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FUZZY L-OPEN SETS AND FUZZY L-CONTINUOUS FUNCTIONS

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Abstract. Recently in 1997, Sarker in [8] introduced the concept of fuzzy ideal and fuzzy local function between fuzzy topological spaces. In the present paper, we introduce some new fuzzy notions via fuzzy ideals. Also, we generalize the notion of L-open sets due to Jankovic and Homlett [6]. In addition to, we generalize the concept of L-closed sets, L- continuity due to Abd El-Monsef et al. [2]. Relationships between the above new fuzzy notions and other relevant classes are investigated.[1]

1 Introduction

The natural of a set was generalized in 1965 with the introduction of fuzzy subsets by Zadeh in his classical paper [9]. Because the concept of fuzzy subsets corresponding to the physical situation in which there is no precisely defined and increasing applications in various fields, including probability theory, artificial intelligence and economics. The first paper on fuzzy topology was published in 1968 by Chang [6]. After the discovery of fuzzy sets, much attention has been paid to the generalization of basic concepts of classical topology to fuzzy sets and thus developing a theory of fuzzy

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topology. The notion of fuzzy ideal and fuzzy local function where introduced and studied in [8]. Our aim in this paper is to investigate and fuzzy L-open sets, fuzzy L-closed sets, and fuzzy L-continuous functions between fuzzy topological spaces.

2 Terminologies

Throughout this paper (X, τ) and (Y, σ) simply X, Y respectively mean fuzzy topological spaces (fts's for short) in Chang's [4] sense, I^X denotes the collection of all fuzzy sets on X. For a fuzzy set μ in I^X a fuzzy point in X with support $x \in X$ and value ε $(0 < \varepsilon \leq 1)$ is denoted by $x_\varepsilon \in \mu$ [7]. The definition of operations of fuzzy sets, fuzzy topological spaces and other concepts can be found in [5,7,9]. For a fuzzy set μ in I^X, μ^-, μ^0, μ^c will respectively denote the fuzzy closure, fuzzy interior and fuzzy complement of μ. The constant fuzzy sets taking values 0 and 1 on X are denoted by $0_X, 1_X$ respectively. A fuzzy set μ in a fts (X, τ) is called fuzzy open [5] (resp. fuzzy α-open [2], fuzzy semiopen [1], fuzzy preopen [4], fuzzy β-open [4], fuzzy regular open [4]) iff $\mu = \mu^0$ (resp. $\mu \leq \mu^{0-c}$, $\mu \leq \mu^{0-}$, $\mu \leq \mu^{-0}$, $\mu \leq \mu^{-0-}$, $\mu = \mu^{-0}$). The complement of a fuzzy open set (resp. fuzzy semiopen, fuzzy regularopen, fuzzy preopen) is called fuzzy closed set (resp. fuzzy semiclosed, fuzzy regularclosed, fuzzy preclosed). The family of all fuzzy open (resp. fuzzy semiopen, fuzzy β-open, fuzzy α-open, fuzzy preopen, fuzzy closed, fuzzy preclosed) sets will be denoted by $FO(X)$, (resp. $FSO(X)$, $F\beta O(X)$, $F\alpha O(X)$, $FPO(X)$, $FC(X)$, $FSC(X)$, $FPC(X)$).

$$f : (X, \tau) \to (Y, \sigma) \text{ is called fuzzy function [9]}$$

if

$$\mu \in I^X,\ f(\mu)(y) = \begin{cases} \operatorname{Sup}\mu(x) & \text{if } f^{-1}(y) \neq 0_X \\ & x \in f^{-1}(y) \\ 0 & \text{otherwise} \end{cases},$$

for each $y \in Y$ and $\xi \in Y$; $f^{-1}(\xi)$ is defined as $f^{-1}(\xi) = \xi(f(x))$ for each $x \in X$. The fuzzy function f is called fuzzy continuous [9] (resp. fuzzy precontinuous [4]) iff for each $\xi \in \delta$, $f^{-1}(\xi)$ is open (resp. preopen) set of X. A fuzzy continuous function is denoted by F-continuous. A non empty collection of fuzzy sets L of set X is called fuzzy ideal [8] on X iff

(i) $\mu \in L$ and $\zeta \leq \mu \Rightarrow \zeta \in L$ (heredity);

(ii) $\mu \in L$ and $\zeta \in L \Rightarrow \mu v \zeta \in L$ (finite additivity).

The fuzzy local function [8] μ^* of a fuzzy set μ is the union of all fuzzy points x_ϵ such that if $v \in (x_\epsilon)$ and $\zeta \in L$ then there is at least one point $r \in X$ for which $v(r) + \mu(r) - 1 > \zeta(r)$. For a fts (X, τ) with fuzzy ideal L, CL^* $(\mu) = \mu\ \mu^*$ [7] for any set μ of X and $\tau^*(L)$ be the fuzzy topology generated by fuzzy CL^* [8]. For any fuzzy set μ is called fuzzy dense (resp. fuzzy dense - in - itself, fuzzy perfect) if $\mu = 1_X$ (resp. if $\mu \leq \mu^d$, if μ is fuzzy dense - in - itself and fuzzy closed) where μ^d is fuzzy derived set of μ [3]. L_n is a fuzzy ideal of fuzzy nowhere dense sets and L_m is fuzzy ideal of fuzzy meager sets [3].

3 New fuzzy notions via ideals

Definition 3.1. *Given a fts (X, τ) with fuzzy ideal L on X, $\mu \in I^X$. Then μ is said to be fuzzy*

(i) τ^*-closed (or F-* closed) if $\mu^* \leq \mu$ [8];

(ii) L-dense-in-itself (or F^*-dense-in-itself) if $\mu \leq \mu^*$;

(iii) *-perfect if μ is F^*-closed and F^*-dense-in-itself.

Theorem 3.1. *Giving a fts (X, τ) with fuzzy ideal L on X, $\mu \in I^X$ then μ is*

(i) F^*-closed iff $CL^*(\mu) = \mu$;

(ii) F^*-dense-in-itself iff $CL^*(\mu) = \mu^*$;

(iii) F^*-perfect iff $CL^*(\mu) = \mu^* = \mu$.

Proof. Follows directly from the fuzzy closure operator CL^* for a fuzzy topology $\tau^*(L)$ in [8] and Definition 2.1. □

Remark 3.1. One can deduce that

(i) Every F^*-dense-in-itself is fuzzy dense set

(ii) Every fuzzy closed (resp. fuzzy open) set is F^*-closed (resp. $F\tau^*$-open).

Corollary 3.1. *Giving a fts (X, τ) with fuzzy ideal L on X and $\mu \in \tau$ then we have*

(i) If μ is F^*-closed then $\mu^* \leq \mu^o \leq \mu^-$;

(ii) If μ is F^*-dense-in-itself then $\mu^o \leq \mu^*$;

(iii) If μ is F^*-perfect then $\mu^o = \mu^- = \mu^*$.

Proof. Obvious. □

Theorem 3.2. *Given a fts (X, τ) with fuzzy ideal L_n on X, $\mu \in I^X$ then we have the following*

(i) is fuzzy α-closed iff μ is F^*-closed;

(ii) is fuzzy β-open iff μ in F^*-dense-in-itself;

(iii) is fuzzy regular closed iff μ is F^*-perfect.

Proof. It is clear. □

Corollary 3.2. *For a fts (X, τ) with fuzzy ideal L and $\mu \in I^X$, the following holds:*

(i) If $\mu \in FC(X)$ then μ is F^*-closed;

(ii) If $\mu \in F\beta C(X)$ then $\mu^{o*o} \leq \mu$;

(iii) If $\mu \in FSC(X)$ then $\mu^{*o} \leq \mu$.

Proof. Obvious. □

4 Fuzzy L-Open and fuzzy L-Closed sets

Definition 4.1. *Given (X, τ) be a fts with fuzzy ideal L on X. $\mu \in I^X$ is called a fuzzy L-open set iff there exist $\zeta \in \tau$ such that $\mu \leq \zeta \leq \mu^*$. We will denote the family of all fuzzy L-open sets by $FLO(X)$.*

Theorem 4.1. *Let (X, τ) be a fts with fuzzy ideal L. Then $\mu \in FLO(X)$ iff $\mu \leq \mu^{*o}$.*

Proof. Assume that $\mu \in FLO(X)$ then by Definition 3.1 there exists $\zeta \in \tau$ such that $\mu \leq \zeta \leq \mu^*$. But $\mu^{*o} \leq \mu^*$, put $\zeta = \mu^{*o}$. Hence $\mu \leq \mu^{*o}$ conversely $\mu \leq \mu^{*o} \leq \mu^*$. Then there exists $\zeta = \mu^{*0} \in \tau$. Hence $\mu \in FLO(X)$. □

Remark 4.1. For a fts (X, τ) with fuzzy ideal L on X and $\mu \in I^X$, the following holds:

(i) If $\mu \in FLO(X)$ then $\mu^o \leq \mu^*$;

(ii) Every fuzzy L-open set is fuzzy *-dense-in-itself.

Theorem 4.2. *Given (X, τ) be a fts with fuzzy ideal L on X and $\mu, \zeta \in I^X$ such that $\mu \in FLO(X)$, $\zeta \in \tau$ then $\mu \wedge \zeta \in FLO(X)$.*

Proof. From the assumption $\mu \wedge \zeta \leq \mu^{*o} \wedge \zeta = (\mu^* \wedge \zeta)^o$ and by using Theorem 3.4 [8], we have $\mu \wedge \zeta \leq (\mu \wedge \zeta)^{*o}$ and this completes the proof. □

Corollary 4.1. *Let $\{\mu_j : j \in J\}$ be a fuzzy L-open set in fts (X, τ) with fuzzy ideal L. Then*

$$\vee\{\mu_j : \ j \in J\} \textit{ is a fuzzy } L - \textit{open set.}$$

Corollary 4.2. *For a fts (X, τ) with fuzzy ideal L, μ, $\zeta \in I^X$ and $\mu \in FLO(X)$ then*

$$\mu^* = \mu^{*o*} \quad \textit{and} \quad (CL^*(\mu)) = \mu^{*o}.$$

Proof. It is clear. □

By utilizing the new fuzzy notions in article [2], we give the relationship between fuzzy L-open sets and some known fuzzy openness.

Theorem 4.3. *Given a fts (X, τ) with fuzzy ideal L on X, $\mu \in I^X$ then the following holds*

(i) If μ is both fuzzy L-open and F^*-perfect then μ is fuzzy open;

(ii) If μ is both fuzzy open and F^*-dense-in-itself then μ is fuzzy L-open.

Proof. Follows from the definitions. □

Corollary 4.1. *For a fuzzy subset μ of a fts (X, τ) with fuzzy ideal L on X, we have:*

(i) If μ is F^*-closed and FL-open then $\mu^o = \mu^{*o}$;

(ii) If μ is F^*-perfect and FL-open then $\mu = \mu^{*o}$.

Remark 4.2. The class of fuzzy L-openness and fuzzy openness are independent concepts as shown by the following example.

Example 4.1. Let $X = \{x\}$ with fuzzy topology $\tau = \{1_X, 0_X, \mu, \zeta\}$, where $\mu(x) = 0.6$, $\zeta(x) = 0.3$ and fuzzy ideal $L = \{0_X, \xi\} \vee \{x_\epsilon : \epsilon \leq 0.2\}$, $\xi(x) = 0.2$, then $\mu \in \tau$ while $\mu \notin FLO(X)$.

Example 4.2. Let (X, τ) be a fts with fuzzy ideal L in example 4.1. If $\zeta(x) = 0.25$ then we can notice that $\zeta \notin \tau$ while $\zeta \in FLO(X)$.

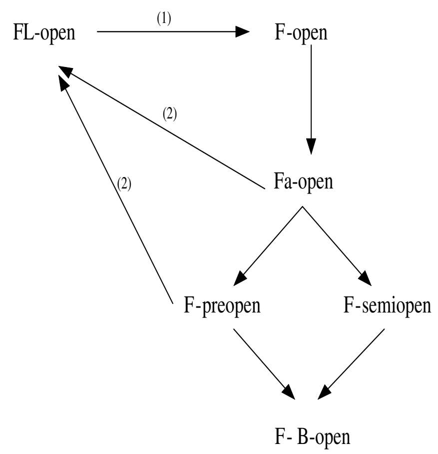
Remark 4.3. One can deduce that

(i) $FLO(X) \leq FPO(X)$ and the converse is not true, in general (Example 4.3);

(ii) The intersection of two fuzzy L-open sets is fuzzy L-open.

Example 4.3. Let (X, τ) be a fts with fuzzy ideal L in example 4.1. then we can deduce that $\mu \in FPO(X)$, but $\mu \neq FLO(X)$.

Remark 4.4. The next diagram show the implications between fuzzy L-open set and some other corresponding types.



The implications (1) and (2) take place under the following conditions

(i) Every fuzzy L-open set is fuzzy *-perfect set;

(ii) Every fuzzy open set is fuzzy *-dense-in -itself.

Corollary 4.3. *Given (X, τ) be a fts with fuzzy ideal L and $\mu \in I^X$. The following holds:*

(i) If $L = \{0_X\}$, then $\mu^*(L) = \mu^-$ and hence each of fuzzy L-open set and fuzzy preopen are conicids;

(ii) If $L = \{I^X\}$, then $\mu^*(L) = 0_X$ and hence μ is fuzzy L-open iff $\mu = 0_X$;

(iii) If $L = L_m$ and $\mu \in FLO(X)$ then $\mu^* \in F\alpha O(X)$;

(iv) If $L = L_n$ and μ both fuzzy L-open and fuzzy semiclosed, then μ is fuzzy regularopen;

(v) If $\mu \in FLO(X)$, then $\mu \wedge \zeta(\mu \wedge \zeta)^*$ for every $\zeta \in FSO(X)$.

Proof. (i) and (ii) follows directly from Theorem 4.1.

(iii) and (iv) obvious from Example 4.1.

(v) its clear. □

Definition 4.3. *Given a fts (X, τ) with fuzzy ideal L and $\mu \in I^X$ then fuzzy ideal interior of μ is defined as largest fuzzy L-open set contained in μ, denoted by FL-int(μ).*

Theorem 4.4. *If (X, τ) is a fts with fuzzy ideal L and $\mu \in I^X$ then*

(i) $\mu \wedge \mu^{*o}$ is fuzzy L-open set;

(ii) FL-int$(\mu) = 0_X$ iff $\mu^{*o} = 0_X$.

Proof. (i) Since $\mu^{*o} = \mu^* \wedge \mu^{*o}$, then $\mu^{*o} = \mu^* \wedge \mu^{*o} \leq (\mu \wedge \mu^*)^*$ thus $\mu \wedge \mu^* \leq (\mu \wedge (\mu \wedge \mu^{*o}))^* \leq (\mu \wedge \mu^{*o})^{*o}$. Hence $\mu \wedge \mu^{*o} \in FLO(X)$.

(ii) Let FL-$int(\mu) = 0_X$, then $\mu \wedge \mu^* = 0_X$, implies $(\mu \wedge \mu^{*o})^- = 0_X$ and so $\mu \wedge \mu^{*o} = 0_X$, by using theorem [8]. Conversely assume that $\mu^{*o} = 0_X$, then $\mu \wedge \mu^{*o} = 0_X$. Hence FL-$int(\mu) = 0_X$. □

Theorem 4.5. *If (X, τ) is a fts with fuzzy ideal L on X and $\mu \in I^X$ then FL-int$(\mu) = \mu \wedge \mu^{*o}$.*

Proof. The first implication follows from Theorem 4.4 that is

$$\mu \wedge \mu^* \leq FL\text{-}int(\mu) \tag{1}$$

For the reverse inclusion, if $\zeta \in FLO(X)$ and $\zeta \leq \mu$ then $\zeta^* \leq \mu^*$ and hence $\zeta^{*o} \leq \mu^{*o}$. This implies $\zeta = \zeta \wedge \zeta^{*o} \leq \mu \wedge \mu^*$. Thus

$$FL\text{-}int(\mu) \leq \mu \wedge \mu^{*o}. \tag{2}$$

From (1) and (2) we have the result. □

Corollary 4.4. *For a fts (X, τ) with fuzzy ideal L on X, $\mu \in I^X$ then the following holds:*

(i) If μ is F^*-closed then $FL\text{-}int(\mu) \leq \mu$;

(ii) If μ is F^*-dense-in-itself then $FL\text{-}int(\mu) \leq \mu^*$;

(iii) If μ is $F - * -$perfect set then $FL\text{-}int(\mu) \leq \mu^*$.

Definition 4.4. *Given (X, τ) be a fts with fuzzy ideal L on X and $\zeta \in I^X$, ζ is called fuzzy L-closed set if its complement is fuzzy L-open set. We will denote the family of fuzzy L-closed sets by $FLC(X)$.*

Theorem 4.6. *If (X, τ) is a fts with fuzzy ideal L on X and $\zeta \in I^X$, with ζ fuzzy L-closed, then $(\zeta^0)^* \leq \zeta$.*

Proof. It is clear. □

Theorem 4.7. *Let (X, τ) be a fts with fuzzy ideal L on X and $\zeta \in I^X$ such that $\zeta^{o*c} = \zeta^{c*o}$. Then $\zeta \in F - L - C(X)$ iff $\zeta^{0*} \leq \zeta$.*

Proof. (Necessity) Follows immediately from the above theorem. □

(Sufficiency) Let $\zeta^{o*} \leq \zeta$ then $\zeta^c \leq \zeta^{o*c} = \zeta^{c*o}$ from the hypothesis. Hence $\zeta^c \in FLO(X)$. Thus $\zeta \in FLC(X)$. □

Corollary 4.4. *For a fts (X, τ) with ideal L on X the following holds:*

(i) The union of fuzzy L-closed set and fuzzy closed set is fuzzy L-closed set;

(ii) The union of fuzzy L-closed and fuzzy closed is fuzzy perfect.

5 Fuzzy L-Continuous functions

By utilizing the notion of FL-open sets, we establish in this section a class of fuzzy functions called fuzzy L-continuous which in the class of fuzzy precontinuous function [2].

Each of fuzzy L-continuous and fuzzy continuous function are independent concepts.

Definition 5.1. *A fuzzy function $f : (X, \tau) \to (Y, \sigma)$ with fuzzy ideal L on X is said to be fuzzy L-continuous if, for every $\zeta \in \sigma$, $f^{-1}(\zeta) \in FLO(X)$.*

Theorem 5.1. *Every fuzzy L-continuous function is fuzzy precontinuous but the converse is not true in general as can be seen Example 5.1.*

Proof. It is clear. □

Example 5.1. Let $X = Y = \{x\}$, τ is the fuzzy indiscrete topology, σ is the fuzzy discrete topology and $L = \{0_X, \mu\} \vee \{x_\varepsilon : \varepsilon \leq 0.3\}$, $\mu(x) = 0.3$. The fuzzy identity function $f : (X, \tau) \to (Y, \sigma)$ is fuzzy precontinuous but not fuzzy L-continuous, since $\mu \in \sigma$, while $F^{-1}(\mu) \in FLO(X)$.

Theorem 5.2. *For a fuzzy function $f : (X, \tau) \to (Y, \sigma)$ with ideal L on X, the following are equivalent:*

(i) f is fuzzy L-continuous;

(ii) For x_ε in X and each $\zeta \in \sigma$ containing $f(x_\varepsilon)$, there exists $\mu \in FLO(X)$ containing x_ε such that $f(\mu) \leq \sigma$;

(iii) For each fuzzy point x in X and $\zeta \in \sigma$ containing $f(x_\varepsilon)$, $(f^{-1}(\zeta))^*$ is fuzzy nbd of x_ε;

(iv) The inverse image of each fuzzy closed set in Y is fuzzy L-closed.

Proof. (i)⇒ (ii). Since $\zeta \in \sigma$ containing $f(x_\varepsilon)$, then by (i), $f^{-1}(\zeta) \in FLO(X)$, by putting $\mu = f^{-1}(\zeta)$ which contains x_ε, we have $f(\mu) \leq \sigma$.

(ii)⇒(iii). Let $\zeta \in \sigma$ containing $f(x_\varepsilon)$. Then by (ii) there exists $\mu \in FLO(X)$ containing x_ε such that $f(\mu) \leq \sigma$, so $x_\varepsilon \in \mu \leq \mu^{*o} \leq (f^{-1}(\zeta))^{*o} \leq (f^{-1}(\zeta))^*$. Hence $(f^{-1}(\zeta))^*$ is a fuzzy nbd of x_ε.

(iii)⇒(i). Let $\zeta \in \sigma$. Since $(f^{-1}(\zeta))$ is fuzzy nbd, for any point $f^{-1}(\zeta)$, every point $x_\varepsilon \in (f^{-1}(\zeta))^*$ is a fuzzy interior point of $f^{-1}(\zeta)^*$. Then $f^{-1}(\zeta) \leq (f^{-1}(\zeta))^{*o}$ and hence f is fuzzy L-continuous.

(i)⇒(iv). Let $\xi \in y$ be a fuzzy closed set. Then ξ^c is fuzzy open set by (i) $f^{-1}(\xi^c) = (f^{-1}(\xi))^c \in FLO(x)$. Thus $f^{-1}(\xi)$ is fuzzy L-closed. □

The following theorem establish the relationship between fuzzy L-continuity and fuzzy continuity by using the new fuzzy notions in section 2.

Theorem 5.3. *If $f : (X, \tau) \to (Y, \sigma)$ is a fuzzy function with ideal L on X then we have: If f is fuzzy L-continuous for each fuzzy $*$-perfect set in X, then f is fuzzy continuous.*

Proof. Obvious. □

Corollary 5.1. *Let $f : (X, \tau) \to (Y, \sigma)$ be a fuzzy function such that each member of X is fuzzy $*$- dense-in-itself.*

Then we have:

(i) Every fuzzy continuous function is fuzzy L-continuous;

(ii) A fuzzy function is precontinuous iff it is fuzzy L-continuous.

Proof. It is clear. □

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