}}\rfloor}^{\lfloor\sqrt[3]{p_{r2}}\rfloor}\frac{1}{3{p_i}^{3s}} + \ldots + \sum_{p_i=\lfloor\sqrt[L]{p_{r1}}\rfloor}^{\lfloor\sqrt[L]{p_{r2}}\rfloor}\frac{1}{L{p_i}^{Ls}}, \quad (45)$$

where $L=\lfloor\log_2 p_{r2}\rfloor+1$ and $\lfloor x\rfloor$ is the integer value of x. The order of Δ is determined by the order of the first term $\sum_{p_i=\lfloor\sqrt{p_{r1}}\rfloor}^{\lfloor\sqrt{p_{r2}}\rfloor}0.5/{p_i}^{2s}$. Thus, the order of Δ can be computed (in the same way the order of δ was computed) to obtain $\Delta = O((\sqrt{p_{r1}})^{1-2\sigma}/(2\sigma-1)) = O(p_{r1}^{0.5-\sigma}/(2\sigma-1))$. Furthermore, if $2\sigma-1$ is a fixed positive number, then $\Delta = O(p_{r1}^{0.5-\sigma})$. It should be pointed out that for $\sigma=0.5$ and $t\neq 0$, Δ is convergent to a finite number by the virtue of PNT.

Since the Chebyshev function $\psi(x)$ is given by the following sum

$$\psi(x) = \sum_{n=1}^{x}\Lambda(n)$$

therefore, using the Stieltjes integral, one may write the sum of Equation (44) as the following integral

$$\sum_{i=r1}^{r2}\frac{1}{p_i^s} = \int_{p_{r1}}^{p_{r2}}\frac{1}{x^s\log x}d\psi(x) + \Delta, \quad (46)$$

where $\psi(x)$ is also given by [1]

$$\psi(x) = x - \sum_\rho \frac{x^\rho}{\rho} + \sum_n \frac{x^{-2n}}{2n} - \frac{\zeta'(0)}{\zeta(0)} \tag{47}$$

It should be pointed out that the first term x in Equation (47) is attributed to the pole of $\zeta(s)$ at $s = 1$, the sum over ρ (or non-travail zeros) is attributed to the non-trivial zeros in the critical strip and the sum over n is attributed to the trivial zeros. Hence, Equation (46) can be written as

$$\sum_{i=r1}^{r2} \frac{1}{p_i^s} = \int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx - \int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} d\left(\sum_\rho \frac{x^\rho}{\rho}\right) + \Delta \tag{48}$$

where the contribution by the last two terms of Equation (47) is negligible compared with the term Δ. In Appendix (3), we have shown that

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}). \tag{49}$$

For the integral with the sum over ρ, we first compute the integral over the ρ's with $|\Im(\rho)| < T$. Thus, we have

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} d\left(\sum_{|\Im(\rho)|<T} \frac{x^\rho}{\rho}\right) = \sum_{|\Im(\rho)|<T} \left(\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} d\left(\frac{x^\rho}{\rho}\right)\right). \tag{50}$$

For the above integral, for each ρ, $|x^\rho/\rho|$ is a continuous function and bounded over the range $p_{r1} \le x \le p_{r2}$, therefore the interchange between the differentiation and summation is justified (alternatively, one may integrate by parts to get the same results, where the sum becomes the integrand and the differentiation is applied to the term $1/(x^s \log x)$ instead of the sum). Furthermore, for each ρ, $\Re(s)$ is higher than $\Re(\rho)$, therefore $\int_{p_{r1}}^{p_{r2}} |x^{\rho-1}/(x^s \log x)| dx$ is convergent as p_{r2} approaches infinity. Hence, the interchange between the integral and the sum is justified. Therefore, Equation (50) can be written as

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} d\left(\sum_{|\Im(\rho)|<T} \frac{x^\rho}{\rho}\right) = \sum_{|\Im(\rho)|<T} \left(E_1((s-\rho)\log p_{r1}) - E_1((s-\rho)\log p_{r2})\right). \tag{51}$$

In Appendix 4, we have shown that the sum on the right side of (51) is convergent as T approaches infinity. Thus,

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} d\left(\sum_\rho \frac{x^\rho}{\rho}\right) = \sum_\rho \left(E_1((s-\rho)\log p_{r1}) - E_1((s-\rho)\log p_{r2})\right). \tag{52}$$

Consequently,

$$\sum_{i=r1}^{r2} \frac{1}{p_i^s} = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}) - \sum_\rho \left(E_1((s-\rho)\log p_{r1}) - E_1((s-\rho)\log p_{r2})\right) + \Delta, \tag{53}$$

where $\Delta = O(p_{r1}^{0.5-\sigma})$. If the function $J(s, p_{r1}, p_{r2})$ is defined as follows

$$J(s, p_{r1}, p_{r2}) = \sum_{i=r_1}^{r_2} \frac{1}{p_i^s} - E_1((s-1)\log p_{r1}) + E_1((s-1)\log p_{r2}), \tag{54}$$

then

$$J(s,p_{r1},p_{r2}) = \sum_{\rho} \left(E_1((s-\rho)\log p_{r1}) - E_1((s-\rho)\log p_{r2}) \right) + \Delta. \tag{55}$$

We notice that the function $J(s,p_{r1},p_{r2})$ is analytic for every p_{r1}, p_{r2} and s. This follows from the fact that although the functions $E_1((s-1)\log p_{r1})$ and $E_1((s-1)\log p_{r2})$ have a branch cut on the negative real axis, the difference does not have a branch cut. Moreover, although the functions $E_1((s-1)\log p_{r1})$ and $E_1((s-1)\log p_{r2})$ have a singularity at $s=1$, the difference has a removable singularity at $s=1$. This follows from the fact that as s approaches 1, the difference can be written as

$$E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}) = -\log\left((1-s)\log p_{r1}\right) - \gamma + \log\left((1-s)\log p_{r2}\right) + \gamma$$

or,

$$E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}) = -\log\log p_{r1} + \log\log p_{r2} \tag{56}$$

Therefore, the function $J(s,p_{r1},p_{r2})$ is analytic for every p_{r1}, p_{r2} and s.

Referring to Appendix (4), we notice that for every s with $\Re(s) > \max \Re(\rho)$, the term $\sum_\rho \left(E_1((s-\rho)\log p_{r1}) - E_1((s-\rho)\log p_{r2})\right)$ approaches zero as p_{r1} approaches infinity. Thus, for $\Re(s) > \max\Re(\rho)$, we have

$$\sum_{i=r1}^{r2} \frac{1}{p_i^s} = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}) + O(p_{r1}^{-\sigma+\max\Re(\rho)}). \tag{57}$$

To compute $\log\zeta(s)$ using Equation (47), we recall Equation (41) of Theorem 1. Thus, for every s with $\Re(s) > \max\Re(\rho)$, we have

$$\log\zeta(s) = E_1\left((s-1)\log p_{r2+1}\right) - \sum_{i=1}^{r2} \log\left(1 - \frac{1}{p_i^s}\right),$$

where the equality of both sides is attained as p_{r2} approaches infinity. Alternatively,

$$\log\zeta(s) = E_1\left((s-1)\log p_{r2+1}\right) - \sum_{i=1}^{r} \log\left(1 - \frac{1}{p_i^s}\right) - \sum_{i=r+1}^{r2} \log\left(1 - \frac{1}{p_i^s}\right).$$

Hence,

$$\log\zeta(s) = E_1\left((s-1)\log p_{r2+1}\right) - \sum_{i=1}^{r} \log\left(1 - \frac{1}{p_i^s}\right) + \sum_{i=r+1}^{r2} \frac{1}{{p_i}^s} + \delta.$$

Consequently, using Equations (46), (48), (49) and (52) (and noting that when $\Re(s-\rho) > 0$ for every ρ, the sum $\sum_\rho E_1\left((s-\rho)\log p_{r2}\right)$ approaches zero as p_{r2} approaches infinity), we have the following theorem

Theorem 5 *If $\Re(s-\rho) > 0$ for every non-trivial zero ρ, then*

$$\log\zeta(s) = -\sum_{i=1}^{r} \log\left(1 - \frac{1}{p_i^s}\right) + E_1\left((s-1)\log p_{r+1}\right) - \sum_\rho E_1\left((s-\rho)\log p_{r+1}\right) + O\left({p_r}^{0.5-\sigma}\right). \tag{58}$$

where $\sigma = \Re(s)$ and the O term is given by $\delta + \Delta$.

The differentiation of $\log \zeta(s)$ or $\zeta'(s)/\zeta(s)$ has been extensively used in the analysis of the Riemann zeta function. Using Equation (58), we may obtain a functional representation of $\zeta'(s)/\zeta(s)$ in terms of the partial Euler product of $\zeta(s)$.

Theorem 6 *If $\Re(s-\rho) > 0$ for every non-trivial zero ρ, then*

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\log \prod_{i=1}^{r-1}\left(1-\frac{1}{p_i^s}\right)\right) - \frac{{p_r}^{-(s-1)}}{s-1} + \sum_{\rho}\frac{{p_r}^{-(s-\rho)}}{s-\rho} + O\left({p_r}^{0.5-\sigma}\right). \tag{59}$$

where $\sigma = \Re(s)$ and the O term is given by $d(\delta+\Delta)/ds$.

Although Theorems (4), (5) and (6) provide a functional representation for $\zeta(s)$ in terms of it partial Euler product, our attempts to prove or disprove the Riemann hypothesis using these representations in conjunction with other properties (such as the growth of $\zeta(1+iT)$ with T) have failed. However, the sum $\sum_{p_{r1}\le p_i \le p_{r2}} 1/{p_i}^{\sigma}$ for $\sigma < 1$ (that was computed using these theorems) has been successfully used to examine the convergence of the series $M(\sigma)$ for $\sigma < 1$ as described in the next section.

6 The convergence of the series $M(\sigma, p_r)$ and $M(\sigma)$ for $\sigma \le 1$.

In this section, we will first provide an estimate for the partial sum $M(1, p_r; 1, {p_r}^a)$ as a approaches infinity. This estimate will be computed by using Equation (57) and noting that $M(1, p_r)$ equals zero for every p_r. Therefore, for every p_r, $M(1, p_r; 1, {p_r}^a)$ approaches zero as a approaches infinity. We have also shown in Appendix 5 that, for every p_r and N, we have

$$|M(1, p_r; 1, N)| = \left|\sum_{n=1}^{N}\frac{\mu(n, p_r)}{n}\right| \le 2.$$

The estimation of the partial sum $M(1, p_r; 1, {p_r}^a)$ as a function of a will then be used to establish a relationship between $M(1, p_r; 1, {p_r}^a)$ and $M(\sigma, p_r; 1, {p_r}^a)$. This relationship is then used to show that $M(\sigma, p_r)$ and $M(\sigma)$ diverge for $\sigma < 1$. We will describe the details of our method in the following three steps.

- In the first step, we will show that, for every a and as p_r approaches infinity, the partial sum $M(1, p_r; 1, {p_r}^a)$ is a function of only a (independent of p_r).

Toward this end, we define the function $f(a, p_r)$ as

$$f(a, p_r) = M(1, p_r; 1, {p_r}^a) = \sum_{n=1}^{{p_r}^a}\frac{\mu(n, p_r)}{n}.$$

We will then show that, for every a and as p_r approaches infinity, the function $f(a, p_r)$ approaches a deterministic function $F(a)$. In other words; if we plot $M(1, p_r; 1, N)$ (where $N = {p_r}^a$) as a function of $a = \log N/\log p_r$, then for each value of a and as p_r approaches infinity, $f(a, p_r)$ approaches a unique value $F(a)$. This is equivalent to the statement

$$F(a) = \lim_{p_r\to\infty} f(a, p_r) = \lim_{p_r\to\infty} M(1, p_r; 1, {p_r}^a).$$

This result can be achieved by first noting that the partial sum $M(1, p_r; 1, {p_r}^a)$ for $1 < a < 2$ is given by

$$M(1, p_r; 1, {p_r}^a) = 1 - \sum_{p_r \le p_i < {p_r}^a} \frac{1}{p_i}.$$

If we define $M_1(1, p_r; 1, {p_r}^a)$ as

$$M_1(1, p_r; p_r, {p_r}^a) = \sum_{p_r \le p_i < {p_r}^a} \frac{1}{p_i},$$

then, using Stieltjes integral, we obtain

$$M(1, p_r; 1, {p_r}^a) = 1 - M_1(1, p_r; p_r, {p_r}^a) = 1 - \int_{p_r}^{{p_r}^a} \frac{d\pi(x)}{x} = 1 - \int_1^a \frac{d\pi({p_r}^y)}{{p_r}^y}.$$

On RH, we have

$$d\pi({p_r}^y) = d\text{Li}({p_r}^y) + dO(\sqrt{{p_r}^y}\log({p_r}^y)),$$

or

$$d\pi({p_r}^y) = \frac{1}{\log({p_r}^y)} \frac{\log p_r}{{p_r}^y} dy + dO(\sqrt{{p_r}^y}\log({p_r}^y)) = \frac{dy}{y{p_r}^y} + dO(\sqrt{{p_r}^y}\log({p_r}^y)).$$

Hence, for $1 < a < 2$, we have

$$M(1, p_r; 1, {p_r}^a) = 1 - \int_1^a \frac{dy}{y} + \int_1^a \frac{dO(\sqrt{{p_r}^y}\log({p_r}^y))}{{p_r}^y} = 1 - \log(a) + O(g_1(p_r, a)),$$

where

$$O(g_1(p_r, a)) = \int_1^a \frac{dO(\sqrt{{p_r}^y}\log({p_r}^y))}{{p_r}^y}.$$

As p_r approaches infinity, $O(g_1(p_r, a)$ approaches zero. Consequently,

$$\lim_{p_r, N \to \infty} M(1, p_r; 1, {p_r}^a) = 1 - \log a.$$

The terms of the partial sum $M(1, p_r; 1, {p_r}^a)$ in the range $p_r \le x < {p_r}^3$ are either a reciprocal of a prime or a reciprocal of the product of two primes. Therefore, for $1 < a < 3$, we have

$$M(1, p_r; 1, {p_r}^a) = 1 - \sum_{p_r \le p_i < {p_r}^a} \frac{1}{p_i} + \sum_{p_r \le p_i p_j < {p_r}^a} \frac{1}{p_i p_j}$$

Let $M_2(1, p_r; 1, {p_r}^a)$ be defined as

$$M_2(1, p_r; 1, {p_r}^a) = \sum_{p_r \le p_i p_j < {p_r}^a} \frac{1}{p_i p_j} = \frac{1}{2} \sum_{p_r \le p_i < {p_r}^{a-1}} \frac{1}{p_i} M_1(1, p_r; p_r, p_r^a/p_i) + R_2,$$

where the factor of half was added since each term of the form $1/(p_i p_j)$ is repeated twice. It should be also noted that the second sum of the above equation includes non free-square terms. The term R_2 was added to offset the contribution by these non square-free terms. We will show later that the contribution by these terms (or R_2) approaches zero as p_r approaches infinity. Using Stieltjes integral, we then have

$$M_2(1, p_r; 1, {p_r}^a) = \frac{1}{2}\int_1^{a-1} \frac{d\pi({p_r}^y)}{{p_r}^y} \left(\log(a-y) + O(g_1(p_r, a-y))\right) + R_2.$$

Hence

$$M(1, p_r; 1, {p_r}^a) = 1 - \log(a) + O(g_1(p_r, a)) + \frac{1}{2}\int_1^{a-1} \frac{\log(a-y)}{y} dy + O(g_2(p_r, a-1)),$$

where

$$O(g_2(p_r, a)) = \int_1^{a-1} \frac{O(g_1(p_r, a-y))}{y} dy + \int_1^{a-1} \log(a-y) \frac{dO(\sqrt{{p_r}^y}\log({p_r}^y))}{p_r^y} +$$
$$\int_1^{a-1} O(g_1(p_r, a-y)) \frac{dO(\sqrt{{p_r}^y}\log({p_r}^y))}{p_r^y} + R_2.$$

It can be easily shown that the three integrals on the right side of the above equation approach zero as p_r approaches infinity. We will also show later that R_2 approaches zero as p_r approaches infinity. Thus, for $1 \le a < 3$, we have

$$\lim_{p_r, N \to \infty} M(1, p_r; 1, {p_r}^a) = 1 - \log a + \int_1^{a-1} \frac{\log(a-y)}{y} dy$$

Therefore, as p_r approaches infinity, $M(1, p_r; 1, {p_r}^a)$ is only dependent on a.

Repeating the previous process $a-1$ times and by using the induction method, we can show that, as p_r approaches infinity, the partial sum $M(1, p_r; 1, {p_r}^a)$ is dependent on only a. Specifically, we first write the partial sum $M(1, p_r; 1, {p_r}^a)$ as follows

$$M(1, p_r; 1, {p_r}^a) = 1 - M_1(1, p_r; 1, {p_r}^a) + M_2(1, p_r; 1, {p_r}^a) - ... + (-1)^j M_j(1, p_r; 1, {p_r}^a) + ...+$$
$$(-1)^{a-1} M_{a-1}(1, p_r; 1, {p_r}^a) + (-1)^a M_a(1, p_r; 1, {p_r}^a),$$

where

$$M_j(1, p_r; 1, {p_r}^a) = \sum_{p_r \le p_{i1} p_{i2} .. p_{ij} < {p_r}^a} \frac{1}{p_{i1} p_{i2} ... p_{ij}}.$$

If we assume that $M_{j-1}(1, p_r; 1, {p_r}^a)$ is given by

$$M_{j-1}(1, p_r; 1, {p_r}^a) = h_{j-1}(a) + O(g_{j-1}(p_r, a))$$

where $h_{j-1}(a)$ is a function of a and $O(g_{j-1}(p_r, a-1))$ approaches zero as p_r approaches infinity, then

$$M_j(1, p_r; 1, {p_r}^a) = \sum_{p_r \le p_{i1} p_{i2} .. p_{ij} < {p_r}^a} \frac{1}{p_{i1} p_{i2} ... p_{ij}} = \frac{1}{2} \sum_{p_r \le p_i < {p_r}^{a-1}} \frac{1}{p_i} M_{j-1}(1, p_r; p_r, p_r^a/p_i) + R_j,$$

where the factor of half was added since each term of the form $1/(p_{i1}p_{i2}...p_{ij})$ is repeated twice. It should be also noted that the second sum of the above equation includes non free-square terms. The term R_j was added to offset the contribution by these non square-free terms. We will show later that the contribution by these terms (or R_j) approaches zero as p_r approaches infinity. Using Stieltjes integral, we then have

$$M_j(1, p_r; 1, {p_r}^a) = \frac{1}{2}\int_1^{a-1} \frac{d\pi({p_r}^y)}{{p_r}^y}\left(h_{j-1}(a-y) + O(g_{j-1}(p_r, a-y))\right) + R_j.$$

Hence

$$M_j(1, p_r; 1, {p_r}^a) = \frac{1}{2}\int_1^{a-1} \frac{h_{j-1}(a-y)}{y}dy + O(g_j(p_r, a)),$$

where the first term is a definite integral with only one variable a. Thus, the definite integral is a function of a. We define this function as $h_j(a)$. The second term is given by

$$O(g_j(p_r, a)) = \int_1^{a-1} \frac{O(g_{j-1}(p_r, a-y))}{y}dy + \int_1^{a-1} h_{j-1}(a-y)\frac{dO(\sqrt{{p_r}^y}\log({p_r}^y))}{p_r^y} +$$
$$\int_1^{a-1} O(g_{j-1}(p_r, a-y))\frac{dO(\sqrt{{p_r}^y}\log({p_r}^y))}{p_r^y} + R_j.$$

It can be easily shown that the three integrals on the right side of the above equation approach zero as p_r approaches infinity. We will also show later that R_j approaches zero as p_r approaches infinity. Hence, as p_r approaches infinity, we have

$$\lim_{p_r, N\to\infty} M_j(1, p_r; 1, {p_r}^a) = \frac{1}{2}\int_1^{a-1} \frac{h_{j-1}(a-y)}{y}dy = h_j(a)$$

where $h_1(a) = \log(a)$. Hence, for every a and as p_r approaches infinity, we have

$$\lim_{p_r\to\infty} M(1, p_r; 1, {p_r}^a) = 1 - h_1(a) + h_2(a) - h_3(a) + ... + (-1)^a h_a(a) = F(a). \tag{60}$$

It should be pointed out that the above equation implies that the partial sums $M(1, p_r; 1, {p_r}^a)$ and $M(1, p_r^y; 1, {p_r}^{ay})$ (where, p_r^y is a prime number) have the same limit as p_r approaches infinity. Hence,

$$\lim_{p_r\to\infty} M(1, p_r; 1, {p_r}^a) = \lim_{p_r\to\infty} M(1, p_r^y; 1, {p_r}^{ay}) = F(a). \tag{61}$$

Equation (61) will be used in the next step to estimate the asymptotic behavior of the function $F(a)$ as a approaches infinity.

As it mentioned earlier, the partial sum $M(1, p_r; 1, {p_r}^a)$ constructed by this process included non square-free terms (i.e R_i's). In the following, we will show that, for every a and as p_r approaches infinity, the total contribution by these non square-free terms approaches zero as well. Toward this end, let S_0 be the sum of the terms with the factor $1/p_r^2$. Let S_1 be the sum of the remaining terms with the factor $1/(p_{r+1})^2$, S_2 be the sum of the remaining terms with the factor $1/(p_{r+2})^2$, and so on. Let R be sum of all the terms associated with non square-free terms. Thus, R is given by

$$R = \frac{1}{{p_r}^2} S_0 + \frac{1}{{p_{r+1}}^2} S_1 + ... + \frac{1}{{p_{r+l}}^2} S_L,$$

where p_{r+l} is the largest prime that its square is less than ${p_r}^a$. However,

$$|S_0|, |S_1|, ..., |S_l| < 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{{p_r}^a}.$$

Thus,

$$|S_0|, |S_1|, ..., |S_l| = O(a \log p_r).$$

Therefore

$$R = \left(\frac{1}{{p_r}^2} + \frac{1}{{p_{r+1}}^2} + ... + \frac{1}{{p_{r+l}}^2} \right) O(a \log p_r).$$

Hence, the contribution by the non square-free terms R is given by,

$$R = O(a \log p_r / p_r).$$

Consequently, for every a and as p_r approaches infinity, R (or the contribution by the non square-free terms) approaches zero.

- In the second step, we write the partial sum $M(1, p_r; 1, {p_r}^a)$ as the sum of two components. The first is the deterministic or regular component and it is given by $F(a)$. The second one is the irregular component $R(1, p_r; 1, {p_r}^a)$ given by $M(1, p_r; 1, {p_r}^a) - F(a)$. We will then show that as a approaches infinity, the function $F(a)$ approaches the function c/a for some constant c (this is the key step to disproving the Riemann Hypotheses as we will show in the third step that RH requires the exponential decay of $F(a)$ or $M(1, p_r; 1, p_r^a)$).

Toward this end, we write the partial sum $M(1, p_r; 1, {p_r}^a)$ as the following sum

$$M(1, p_r; 1, {p_r}^a) = 1 - \sum_{p_r \le p_i < {p_r}^{a/2}} \frac{1}{p_i} M(1, p_i; 1, {p_r}^a / p_i) - \sum_{{p_r}^{a/2} \le p_i < {p_r}^a} \frac{1}{p_i}. \quad (62)$$

Notice that the above equation is justified by the virtue that $M(1, p_i; 1, {p_r}^a / p_i)$ is comprised of 1 and the terms of the form $1/n$ where $p_i < n < p^a / p_i$ and every factor of n is greater than p_i. Using Stieltjes integral, we can write the above equation as follows

$$M(1, p_r; 1, {p_r}^a) = 1 - \int_1^{a/2} \frac{d\pi({p_r}^y)}{p_r^y} M(1, {p_r}^y; 1, p_r^a / p_r^y) - \int_{a/2}^{a} \frac{d\pi({p_r}^y)}{p_r^y}, \quad (63)$$

where, on RH, $d\pi({p_r}^y)$ is given by $d\mathrm{Li}({p_r}^y) + dO(\sqrt{{p_r}^y} \log({p_r}^y))$. As p_r approaches infinity, $M(1, {p_r}^y; 1, p_r^{a-y})$ approaches $F(a/y - 1)$ (refer to Equation (61)). Therefore, as p_r approaches infinity, we have

$$F(a) = 1 - \int_1^{a/2} \frac{F\left(\frac{a}{y} - 1\right)}{y} dy - \int_{a/2}^{a} \frac{dy}{y}. \quad (64)$$

It is shown in Appendix 5 that $|M(1,p_r;1,{p_r}^a)|\leq 2$ for every p_r and a. Hence, $|F(a)|\leq 2$. Consequently, $F(a)$ approaches zero as a approaches infinity (this follows from the fact that if $F(a)$ does not converge to zero, then the first integral of the above equation diverges as a approaches infinity which then leads to the divergence of $F(a)$. This contradicts our earlier statement that $|F(a)|\leq 2$). Thus, as a approaches infinity, we have

$$\int_1^{a/2}\frac{F\left(\frac{a}{y}-1\right)}{y}dy = 1-\log 2. \tag{65}$$

The most important step in our method to disproof the Riemann Hypothesis is the computation of the rate at which $F(a)$ (and consequently $M(1,p_r;1,p_r^a)$) decays to zero. It will be shown in the next step that for the Riemann Hypothesis to be valid, $M(1,p_r;1,p_r^a)$ must have an exponential decay to zero. In the following, we will show that $F(a)$ decays at at much slower rate. We will achieve this result by using Equation (64) to compute the difference $F(a+\Delta a)-F(a)$ (where, Δa is an arbitrary small number) to obtain

$$F(a+\Delta a)-F(a) = -\int_1^{(a+\Delta a)/2}\frac{F\left(\frac{a+\Delta a}{y}-1\right)}{y}dy+\int_1^{a/2}\frac{F\left(\frac{a}{y}-1\right)}{y}dy-\int_{(a+\Delta a)/2}^{(a+\Delta a)}\frac{dy}{y}+\int_{a/2}^{a}\frac{dy}{y}.$$

Hence,

$$F(a+\Delta a)-F(a) = -\int_1^{(a+\Delta a)/2}\frac{F\left(\frac{a+\Delta a}{y}-1\right)}{y}dy+\int_1^{a/2}\frac{F\left(\frac{a}{y}-1\right)}{y}dy.$$

If we define $y=(1+\Delta a/a)z$, then we have

$$F(a+\Delta a)-F(a) = -\int_{1/(1+\Delta a/a)}^{(a/2)(1+\Delta a/a)/(1+a/\Delta a)}\frac{F\left(\frac{a}{z}-1\right)}{z}dz+\int_1^{a/2}\frac{F\left(\frac{a}{y}-1\right)}{y}dy.$$

Thus,

$$F(a+\Delta a)-F(a) = -\int_{1/(1+\Delta a/a)}^{1}\frac{F\left(\frac{a}{z}-1\right)}{z}dz.$$

Dividing both sides of the above equation by Δa and letting Δa approach zero, we then obtain

$$\frac{dF(a)}{da} = -\frac{F(a-1)}{a}, \tag{66}$$

where $F(a)=1-\log(a)$ for $1\leq a\leq 2$. Hence, $F'(a)/F(a-1)=-1/a$ and as a approaches infinity, $(F(a)-F(a-1))/F(a-1)$ approaches zero. Hence, as a approaches infinity, we then have

$$\frac{dF(a)}{da} = -\frac{F(a)}{a},$$

Thus, as a approaches infinity, $F(a)$ approaches c/a for some constant c. In other words; as a approaches infinity, the regular component of the partial sum $M(1,p_r;1,p_r^a)$ decays to zero as c/a.

Furthermore, if we write the partial sum of $M(1, p_r)$ as $M(1, p_r, 1, N)$ (where $N = p_r^a$), then as a approaches infinity, the regular component of $M(1, p_r, 1, N)$ will be asymptotic to $1/\log N$.

Similarly, we can show that, for every value of a_0, there are values of $a > a_0$ such that $M(1, p_r; 1, {p_r}^a) > 1/a^{1+\epsilon}$ (i.e. $M(1, p_r; 1, {p_r}^a) = \Omega(1/a^{1+\epsilon})$, for any arbitrary small number ϵ). This task can be achieved by noting that the irregular component of $M(1, p_r; 1, {p_r}^a)$ is given by subtracting Equation (64) from Equation (63) to obtain

$$R(1, p_r; 1, {p_r}^a) = \int_1^{a/2} \frac{M(1, {p_r}^y; 1, p_r^{a-y})}{p_r^y} dO(\sqrt{{p_r}^y} \log({p_r}^y)) - \int_1^{a/2} \frac{R(1, {p_r}^y; 1, p_r^{a-y})}{y p_r^y} dy - \int_{a/2}^{a} \frac{dO(\sqrt{{p_r}^y} \log({p_r}^y))}{p_r^y}. \quad (67)$$

Therefore, if we assume that $M(1, p_r; 1, {p_r}^a) = O(g(a))$, where $g(a) = 1/a^{1+\epsilon}$ (for some arbitrary small number ϵ), then as a approaches infinity, $M(1, p_r; 1, {p_r}^a)$ becomes arbitrarily smaller than $F(a)$. Thus

$$\lim_{a\to\infty} M(1, p_r; 1, {p_r}^a)/F(a) = 0,$$

or,

$$\lim_{a\to\infty} R(1, p_r; 1, {p_r}^a)/F(a) = -1.$$

In other words; if $g(a) = 1/a^{1+\epsilon}$, then for sufficiently large a, we have

$$R(1, p_r; 1, {p_r}^a) \approx -F(a).$$

In the next step, we will show that the Riemann Hypothesis requires the exponential decay to zero for $M(1, p_r; 1, {p_r}^a)$ (or, $g(a) = e^{-\alpha a}$ for some constant α). If we assume that $g(a) = e^{-\epsilon a}$ (for any arbitrary small number ϵ), then it can be easily shown that, for sufficiently large p_r and a, the first and third integrals of equation (67) are negligible compared with $F(a)$. Hence

$$R(1, p_r; 1, {p_r}^a) = \int_1^{a/2} \frac{R(1, {p_r}^y; 1, p_r^{a-y})}{y p_r^y} dy \approx -F(a),$$

or

$$-F(a) \approx \int_1^{a/2} \frac{M(1, {p_r}^y; 1, p_r^{a-y}) - F(a/y - 1)}{y p_r^y} dy.$$

Since the integral with the function $M(1, {p_r}^y; 1, p_r^{a-y})$ is negligible with respect to the integral with the function $F(a/y - 1)$, therefore,

$$|F(a)| \approx \left| \int_1^{a/2} \frac{F(a/y - 1)}{y p_r^y} dy \right| < \frac{1}{p_r} \left| \int_1^{a/2} \frac{F(a/y - 1)}{y} dy \right| = \frac{|F(a)|}{p_r}.$$

Thus, the function $M(1, p_r; 1, {p_r}^a)$ can't be given by $O(e^{-\epsilon a})$. In other words; $M(1, p_r; 1, {p_r}^a) = \Omega(e^{-\epsilon a})$. While this result is sufficient to disprove the Riemann Hypothesis, it should pointed out that following the previous method, one can show that $M(1, p_r; 1, {p_r}^a) = \Omega(1/a^{1+\epsilon})$. This result may be useful in estimating when the distribution of the prime numbers deviates from that predicted by the Riemann Hypothesis. This task requires accurate computation of the function $F(a)$ to determine at what values of a, $F(a)$ approaches its limit of c/a.

- For the third step, we will compute the partial sum $M(\sigma, p_r; 1, {p_r}^a)$ for $\sigma < 1$ and show that it diverges as a approaches infinity.

This claim can be justified by the following statement. First, we write the partial sum $M(\sigma, p_r; {p_r}^a, {p_r}^{a+\delta})$ as the following sum

$$M(\sigma, p_r; {p_r}^a, {p_r}^{a+\delta}) = -\sum_{p_r \le p_i < {p_r}^{a+\delta-1}} \frac{1}{{p_i}^\sigma} M(\sigma, p_i; {p_r}^a/p_i, {p_r}^{a+\delta}/p_i) - \sum_{{p_r}^a \le p_i < {p_r}^{a+\delta}} \frac{1}{{p_i}^\sigma}. \tag{68}$$

Using Stieltjes integral, we can write the above equation as follows

$$M(\sigma, p_r; {p_r}^a, {p_r}^{a+\delta}) = \int_1^{a+\delta-1} \frac{d\pi({p_r}^y)}{p_r^{\sigma y}} M(\sigma, {p_r}^y; p_r^a/p_r^y, p_r^{a+\delta}/p_r^y) - \int_a^{a+\delta} \frac{d\pi({p_r}^y)}{p_r^{\sigma y}},$$

or

$$M(\sigma, p_r; {p_r}^a, {p_r}^{a+\delta}) = \int_1^{a+\delta-1} \frac{d\pi({p_r}^y)}{p_r^{\sigma y}} M(\sigma, {p_r}^y; p_r^{a-y}, p_r^{a+\delta-y}) - \int_a^{a+\delta} \frac{d\pi({p_r}^y)}{p_r^{\sigma y}}. \tag{69}$$

However, for $\delta \log p_r << 1$, we may write

$$M(\sigma, {p_r}^y; p_r^{a-y}, p_r^{a-y+\delta}) = {p_r}^{(1-\sigma)(a-y)} M(1, {p_r}^y; p_r^{a-y}, p_r^{a-y+\delta}). \tag{70}$$

Hence

$$M(\sigma, p_r; {p_r}^a, {p_r}^{a+\delta}) = p_r^{a(1-\sigma)} \int_1^{a+\delta-1} \frac{d\pi({p_r}^y)}{p_r^y} M(1, {p_r}^y; p_r^{a-y}, p_r^{a-y+\delta}) - p_r^{a(1-\sigma)} \int_a^{a+\delta} \frac{d\pi({p_r}^y)}{p_r^y}. \tag{71}$$

Consequently

$$M(\sigma, p_r; p_r^a, p_r^{a+\delta}) = {p_r}^{(1-\sigma)a} M(1, p_r; p_r^a, p_r^{a+\delta}). \tag{72}$$

Since the partial sum $M(1, p_r; 1, p_r^a) = \Omega(e^{-\epsilon a})$, therefore the sum $M(\sigma, p_r; p_r^a, p_r^{a+\delta})$ does not converge to zero as a approaches infinity. Consequently, the series $M(\sigma, p_r)$ and $M(\sigma)$ diverge for $\sigma < 1$. This implies that the Riemann Hypothesis is invalid and the zeros can be found arbitrary close to line $\Re(s) = 1$.

Alternatively, one may redefine the Mertin's function $M(0, p_r; 1, N)$ as

$$M_t(x, p_r) = M(0, p_r; 1 : x) = \sum_{n=1}^{x} u(n, p_r). \tag{73}$$

Thus,

$$M(\sigma, p_r; N_1, N_2) = \sum_{n=N_1}^{N_2} \frac{u(n,p_r)}{n^\sigma} = \sum_{n=N_1}^{N_2} \frac{M_t(n,p_r) - M_t(n-1,p_r)}{n^\sigma},$$

or

$$M(\sigma, p_r; N_1, N_2) = -\frac{M_t(N_1-1,p_r)}{N_1^\sigma} + \sum_{n=N_1}^{N_2} M_t(n,p_r)\left\{\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma}\right\} + \frac{M_t(N_2,p_r)}{N_2^\sigma}.$$

Since

$$\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} = \sigma \int_n^{n+1} \frac{dx}{x^{\sigma+1}},$$

hence

$$M(\sigma, p_r; N_1, N_2) = \sigma \int_{N_1}^{N_2} \frac{M_t(x,p_r)dx}{x^{\sigma+1}} - \frac{M_t(N_1-1,p_r)}{N_1^\sigma} + \frac{M_t(N_2,p_r)}{N_2^\sigma}. \quad (74)$$

In the following, we will show that $M_t(x, p_r)$ grows faster than than $x^{1-\epsilon}$, where ϵ is an arbitrary small number. This task can be achieved by using $\sigma = 1$ in Equation (74) to obtain

$$M(1, p_r; p_r^{a1}, p_r^{a2}) = \int_{p_r^{a1}}^{p_r^{a2}} \frac{M_t(x,p_r)dx}{x^2} - \frac{M_t(p_r^{a1}-1,p_r)}{p_r^{a1}} + \frac{M_t(p_r^{a2},p_r)}{p_r^{a2}}.$$

If $M_t(p_r^y, p_r) = O(p_r^{(1-\epsilon)y})$, then

$$M(1, p_r; p_r^{a1}, p_r^{a2}) = \int_{a1}^{a2} \frac{O(p_r^{(1-\epsilon)y})dp_r^y}{p_r^{2y}} - O(p_r^{-\epsilon a_1}) + O(p_r^{-\epsilon a_2}).$$

As a_2 approaches infinity, we then have

$$|M(1, p_r; 1, p_r^{a1})| = O(p_r^{-\epsilon a_1}).$$

However, this contradicts our earlier results in step 2 that $M(1, p_r, 1, p_r^a) = \Omega(e^{-\epsilon a})$ for any arbitrary small number ϵ. Hence, $M_t(x, p_r)$ grows faster than than $x^{1-\epsilon}$ which infers that the non-trivial zeros can be found arbitrary close to zero.

The previous process can be used to show that the partial sum $M(\sigma, p_r, 1, x)$ grows faster than $x^{(1-\sigma)\epsilon}$ which means that the the series $M(\sigma, p_r)$ diverges for $\sigma < 1$. This task can be achieved by first writing the following relationship between the partial sums $M(\sigma_1, p_r, 1, x)$ and $M(\sigma_2, p_r, 1, x)$ as follows

$$M(\sigma_1, p_r; N_1, N_2) = \sum_{n=N_1}^{N_2} \frac{u(n,p_r)}{n^{\sigma_1}} = \sum_{n=N_1}^{N_2} \frac{M(\sigma_2,p_r;1,n) - M(\sigma_2,p_r;1,n-1)}{n^{\sigma_1-\sigma_2}},$$

or

$$M(\sigma_1, p_r; N_1, N_2) = \sum_{n=N_1}^{N_2} M(\sigma_2, p_r; N_1, N_2)\left\{\frac{1}{n^{\sigma_1-\sigma_2}} - \frac{1}{(n+1)^{\sigma_1-\sigma_2}}\right\} -$$

$$\frac{M(\sigma_2, p_r, 1, N_1-1)}{N_1^{\sigma_1-\sigma_2}} + \frac{M(\sigma_2, p_r; 1, N_2)}{N_2^{\sigma_1-\sigma_2}}.$$

Hence

$$M(\sigma_1, p_r; N_1, N_2) = (\sigma_1 - \sigma_2) \int_{N_1}^{N_2} \frac{M(\sigma_2, p_r; 1, x)dx}{x^{\sigma_1-\sigma_2+1}} - \frac{M(\sigma_2, p_r, 1, N_1 - 1)}{N_1^{\sigma_1-\sigma_2}} + \frac{M(\sigma_2, p_r; 1, N_2)}{N_2^{\sigma_1-\sigma_2}}. \tag{75}$$

In the following, we will show that $M(\sigma, p_r; 1, x)$ grows faster than than $x^{1-\sigma-\epsilon}$, where ϵ is an arbitrary small number. This task can be achieved by setting $\sigma_1 = 1$ and $\sigma_2 = \sigma$ in Equation (75) to obtain

$$M(1, p_r; p_r^{a1}, p_r^{a2}) = (1 - \sigma) \int_{p_r^{a1}}^{p_r^{a2}} \frac{M(\sigma, p_r; 1, x)}{x^{2-\sigma}} dx - \frac{M(\sigma, p_r; 1, p_r^{a1} - 1)}{p_r^{(1-\sigma)a1}} + \frac{M(\sigma, p_r; 1, p_r^{a2})}{p_r^{(1-\sigma)a2}}.$$

If $M(\sigma, p_r; 1, p_r^y) = O(p_r^{(1-\sigma-\epsilon)y})$, then

$$M(1, p_r; p_r^{a1}, p_r^{a2}) = \int_{a1}^{a2} \frac{O(p_r^{(1-\sigma-\epsilon)y})}{p_r^{(2-\sigma)y}} dp_r^y - O(p_r^{-\epsilon a_1}) + O(p_r^{-\epsilon a_2}).$$

As a_2 approaches infinity, we then have

$$|M(1, p_r; 1, p_r^{a1})| = O(p_r^{-\epsilon a_1}).$$

However, this contradicts our earlier results in step 2 that $M(1, p_r, 1, p_r^a) = \Omega(e^{-\epsilon a})$ for any arbitrary small number ϵ. Hence, $M(\sigma, p_r; 1, x)$ grows faster than than $x^{1-\sigma-\epsilon}$. Therefore, the series $M(\sigma, p_r)$ and $M(\sigma)$ diverge for $\sigma < 1$.

Appendix 1

To prove the first part of Theorem 1 (i.e. for $s = \sigma + i0$ and $0.5 < \sigma \leq 1$, the series $M(\sigma, p_r)$ converges conditionally if $M(\sigma)$ converges conditionally), we first start with proving that $M(\sigma, 2)$ is conditionally convergent if $M(\sigma)$ is convergent. Since $M(\sigma)$ is convergent, then for any arbitrary small number δ, there exists an integer N_0 such that for every integer $N > N_0$

$$|M(\sigma; N, \infty)| = \left| \sum_{n=N}^{\infty} \frac{\mu(n)}{n^\sigma} \right| < \delta. \tag{76}$$

Let the sums $M(\sigma; 1, N)$, $M(\sigma; N+1, 2N)$, $M(\sigma; 2N+1, 2^2N)$, $M(\sigma; 2^2N+1, 2^3N)$, ..., $M(\sigma; 2^{L-1}N+1, 2^L N)$ be defined as

$$M(\sigma; 1, N) = \sum_{n=1}^{N} \frac{\mu(n)}{n^\sigma} = A_1,$$

$$M(\sigma; N+1, 2N) = \sum_{n=N+1}^{2N} \frac{\mu(n)}{n^\sigma} = \delta_1,$$

$$M(\sigma; 2N+1, 2^2N) = \sum_{n=2N+1}^{2^2N} \frac{\mu(n)}{n^\sigma} = \delta_2,$$

$$M(\sigma; 2^2N+1, 2^3N) = \sum_{n=2^2N+1}^{2^3N} \frac{\mu(n)}{n^\sigma} = \delta_3,$$

$$M(\sigma; 2^{L-1}N+1, 2^LN) = \sum_{n=2^{L-1}N+1}^{2^LN} \frac{\mu(n)}{n^\sigma} = \delta_L.$$

Throughout the analysis in this appendix, N will be a fixed number (that is larger than N_0) while the test for the convergence will be achieved by letting L approach infinity.

Let $\delta(l)$ be defined as the maximum of $|\delta_l|, |\delta_{l+1}|, |\delta_{l+2}|, ..., |\delta_L|, |\delta_l + \delta_{l+1}|, |\delta_l + \delta_{l+1} + \delta_{l+2}|, ..., |\delta_l + \delta_{l+1} + ... + \delta_L|$, then by the virtue of the convergence of $M(\sigma)$,

$$|\delta_1|, |\delta_2|, |\delta_3|, ..., |\delta_L|, |\delta_1+\delta_2|, |\delta_1+\delta_2+\delta_3|, ..., |\delta_1+\delta_2+\delta_3+...+\delta_L| \leq \delta(1) \leq 2\delta.$$

We also have

$$|\delta_l|, |\delta_{l+1}|, |\delta_{l+2}|, ..., |\delta_L|, |\delta_l+\delta_{l+1}|, |\delta_l+\delta_{l+1}+\delta_{l+2}|, ..., |\delta_l+\delta_{l+1}+...+\delta_L| \leq \delta(l),$$

where by the virtue of the convergence of $M(\sigma)$, $\delta(l)$ can be set arbitrary close to zero (since δ, defined in Equation 76, can be set arbitrary close to zero).

Furthermore, let the sums $M(\sigma, 2; 1, N)$, $M(\sigma, 2; N+1, 2N)$, $M(\sigma, 2; 2N+1, 2^2N)$, $M(\sigma, 2; 2^2N+1, 2^3N)$, ..., $M(\sigma, 2; 2^{L-1}N+1, 2^LN)$ be defined as

$$M(\sigma, 2; 1, N) = \sum_{n=1}^{N} \frac{\mu(n,2)}{n^\sigma} = B_1,$$

$$M(\sigma, 2; N+1, 2N) = \sum_{n=N+1}^{2N} \frac{\mu(n,2)}{n^\sigma} = \epsilon_1,$$

$$M(\sigma, 2; 2N+1, 2^2N) = \sum_{n=2N+1}^{2^2N} \frac{\mu(n,2)}{n^\sigma} = \epsilon_2,$$

$$M(\sigma, 2; 2^2N+1, 2^3N) = \sum_{n=2^2N+1}^{2^3N} \frac{\mu(n,2)}{n^\sigma} = \epsilon_3,$$

$$M(\sigma, 2; 2^{L-1}N+1, 2^LN) = \sum_{n=2^{L-1}N+1}^{2^LN} \frac{\mu(n,2)}{n^\sigma} = \epsilon_L,$$

Since

$$\sum_{n=1}^{2N} \frac{\mu(n)}{n^\sigma} = \sum_{n=1}^{2N} \frac{\mu(n,2)}{n^\sigma} - \sum_{n=1}^{N} \frac{\mu(n,2)}{(2n)^\sigma},$$

thus

$$M(\sigma; 1, 2N) = M(\sigma, 2; 1, 2N) - \frac{1}{2^\sigma} M(\sigma, 2; 1, N).$$

Similarly, since

$$\sum_{n=2^lN+1}^{2^{l+1}N}\frac{\mu(n)}{n^\sigma}=\sum_{n=2^lN+1}^{2^{l+1}N}\frac{\mu(n,2)}{n^\sigma}-\sum_{n=2^{l-1}N+1}^{2^lN}\frac{\mu(n,2)}{(2n)^\sigma},$$

thus

$$M(\sigma;2^lN+1,2^{l+1}N)=M(\sigma,2;2^lN+1,2^{l+1}N)-\frac{1}{2^\sigma}M(\sigma,2;2^{l-1}N+1,2^lN).$$

Rearranging the previous equations, we then have

$$A_1+\delta_1=B_1+\epsilon_1-\frac{1}{2^\sigma}B_1, \tag{77}$$

$$\delta_2=\epsilon_2-\frac{1}{2^\sigma}\epsilon_1,$$

$$\delta_3=\epsilon_3-\frac{1}{2^\sigma}\epsilon_2,$$

$$\delta_L=\epsilon_L-\frac{1}{2^\sigma}\epsilon_{L-1},$$

where $|\delta_1|,|\delta_2|,|\delta_3|,...,|\delta_L|,|\delta_1+\delta_2|,|\delta_1+\delta_2+\delta_3|,|\delta_1+\delta_2+\delta_3+...+\delta_L|\leq\delta(1)\leq 2\delta$ and δ can be set arbitrary close to zero. Hence

$$\epsilon_2=\frac{1}{2^\sigma}\epsilon_1+\delta_2,$$

$$\epsilon_3=\frac{1}{2^\sigma}\epsilon_2+\delta_3=\frac{1}{2^{2\sigma}}\epsilon_1+\frac{1}{2^\sigma}\delta_2+\delta_3,$$

$$\epsilon_4=\frac{1}{2^\sigma}\epsilon_3+\delta_4=\frac{1}{2^{3\sigma}}\epsilon_1+\frac{1}{2^{2\sigma}}\delta_2+\frac{1}{2^\sigma}\delta_3+\delta_4,$$

$$\epsilon_L=\frac{1}{2^\sigma}\epsilon_{L-1}+\delta_L=\frac{1}{2^{(L-1)\sigma}}\epsilon_1+\frac{1}{2^{(L-2)\sigma}}\delta_2+\frac{1}{2^{(L-3)\sigma}}\delta_3+...+\delta_L.$$

Therefore,

$$\epsilon_1+\epsilon_2+\epsilon_2+...+\epsilon_L=\left(1+\frac{1}{2^\sigma}+\frac{1}{2^{2\sigma}}+...+\frac{1}{2^{(L-1)\sigma}}\right)\epsilon_1+(\delta_2+\delta_3+...+\delta_L)+$$
$$\frac{1}{2^\sigma}(\delta_2+\delta_3+...+\delta_{L-1})+\frac{1}{2^{2\sigma}}(\delta_2+\delta_3+...+\delta_{L-2})+...+\frac{1}{2^{(L-2)\sigma}}\delta_2.$$

Since $|\delta_2|\leq\delta(1)$, $|\ |\delta_2+\delta_3|\leq\delta(1)$, ..., $|\delta_1+\delta_2+\delta_3+...+\delta_L|\leq\delta(1)$, hence

$$|\delta_2+\delta_3+...+\delta_L|+\frac{1}{2^\sigma}|\delta_2+\delta_3+...+\delta_{L-1}|+...+\frac{1}{2^{(L-2)\sigma}}|\delta_2|\leq\left|\delta(1)+\frac{1}{2^\sigma}\delta(1)+...+\frac{1}{2^{(L-2)\sigma}}\delta(1)\right|,$$

or

$$|\delta_2+\delta_3+...+\delta_L|+\frac{1}{2^\sigma}|\delta_2+\delta_3+...+\delta_{L-1}|+...+\frac{1}{2^{(L-2)\sigma}}|\delta_2|\leq\frac{2^\sigma}{2^\sigma-1}|\delta(1)|.$$

Therefore

$$\epsilon_1+\epsilon_2+\epsilon_3+...+\epsilon_L=\left(1+\frac{1}{2^\sigma}+\frac{1}{2^{2\sigma}}+...+\frac{1}{2^{L\sigma}}\right)\epsilon_1+\gamma_1,$$

where γ_1 is of the same order as that of $\delta(1)$ (where $\delta(1)$ can be set arbitrary close to zero by setting δ, defined in Equation 76, arbitrary close to zero).

As L approaches infinity, we then obtain

$$\sum_{i=1}^{\infty}\epsilon_i=\frac{2^\sigma}{2^\sigma-1}\epsilon_1+\gamma_1.$$

Therefore, the sum $M(\sigma,2;N+1,\infty)$ (which is equal to $\epsilon_1+\epsilon_2+\epsilon_3+...$) is bounded by the sum $M(\sigma,2;N+1,2N)$ (which is equal to ϵ_1).

The previous process can be repeated with the substitution of A_1 and B_1 in Equation Equation (77) with A_2 and B_2 where where $A_2=M(\sigma;1,2N)$ and $B_2=M(\sigma,2;1,2N)$ to obtain

$$A_2+\delta_2=B_2+\epsilon_2-\frac{1}{2^\sigma}B_2.$$

Thus,

$$A_2=B_2-\frac{1}{2^\sigma}B_2+\frac{1}{2^\sigma}\epsilon_1.$$

Following the same process, we can show that the sum $M(\sigma,2;2N+1,\infty)$ is given by

$$\sum_{i=2}^{\infty}\epsilon_i=\frac{1}{2^\sigma-1}\epsilon_1+\gamma_2.$$

where γ_2 is of the same order as that of $\delta(2)$ (where $\delta(2)$ can be set arbitrary close to zero by setting δ, defined in Equation 76, arbitrary close to zero).

If we repeat the process l times, we obtain

$$A_l=B_l-\frac{1}{2^\sigma}B_l+\frac{1}{2^{(l-1)\sigma}}\epsilon_1,$$

where $A_l=M(\sigma;1,2^lN)$ and $B_l=M(\sigma,2;1,2^lN)$ and the sum $M(\sigma,2;2^lN+1,\infty)$ is given by

$$\sum_{i=l}^{\infty}\epsilon_i=\frac{1}{2^{(l-2)\sigma}}\frac{1}{2^\sigma-1}\epsilon_1+\gamma_l.$$

where γ_l is of the same order as that of $\delta(l)$. Since by the virtue of the convergence of $M(\sigma)$, $\delta(l)$ tends to zero as l approaches infinity, therefore γ_l and the above sum approach zero as l approaches infinity.

Thus, we conclude that $M(\sigma, 2; 2^l N+1, \infty)$ (given by $\sum_{i=l}^{\infty} \epsilon_i$) approaches zero as l approaches infinity. Furthermore, as l approaches infinity, $B = \lim_{l\to\infty} B_l$ approaches its limit given by

$$\left(1-\frac{1}{2^\sigma}\right) B = M(\sigma; 1, \infty).$$

Hence,

$$\left(1-\frac{1}{2^\sigma}\right) M(\sigma, 2) = M(\sigma).$$

Similarly, following the same steps, we can show that

$$\left(1-\frac{1}{3^\sigma}\right) M(\sigma, 3; 1, \infty) = M(\sigma, 2; 1, \infty).$$

or

$$\left(1-\frac{1}{2^\sigma}\right)\left(1-\frac{1}{3^\sigma}\right) M(\sigma, 3; 1, \infty) = M(\sigma; 1, \infty).$$

This task can be achieved by first defining

$$M(\sigma, 2; 1, N) = \sum_{n=1}^{N} \frac{\mu(n,2)}{n^\sigma} = A_1,$$

$$M(\sigma, 2; N+1, 3N) = \sum_{n=N+1}^{3N} \frac{\mu(n,2)}{n^\sigma} = \delta_1,$$

$$M(\sigma, 2; 3N+1, 3^2N) = \sum_{n=3N+1}^{3^2N} \frac{\mu(n,2)}{n^\sigma} = \delta_2,$$

$$M(\sigma, 2; 3^{L-1}N+1, 3^LN) = \sum_{n=3^{L-1}N+1}^{3^LN} \frac{\mu(n,2)}{n^\sigma} = \delta_L,$$

and

$$M(\sigma, 3; 1, N) = \sum_{n=1}^{N} \frac{\mu(n,3)}{n^\sigma} = B_1,$$

$$M(\sigma, 3; N+1, 3N) = \sum_{n=N+1}^{3N} \frac{\mu(n,3)}{n^\sigma} = \epsilon_1,$$

$$M(\sigma, 3; 3N+1, 3^2N) = \sum_{n=3N+1}^{3^2N} \frac{\mu(n,3)}{n^\sigma} = \epsilon_2,$$

$$M(\sigma, 3; 3^{L-1}N+1, 3^LN) = \sum_{n=3^{L-1}N+1}^{3^LN} \frac{\mu(n,3)}{n^\sigma} = \epsilon_L,$$

Since

$$\sum_{n=1}^{3N} \frac{\mu(n,2)}{n^\sigma} = \sum_{n=1}^{3N} \frac{\mu(n,3)}{n^\sigma} - \sum_{n=1}^{N} \frac{\mu(n,3)}{(3n)^\sigma},$$

thus

$$M(\sigma,2;1,3N) = M(\sigma,3;1,3N) - \frac{1}{3^\sigma} M(\sigma,3;1,N)$$

Similarly,

$$M(\sigma,2;3^l N+1,3^{l+1}N) = M(\sigma,3;3^l N+1,3^{l+1}N) - \frac{1}{3^\sigma} M(\sigma,3;3^{l-1}N+1,3^l N)$$

Following the same process, we can show that

$$\sum_{i=1}^{\infty} \epsilon_i = \frac{3^\sigma}{3^\sigma - 1}\epsilon_1 + \gamma_1,$$

where γ_1 is of the same order as that of $\delta(1)$ ($\delta(l)$ is defined as the maximum of $|\delta_l|, |\delta_{l+1}|, |\delta_{l+2}|, ..., |\delta_L|, |\delta_l + \delta_{l+1}|, |\delta_l + \delta_{l+1} + \delta_{l+2}|, ..., |\delta_l + \delta_{l+1} + ... + \delta_L|$).

Similarly, if we define $A_2 = M(\sigma,2;1,3N)$ and $B_2 = M(\sigma,3;1,3N)$, then

$$A_2 = B_2 - \frac{1}{3^\sigma}B_2 + \frac{1}{3^\sigma}\epsilon_1.$$

Therefore

$$\sum_{i=2}^{\infty} \epsilon_i = \frac{1}{3^\sigma - 1}\epsilon_1 + \gamma_2.$$

where γ_2 is of the same order as that of $\delta(2)$.

Repeating the steps l times, we then obtain

$$\sum_{i=l}^{\infty} \epsilon_i = \frac{1}{3^{(l-2)\sigma}} \frac{1}{3^\sigma - 1}\epsilon_1 + \gamma_l.$$

where γ_l is of the same order as that of $\delta(l)$. Hence the above sum approaches zero as l approaches infinity

Thus, we conclude that $M(\sigma,3;3^l N+1,\infty)$ (given by $\sum_{i=l}^{\infty} \epsilon_i$) approaches zero as l approaches infinity. Furthermore, as l approaches infinity, $B = \lim_{l\to\infty} B_l$ approaches its limit given by

$$\left(1 - \frac{1}{3^\sigma}\right) B = M(\sigma,2;1,\infty).$$

Hence,
$$\left(1-\frac{1}{3^\sigma}\right)M(\sigma,3)=M(\sigma,2).$$

Repeating the process r times, we then conclude
$$M(\sigma)=M(\sigma,p_r)\prod_{i=1}^{r}\left(1-\frac{1}{{p_i}^\sigma}\right).$$

The second part of the theorem can be proved by recalling
$$M(s,p_{r-1};1,Np_r)=M(s,p_r;1,Np_r)-\frac{1}{p_r^s}M(s,p_r;1,N).$$
If both series $M(s,p_{r-1})$ and $M(s,p_r)$ are convergent, then as N approaches infinity, we obtain
$$M(s,p_{r-1})=M(s,p_r)\left(1-\frac{1}{p_i^s}\right).$$

Repeating the process r times, we then conclude
$$M(\sigma)=M(\sigma,p_r)\prod_{i=1}^{r}\left(1-\frac{1}{{p_i}^\sigma}\right).$$

Appendix 2

Assuming RH is valid and for $\sigma>0.5$, to show that
$$\sum_{i=r1}^{r2}\frac{1}{{p_i}^\sigma}=E_1((\sigma-1)\log p_{r1})-E_1((\sigma-1)\log p_{r2})+\varepsilon$$
where, $\varepsilon=O\left(\frac{t}{(\sigma-0.5)^2}\,{p_{r1}}^{1/2-\sigma}\log p_{r1}\right)$, we first recall that
$$\sum_{i=r1}^{r2}\frac{1}{p_i^\sigma}=\int_{p_{r1}}^{p_{r2}}\frac{d\pi(x)}{x^\sigma}=\int_{p_{r1}}^{p_{r2}}\frac{1}{x^\sigma\log x}dx+\int_{p_{r1}}^{p_{r2}}\frac{1}{x^\sigma}dO\left(\sqrt{x}\log x\right).$$

We will first compute the integral with the O notation. This can be done by integration by parts to obtain
$$\int_{p_{r1}}^{p_{r2}}\frac{1}{x^\sigma}dO\left(\sqrt{x}\log x\right)=\frac{O\left(\sqrt{p_{r2}}\log p_{r2}\right)}{{p_{r2}}^\sigma}-\frac{O\left(\sqrt{p_{r1}}\log p_{r1}\right)}{{p_{r1}}^\sigma}-\int_{p_{r1}}^{p_{r2}}O\left(\sqrt{x}\log x\right)d\left(\frac{1}{x^\sigma}\right)$$
Since $x>0$, thus
$$\int_{p_{r1}}^{p_{r2}}\frac{1}{x^\sigma}dO\left(\sqrt{x}\log x\right)=\frac{O\left(\sqrt{p_{r2}}\log p_{r2}\right)}{{p_{r2}}^\sigma}-\frac{O\left(\sqrt{p_{r1}}\log p_{r1}\right)}{{p_{r1}}^\sigma}-O\left(\int_{p_{r1}}^{p_{r2}}\sqrt{x}\log x\,d\left(\frac{1}{x^\sigma}\right)\right)$$

With the substitution of variables $y = \log x$, we then obtain

$$\int_{p_{r1}}^{p_{r2}} \sqrt{x} \log x \, d\left(\frac{1}{x^\sigma}\right) = -\int_{p_{r1}}^{p_{r2}} \sigma y e^{(\frac{1}{2}-\sigma)y} dy.$$

Since

$$\int x e^{ax} dx = \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax},$$

therefore

$$\int_{p_{r1}}^{p_{r2}} \sqrt{x} \log x \, d\left(\frac{1}{x^\sigma}\right) = -\sigma \left(\frac{\log p_{r2}}{0.5-\sigma} - \frac{1}{(0.5-\sigma)^2}\right) {p_{r2}}^{0.5-\sigma} + \sigma \left(\frac{\log p_{r1}}{0.5-\sigma} - \frac{1}{(0.5-\sigma)^2}\right) {p_{r1}}^{0.5-\sigma}.$$

Hence, for $\sigma > 0.5$, we have

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma} dO\left(\sqrt{x}\log x\right) = O\left(\frac{{p_{r1}}^{0.5-\sigma}\log p_{r1}}{(\sigma-0.5)^2}\right) \tag{78}$$

For $\sigma \geq 1$, the integral $\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx$ can be computed directly from the definition of the Exponential Integral $E_1(r) = \int_r^\infty \frac{e^{-u}}{u} du$ (where $r \geq 0$) to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx = E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2})$$

To compute the integral $\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx$ for $\sigma < 0$, we first use the substantiation $y = \log x$ to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx = \int_{\log p_{r1}}^{\log p_{r2}} \frac{e^{(1-\sigma)y}}{y} dy = \int_{\epsilon}^{\log p_{r2}} \frac{e^{(1-\sigma)y}}{y} dy - \int_{\epsilon}^{\log p_{r1}} \frac{e^{(1-\sigma)y}}{y} dy$$

where, ϵ is an arbitrary small positive number. With the variable substantiations $z_1 = y/\log p_{r1}$ and $z_2 = y/\log p_{r2}$, we then obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx = \int_{\epsilon/\log p_{r2}}^{1} \frac{e^{(1-\sigma)(\log p_{r2})z_2}}{z_2} dz_2 - \int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)(\log p_{r1})z_1}}{z_1} dz_1.$$

With the variable substantiations $w_1 = (1-\sigma)(\log p_{r1})z_1$ and $w_2 = (1-\sigma)(\log p_{r2})z_1$ and by adding and subtracting the terms $-\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} + \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{dw_1}{w_1}$, we then have

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx = \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{e^{w_2}-1}{w_2} dw_2 - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{e^{w_1}-1}{w_1} dw_1 +$$
$$\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{dw_1}{w_1}.$$

Using the following identity [9, page 230]

$$\int_0^a \frac{e^t - 1}{t} dt = -E_1(-a) - \log(a) - \gamma$$

where $a > 0$, we then obtain for $\sigma < 1$,

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx = E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2})$$

Hence, for $\sigma > 0.5$, we have

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^\sigma} = E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2}) + \varepsilon$$

It should be pointed out that in general, if there are no non-trivial zeros for values of s with $\Re(s) > a$, then by following the same steps, we may also show that for $\sigma > a$, we have

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^\sigma} = E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2}) + \varepsilon$$

where, $\varepsilon = O\left(\frac{t}{(\sigma-a)^2} {p_{r1}}^{a-\sigma} \log p_{r1}\right)$.

Appendix 3

Assuming RH is valid and for $\sigma > 0.5$, to show that

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^s} = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}) + \varepsilon$$

where, $\varepsilon = O\left(\frac{t+1}{(\sigma-0.5)^2} {p_{r1}}^{1/2-\sigma} \log p_{r1}\right)$, we first recall that

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^s} = \int_{p_{r1}}^{p_{r2}} \frac{d\pi(x)}{x^s} = \int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx + \int_{p_{r1}}^{p_{r2}} \frac{1}{x^s} dO\left(\sqrt{x}\log x\right).$$

We will first compute the integral with the O notation. This can be done by integration by parts to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s} dO\left(\sqrt{x}\log x\right) = \frac{O\left(\sqrt{p_{r2}}\log p_{r2}\right)}{{p_{r2}}^s} - \frac{O\left(\sqrt{p_{r1}}\log p_{r1}\right)}{{p_{r1}}^s} - \int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) d\left(\frac{1}{x^s}\right)$$

The integral on the right side of the above equation can be then written as

$$\int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) d\left(\frac{1}{x^s}\right) = -s\int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) x^{-s-1} dx.$$

Hence,

$$\left|\int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) d\left(\frac{1}{x^s}\right)\right| \le |s| \int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) |x^{-s-1}| dx.$$

For sufficiently large t, we can write $|s| = t$ and consequently

$$\left|\int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) d\left(\frac{1}{x^s}\right)\right| = O\left(t\ \frac{{p_{r1}}^{0.5-\sigma}\log p_{r1}}{(\sigma-0.5)^2}\right).$$

Hence,

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s} dO\left(\sqrt{x}\log x\right) = O\left((t+1)\ \frac{{p_{r1}}^{0.5-\sigma}\log p_{r1}}{(\sigma-0.5)^2}\right).$$

For $\Re(s) \geq 1$, the integral $\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx$ can be computed directly from the definition of the Exponential Integral $E_1(z) = \int_1^\infty \frac{e^{-tz}}{t} dt$ (where $\Re(z) \geq 0$) to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2})$$

To compute the integral $\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx$ for $\Re(z) < 1$, we first write the integral as follows

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx = \int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x}\cos(t\log x)}{\log x} dx - i\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma\log x}\sin(t\log x)}{\log x} dx.$$

The first integral on the right side $\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma\log x}\cos(t\log x)}{\log x} dx$ can be computed by using the substitution $y = \log x$ to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma\log x}\cos(t\log x)}{\log x} dx = \int_{p_{r1}}^{p_{r2}} \frac{e^{(1-\sigma)y}\cos(ty)}{y} dy,$$

or

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma\log x}\cos(t\log x)}{\log x} dx = \int_{p_{r1}}^{p_{r2}} \frac{e^{(1-\sigma)y}\cos(ty)}{y} dy + \int_{p_{r1}}^{p_{r2}} \frac{e^{(1-\sigma)y}}{y} dy - \int_{p_{r1}}^{p_{r2}} \frac{e^{(1-\sigma)y}}{y} dy.$$

Hence,

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma\log x}\cos(t\log x)}{\log x} dx = \int_{\epsilon}^{p_{r1}} \frac{e^{(1-\sigma)y}(1-\cos(ty))}{y} dy - \int_{\epsilon}^{p_{r2}} \frac{e^{(1-\sigma)y}(1-\cos(ty))}{y} dy -$$

$$\int_{\epsilon}^{p_{r1}} \frac{e^{(1-\sigma)y}}{y} dy + \int_{\epsilon}^{p_{r2}} \frac{e^{(1-\sigma)y}}{y} dy$$

where, ϵ is an arbitrary small positive number. With the variable substantiations $z_1 = y/\log p_{r1}$ and $z_2 = y/\log p_{r2}$, we then obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma\log x}\cos(t\log x)}{\log x} dx = \int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)(\log p_{r1})z_1}(1-\cos(t(\log p_{r1})z_1))}{z_1} dz_1 -$$

$$\int_{\epsilon/\log p_{r2}}^{1} \frac{e^{(1-\sigma)(\log p_{r2})z_2}(1-\cos(t(\log p_{r2})z_2))}{z_2} dz_2 -$$

$$\int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)(\log p_{r1})z_1}}{z_1} dz_1 + \int_{\epsilon/\log p_{r2}}^{1} \frac{e^{(1-\sigma)(\log p_{r2})z_2}}{z_2} dz_2$$

By the virtue of the following identity ([9], page 230)

$$\int_0^1 \frac{e^{at}(1-\cos(bt))}{t} dt = \frac{1}{2}\log(1+b^2/a^2) + \mathrm{Li}(a) + \Re[E_1(-a+ib)],$$

where $a > 0$, we then obtain the following

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \Re[E_1((s-1)\log p_{r1})] + \mathrm{Li}((1-\sigma)\log p_{r1}) - \Re[E_1((s-1)\log p_{r2})] - \mathrm{Li}((1-\sigma)\log p_{r2}) - \int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)(\log p_{r1})z_1}}{z_1} dz_1 + \int_{\epsilon/\log p_{r2}}^{1} \frac{e^{(1-\sigma)(\log p_{r2})z_2}}{z_2} dz_2$$

With the variable substantiations $w_1 = (1-\sigma)(\log p_{r1})z_1$ and $w_1 = (1-\sigma)(\log p_{r1})z_1$ and by adding and subtracting the terms $-\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} + \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{dw_1}{w_1}$, we then have

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \Re[E_1((s-1)\log p_{r1})] + \mathrm{Li}((1-\sigma)\log p_{r1}) - \Re[E_1((s-1)\log p_{r2})] - \mathrm{Li}((1-\sigma)\log p_{r2}) + \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{e^{w_2}-1}{w_2} dw_2 - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{e^{w_1}-1}{w_1} dw_1 + \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{dw_1}{w_1}.$$

Using the following identity [9, page 230]

$$\int_0^a \frac{e^t - 1}{t} dt = \mathrm{Ei}(a) - \log(a) - \gamma$$

where $a > 0$, we then obtain for $\sigma < 1$,

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \Re[E_1((s-1)\log p_{r1})] - \Re[E_1((s-1)\log p_{r2})]$$

Similarly, using the identity [9, page 230]

$$\int_{p_0}^{1} \frac{e^{at} \sin(bt)}{t} dt = \pi - \arctan(b/a) + \Im[E_1(-a+ib)],$$

where $a > 0$, we can show that for $\sigma < 1$, we have

$$-\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \sin(t \log x)}{\log x} dx = \Im[E_1((s-1)\log p_{r1})] - \Im[E_1((s-1)\log p_{r2})].$$

Therefore, for $\Re(s) > 0.5$, we have

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^s} = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}) + \varepsilon$$

where, $\varepsilon = O\left(\frac{t+1}{(\sigma-0.5)^2} {p_{r1}}^{1/2-\sigma} \log p_{r1}\right)$.

Appendix 4

In Appendix 4, we will show that the sum $\sum_\rho E_1\left((s-\rho)\log p_r\right)$ is convergent if $|s-\rho|> 0$ for every ρ. Furthermore, we will show that the sum approaches zeros as p_r approaches infinity. this task will be achieved by noting that, for sufficiently large p_r, $E_1\left((s-\rho)\log p_r\right)$ can be written as

$$E_1\left((s-\rho)\log p_r\right) = \frac{e^{-(s-\rho)\log p_r}}{(s-\rho)\log p_r}\left(1+O\left(\frac{1}{|s-\rho|\log p_r}\right)\right) \tag{79}$$

Therefore, if the sum $\sum_\rho E_1\left((s-\rho)\log p_r\right)$ is convergent, then it will be given by

$$\sum_\rho E_1\left((s-\rho)\log p_r\right) = \sum_\rho \frac{e^{-(s-\rho)\log p_r}}{(s-\rho)\log p_r} + \epsilon, \tag{80}$$

where ϵ is the contribution by the sum of the O terms in Equation (79). It can be easily shown that if $|s-\rho|\geq \varepsilon > 0$ for every ρ, then ϵ in Equation (80) tends to zero as p_r approaches infinity. This result can be deduced by noting that $O(\epsilon) = (p_r^{\min \Re(s-\rho)}/(\log p_r)^2)\sum_\rho 1/|s-\rho|^2$. Since the sum $\sum_\rho 1/|s-\rho|^2$ is bounded, therefore Equation (80) can be further simplified to

$$\sum_\rho E_1\left((s-\rho)\log p_r\right) = \frac{{p_r}^{-s}}{\log p_r}\sum_\rho \frac{{p_r}^{\rho}}{s-\rho} + O(p_r^{\min \Re(s-\rho)}/(\log p_r)^2). \tag{81}$$

To show the sum $\sum_\rho E_1\left((s-\rho)\log p_r\right)$ is convergent, let $s=\sigma+iT$ and $\rho_i = \beta_i + i\gamma_i$. We split ρ_i's into two groups. The first group comprises of the non-trivial zeros with γ_i's less than or equal to mT, where $m>1$. The rest of the non-trivial zeros belong to the second group. Since the first group has a finite number of ρ_i's, thus the sum $\sum_{|\gamma_i|\leq mT} E_1\left((s-\rho)\log p_r\right)$ is bounded. Since $|{p_r}^{-s}{p_r}^{\rho}|< 1$ for every ρ, therefore

$$\left|\sum_{|\gamma_i|\leq mT} E_1\left((s-\rho)\log p_r\right)\right| = (1/\log p_r)\sum_{|\gamma_i|\leq mT}\frac{1}{|s-\rho|}.$$

Hence

$$\sum_{|\gamma_i|\leq mT} E_1\left((s-\rho)\log p_r\right) = O(1/\log p_r).$$

The sum over the second group can be expanded as follows

$$\sum_{|\gamma_i|>mT} E_1\left((s-\rho)\log p_r\right) = -\frac{{p_r}^{-s}}{\log p_r}\left(\sum_{|\gamma_i|>mT}\frac{{p_r}^{\rho_i}}{\rho_i} + s\sum_{|\gamma_i|>mT}\frac{{p_r}^{\rho_i}}{{\rho_i}^2} + s^2\sum_{|\gamma_i|>mT}\frac{{p_r}^{\rho}}{{\rho_i}^3} + ...\right) + \epsilon.$$

The first sum $\sum_{|\gamma_i|>mT} {p_r}^{\rho_i}/\rho_i$ is convergent by the virtue of Equation (47). The upper bound for the second term $({p_r}^{-s}/\log p_r)\, s\, \sum_{|\gamma_i|>mT} {p_r}^{\rho_i}/{\rho_i}^2$can be determined as follows

$$\left|\frac{{p_r}^{-s}s}{\log p_r}\sum_{|\gamma_i|>mT}\frac{{p_r}^{\rho_i}}{{\rho_i}^2}\right| \leq \frac{|{p_r}^{-s}||s|}{\log p_r}\sum_{|\gamma_i|>mT}\frac{|{p_r}^{\rho_i}|}{|{\rho_i}^2|}.$$

Since for sufficiently large T, $|s|$ is given by T and the density of the non-trivial zeros is given by $O(\log t)$ (note that if there are roots off the critical line then their density is given by Bohr Landau theorem [1] and it is less than $O(\log t)$), thus

$$\left| \frac{{p_r}^{-s} s}{\log p_r} \sum_{|\gamma_i| > mT} \frac{{p_r}^{\rho_i}}{{\rho_i}^2} \right| \leq \frac{{p_r}^{-\sigma + \max \beta_i} T}{\log p_r} \int_{mT}^{\infty} \frac{O(\log t)}{t^2} dt.$$

Hence

$$\left| \frac{{p_r}^{-s} s}{\log p_r} \sum_{|\gamma_i| > mT} \frac{{p_r}^{\rho_i}}{{\rho_i}^2} \right| \leq \frac{{p_r}^{-\sigma + \max \beta_i}}{\log p_r} \frac{O(\log T))}{m}.$$

Similarly, we can show that

$$\left| \frac{{p_r}^{-s} s^2}{\log p_r} \sum_{|\gamma_i| > mT} \frac{{p_r}^{\rho_i}}{{\rho_i}^3} \right| \leq \frac{{p_r}^{-\sigma + \max \beta_i}}{\log p_r} \frac{O(\log T)}{m^2},$$

and,

$$\left| \frac{{p_r}^{-s} s^i}{\log p_r} \sum_{|\gamma_i| > mT} \frac{{p_r}^{\rho_i}}{{\rho_i}^{i+1}} \right| \leq \frac{{p_r}^{-\sigma + \max \beta_i}}{\log p_r} \frac{O(\log T)}{m^i}.$$

Therefore,

$$\left| \frac{{p_r}^{-s}}{\log p_r} \left(s \sum_{|\gamma_i| > mT} \frac{{p_r}^{\rho_i}}{{\rho_i}^2} + s^2 \sum_{|\gamma_i| > mT} \frac{{p_r}^{\rho}}{{\rho_i}^3} + \ldots \right) \right| \leq \frac{{p_r}^{-\sigma + \max \beta_i} O(\log T)}{\log p_r} \sum_{i=1}^{\infty} \frac{1}{m^i}.$$

Since $\sum_{i=1}^{\infty} 1/m^i$ is convergent, hence $({p_r}^{-\sigma + \max \beta_i} O(\log T)/\log p_r) \sum_{i=1}^{\infty} 1/m^i$ is convergent and it is given by

$$\left| \frac{{p_r}^{-s}}{\log p_r} \left(s \sum_{|\gamma_i| > mT} \frac{{p_r}^{\rho_i}}{{\rho_i}^2} + s^2 \sum_{|\gamma_i| > mT} \frac{{p_r}^{\rho}}{{\rho_i}^3} + \ldots \right) \right| = O({p_r}^{-\sigma + \max \beta_i} \log(T)/\log p_r).$$

Hence

$$\sum_{|\gamma_i| > mT} E_1 \left((s - \rho) \log p_r \right) = -\frac{{p_r}^{-s}}{\log p_r} \left(\sum_{|\gamma_i| > mT} \frac{{p_r}^{\rho_i}}{\rho_i} \right) + O({p_r}^{-\sigma + \max \beta_i} \log(T)/\log p_r).$$

Thus

$$\sum_{|\gamma_i| > mT} E_1 \left((s - \rho) \log p_r \right) = -\frac{{p_r}^{-s}}{\log p_r} \left(\sum_{|\gamma_i| > mT} \frac{{p_r}^{\rho_i}}{\rho_i} \right) + O({p_r}^{-\sigma + \max \beta_i} \log(T)/\log p_r).$$

Consequently, $\sum_{\rho} E_1 \left((s - \rho) \log p_r \right)$ is convergent and it is given by

$$\sum_{\rho} E_1 \left((s - \rho) \log p_r \right) = \frac{{p_r}^{-s}}{\log p_r} \sum_{\rho} \frac{{p_r}^{\rho}}{s - \rho} + O(1/\log p_r).$$

In the remaining of this Appendix, we will derive a formula to show the dependence of the sum $\sum_\rho E_1\left((s-\rho)\log p_r\right)$ on T (where, $s=\sigma+iT$). On RH, we have

$$\sum_{|\gamma_i|>mT} E_1\left((s-\rho)\log p_r\right) = -\frac{{p_r}^{-s}}{\log p_r}\left(\sum_{|\gamma_i|>mT}\frac{{p_r}^{\rho_i}}{\rho_i}\right) + O\left({p_r}^{0.5-\sigma}\log(T)/\log p_r\right).$$

Thus

$$\left|\sum_{|\gamma_i|>mT} E_1\left((s-\rho)\log p_r\right)\right| = O\left(p_r^{0.5-\sigma}\log p_r\right) + O\left({p_r}^{0.5-\sigma}\log(T)/\log p_r\right).$$

Since the density of the roots on the critical line is given by $\log T$, thus the sum over the roots with $|\gamma_i|\leq mT$ can be given by the following integral

$$\left|\sum_{|\gamma_i|\leq mT} E_1\left((s-\rho)\log p_r\right)\right| = \frac{p_r^{0.5-\sigma}}{\log p_r}\int_{-mT}^{mT}\frac{O(\log t)}{\sqrt{(t-T)^2+(\sigma-0.5)^2}}dt.$$

Thus, for fixed $\sigma>0.5+\epsilon$, we have

$$\left|\sum_{|\gamma_i|\leq mT} E_1\left((s-\rho)\log p_r\right)\right| = p_r^{0.5-\sigma}O((m\log T)^2)/\log p_r.$$

Therefore, on RH, we have

$$\left|\sum_{\rho} E_1\left((s-\rho)\log p_r\right)\right| = O\left({p_r}^{0.5-\sigma}\log p_r(\log T)^2\right). \tag{82}$$

Appendix 5

To show that

$$\left|\sum_{n=1}^{N}\frac{\mu(n,p_r)}{n}\right| \leq 2$$

we first note that

$\sum_{d/n}\mu(d,p_r)=1$, if $n=1$,
$\sum_{d/n}\mu(d,p_r)=1$, if all the prime factor of n are less than p_r,
$\sum_{d/n}\mu(d,p_r)=0$, if any of the prime factor of n is greater than p_r.

Adding all the terms $\sum_{d/n}\mu(d,p_r)$ for $1\leq n\leq N$, we then obtain

$$0<\sum_{n=1}^{N}\mu(n,p_r)\left[\frac{N}{n}\right]\leq N,$$

where $[x]$ refers to the integer value of x. Define r_n as

$$r_n=\frac{N}{n}-\left[\frac{N}{n}\right],$$

where $0 \leq r_n < 1$. Hence, we have

$$\sum_{n=1}^{N} \mu(n,p_r) r_n < \sum_{n=1}^{N} \mu(n,p_r) \left[\frac{N}{n}\right] + \sum_{n=1}^{N} \mu(n,p_r) r_n \leq \sum_{n=1}^{N} \mu(n,p_r) r_n$$

Since

$$-N \leq \sum_{n=1}^{N} \mu(n,p_r) r_n \leq N,$$

thus, for every p_r we have

$$-N < \sum_{n=1}^{N} \mu(n,p_r) \frac{N}{n} \leq 2N,$$

or

$$-1 < \sum_{n=1}^{N} \frac{\mu(n,p_r)}{n} \leq 2.$$

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