The complete classification of self-similar solutions of the Navier-Stokes equations for incompressible flow

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A new classification of self-similar solutions of the Navier-Stokes system of equations is presented here.
The equation of momentum has been split into the sub-system of 2 equations: an irrotational (*curl-free*) one, and a solenoidal (*divergence-free*) one.
The irrotational (*curl-free*) equation used for obtaining of the components of pressure gradient. As a term of such an equation, we used the irrotational (*curl-free*) *vector field of flow velocity* (*potential of which should be a harmonic function*).
Then we consider equation for *solenoidal* (*divergence-free*) field of flow velocity in the proper rotating co-ordinate system. Such a transition transforms the equation of motion to the *Helmholtz* vector differential equation.
The inverse transformation from the rotating system to ordinary Cartesian coordinate system is possible only in case of constant angular velocity of rotation of such a system.
The *Helmholtz* differential equation can be solved by separation of variables in only 11 coordinate systems, so it forms a complete set of all possible cases of self-similar solutions for Navier-Stokes system of equations.

Keywords: Navier-Stokes equations, self-similar solutions, incompressible flow.

1. Introduction, the Navier-Stokes system of equations.

In accordance with [1-2], the Navier-Stokes system of equations for incompressible flow of Newtonian fluids should be presented as below:

$$\nabla \cdot \vec{u} \ = 0 \ , \tag{1.1}$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{\nabla p}{\rho} + \nu \cdot \nabla^2 \vec{u} + \vec{F} \ , \tag{1.2}$$

- where $\boldsymbol{u}$ is the flow velocity, a vector field; ρ is the fluid density, p is the pressure, ν is the kinematic viscosity, and $\boldsymbol{F}$ represents body forces (*per unit of mass in a volume*) acting on the fluid and ∇ is the del (nabla) operator. Let us also choose the Oz axis coincides to the main direction of flow propagation.

In the differential equations, a *local* description of the fields, the nabla symbol ∇ denotes the three-dimensional gradient operator, and from it $\nabla\cdot$ is the divergence operator and $\nabla\times$ is the curl operator.

The system of Navier-Stokes equations is known to be the system of the mixed parabolic and hyperbolic type.

2. The originating system of PDE for Navier-Stokes Eq.

Using the identity $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = (1/2)\nabla(\boldsymbol{u}^2) - \boldsymbol{u}\times(\nabla\times\boldsymbol{u})$, and then using the curl of the curl identity $\nabla\times(\nabla\times\boldsymbol{u}) = \nabla(\nabla\cdot\boldsymbol{u}) - \nabla^2\boldsymbol{u}$, we could present the Navier-Stokes equations in the case of incompressible flow of Newtonian fluids as below:

$$
\begin{aligned}
&\nabla \cdot \vec{u} \;=\; 0\,, \\
&\frac{\partial \vec{u}}{\partial t} \;=\; \vec{u} \times \vec{w} \;+\; \nu \cdot \nabla^{2} \vec{u} \;-\; \left(\frac{1}{2} \nabla (u^{2}) + \frac{\nabla p}{\rho} - \vec{F} \right)
\end{aligned}
\tag{2.1}
$$

- here we denote *the curl field* ***w***, a pseudovector field (*time-dependent*).

The system of equation (2.1) is equivalent to the Navier-Stokes system of equations for incompressible Newtonian fluids (1.1)-(1.2) in the sense of existence and smoothness of a general solution.

Let us denote as below (*according to the Helmholtz fundamental theorem of vector calculus*):

$$
\nabla \times \vec{u} \;\equiv\; \vec{w}\,, \qquad \vec{u} \;\equiv\; \vec{u}_{p} \;+\; \vec{u}_{w}\,,
$$

$$
\nabla \cdot \vec{u}_{w} \;\equiv\; 0\,, \qquad \nabla \times (\vec{u}_{p}) \;\equiv\; 0\,,
$$

- where $\boldsymbol{u_p}$ is *an irrotational* (*curl-free*) field of flow velocity, and $\boldsymbol{u_w}$ - is *a solenoidal* (*divergence-free*) field of flow velocity which generates a curl field ***w***:

$$
\vec{u}_{p} \;\equiv\; \nabla \varphi\,, \qquad \vec{u}_{w} \;\equiv\; \nabla \times \vec{A}\,,
$$

- here φ - is the proper scalar potential, ***A*** – is the appropriate vector potential.

We could obtain from the 1-st equation of (2.1) the equality below

$$
\nabla \cdot (\nabla \varphi + \nabla \times \vec{A}) \;=\; 0\,, \quad \Rightarrow \quad \Delta \varphi \;=\; 0\,, \tag{2.2}
$$

- it means that φ - is the proper *harmonic function* [3].

Thus, the 2-nd equation of (2.1) could be presented as the system of equations below:

$$\begin{cases} \dfrac{\partial(\nabla\varphi)}{\partial t} = (\nabla\varphi)\times(\nabla\times\vec{u}_w) - \left(\dfrac{1}{2}\nabla\{(\nabla\varphi + \vec{u}_w)^2\} + \dfrac{\nabla p}{\rho} - \vec{F}\right), \\ \\ \dfrac{\partial\vec{u}_w}{\partial t} = \vec{u}_w\times(\nabla\times\vec{u}_w) + \nu\cdot\nabla^2\vec{u}_w \ , \end{cases} \tag{2.3}$$

- so, if we solve the second equation of (2.3) for the components of vector $\boldsymbol{u}_w$, we could substitute it into the 1-st equation of (2.3) for obtaining of a proper expression for vector function ∇p:

$$\frac{\nabla p}{\rho} = (\nabla\varphi)\times(\nabla\times\vec{u}_w) + \vec{F} - \frac{\partial(\nabla\varphi)}{\partial t} - \frac{1}{2}\nabla\{(\nabla\varphi + \vec{u}_w)^2\}\ , \tag{2.4}$$

- where φ - is the proper *harmonic function*, see Eq. (2.2).

So, the key equation of Navier-Stokes equations in the case of incompressible flow for Newtonian fluids could be presented in a form below:

$$\frac{\partial\vec{u}_w}{\partial t} = \vec{u}_w\times(\nabla\times\vec{u}_w) + \nu\cdot\nabla^2\vec{u}_w \tag{2.5}$$

Let us consider equation (2.5) in the rotating co-ordinate system by adding of the proper *Coriolis force* to the equation of motion (2.5) as below

$$\frac{\partial\vec{u}_w}{\partial t} - 2\vec{\Omega}\times\vec{u}_w = \vec{u}_w\times(\nabla\times\vec{u}_w) + \nu\cdot\nabla^2\vec{u}_w\ ,$$

- where $\mathbf{\Omega}$ - is the angular velocity, for which the equality below is valid in the case of Newtonian fluids [2]:

$$\mathbf{\Omega} = (\nabla\times \boldsymbol{u}_w)/2$$

So, the equation of motion (2.5) should be presented in the rotating co-ordinate system as below

$$\frac{\partial \vec{u}_w}{\partial t} = \nu\cdot\nabla^2 \vec{u}_w \,, \tag{2.6}$$

The inverse transformation of exact solutions from the rotating system to ordinary Cartesian coordinate system is possible only in case Ω = const.

3. Classification of exact solutions for Navier-Stokes Eq.

For non-stationary solutions $\partial/\partial t \neq 0$, Eq. (2.6) could be solved analytically only in the cases below:

1) $\partial/\partial t \sim \partial/\partial z$ - it means that the Oz axis represents a preferential direction similar to the time arrow in mechanical processes [4];

2) Time-dependent self-similar case, $\boldsymbol{u}_w = \exp(-\omega t)\cdot\boldsymbol{u}_w\,(x, y, z)$, ω = const > 0 (*frequency-parameter*).

For the time-dependent self-similar case, equation (2.6) should be presented as below

$$\nabla^2 \vec{u}_w + \left(\frac{\omega}{\nu}\right)\vec{u}_w = 0 \,, \tag{3.1}$$

- which is the proper *Helmholtz* differential equation for vector fields $\boldsymbol{u}_w$ [2].

The *Helmholtz* differential equation can be solved by separation of variables in only 11 coordinate systems, 10 of which (*with the exception of confocal paraboloidal coordinates*) are particular cases of the confocal ellipsoidal system: Cartesian, confocal ellipsoidal, confocal paraboloidal, conical, cylindrical, elliptic cylindrical, oblate spheroidal, paraboloidal, parabolic cylindrical, prolate spheroidal, and spherical coordinates [5-6].

Thus, above 11 classes form a complete set of all possible cases of self-similar solutions for Navier-Stokes system of equations.

4. Conclusion.

A new classification of self-similar solutions of the Navier-Stokes system of equations is presented here. The equation of momentum should be split into the sub-system of 2 equations: an irrotational (*curl-free*) one, and a solenoidal (*divergence-free*) one.

The irrotational (*curl-free*) equation used for obtaining of the components of pressure gradient ∇p. As a term of such an equation, we used the irrotational (*curl-free*) *vector field of flow velocity*, which is given by the proper potential (*besides, the continuity equation determines such a potential as a harmonic function*).

Then we consider the equation for *solenoidal* (*divergence-free*) field of flow velocity in the rotating co-ordinate system. Such a transition transforms the equation of motion to the *Helmholtz* vector differential equation. The inverse transformation from the rotating system to ordinary Cartesian coordinate system is possible only in the case of constant angular velocity of rotation. The *Helmholtz* differential equation can be solved by separation of variables in only 11 coordinate systems, so it forms a complete set of all possible cases of self-similar solutions for Navier-Stokes system of equations.

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