THE CONTINUUM HYPOTHESIS

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In this paper we prove the continuum hypothesis with categorical logic by proving that the theory of initial ordinals and the theory of cardinals are isomorphic. To prove that cardinals and initial ordinals are isomorphic structures, and so, since isomorphic structures are isomorphic theories by the fundamental theorem of mathematical logic, the theorems of the theory of cardinals are theorems of the theory of initial ordinals, and conversely the theorems of the theory of initial ordinals are theorems of the theory of cardinals, we use the definition of a theory, the definition of an isomorphism of structures, in its equivalent form, the definition of an isomorphism of categories, the definition of a structure, the definition of a formal language, the definition of a functor, the definition of a category, the axioms of mathematical logic and the axioms of the theory of categories, which include the Gödel-Bernays-von Neumann axioms for classes and sets. And thus, applying both the theorem on the comparablity of ordinals to the theory of cardinals, and the fundamental theorem of cardinal arithmetic to the theory of ordinals, we prove the theorem.

Theorem "generalized continuum hypothesis" *For every transfinite cardinal number α, there is no cardinal number between α and 2^{α}.*

Proof. Let **Card** be the class of cardinals and let **Ord** be class of initial ordinals. Since, according to the definition of a category and by the axiom of choice, in its equivalent form, the well-ordering principle, every structure of a formal language is a category, namely a preorder, fact that is the foundation of categorical logic, and the classes **Card** and **Ord** are both structures of the formal second-order language of set theory, **Card** and **Ord** are categories, namely well-ordered semirings. We prove that **Card** and **Ord** are isomorphic categories, proving that there is a full and faithful functor $T\colon$ **Card**$\to$**Ord** such that each initial ordinal β is isomorphic to an initial ordinal $T\alpha$ for some cardinal α.

Let $T\colon$ **Card**$\to$**Ord** be the function of categories which assigns to every cardinal α the initial ordinal $T\alpha$ of its equipotence class, $\alpha \mapsto T\alpha$, and to every arrow $f\colon \alpha \to \alpha'$ in **Card** the arrow $Tf\colon T\alpha \to T\alpha'$ in **Ord**, $f \mapsto Tf$, for each pair of cardinals α and α'. The function of categories T is well-defined because each cardinal α lies in a unique equipotence class defined by α, so, it defines uniquely $T\alpha$, and because there is only one arrow Tf in **Ord** for every arrow f in **Card**, since each arrow f in a category C is a pair of objects α and α' for which $f\colon \alpha \to \alpha'$ is an arrow in C, to each pair of cardinals α and α' there is a unique pair of initial ordinals $T\alpha$ and $T\alpha'$, which are isomorphic to α and α', respectively, by definition of T, and so, by the orderings in **Card** and **Ord**, for which the arrow $f\colon \alpha \to \alpha'$ is in **Card** if, and only if, the arrow $g = Tf\colon T\alpha \to T\alpha'$ is in **Ord**, which is unique for **Card** and **Ord** are preorders. The latter condition on T means that T is an order-preserving function of the linear order **Card** to the linear order **Ord**.

Since a preorder has all its monoids, as any category does, by definition of category, the function of categories T is a functor because a functor is a function of categories preserving monoids, that is, preserving identities and composable pair of arrows, $T1_{\alpha} = 1_{T\alpha}$ and $T(f \circ g) = Tf \circ Tg$, for every identity 1_{α} and every composable pair of arrows f and g in the domain of T. For, each identity 1_{α} in **Card** is a cardinal α, the initial ordinal $T\alpha$ of the equipotence class of each cardinal α is the identity $1_{T\alpha}$ in **Ord**, every category has all its identities, the identities 1_{α} in a category C are the objects α in the category C, and **Ord** is a category. And for, in any category C each arrow $f \circ g$ in C is a composable pair of arrows f and g in C, each composable pair of arrows f and g in C is a triad of objects α, α' and α'' in C such that $f\colon \alpha \to \alpha'$, $g\colon \alpha' \to \alpha''$ and $f \circ g\colon \alpha \to \alpha''$ are arrows in C, $Tf\colon T\alpha \to T\alpha'$, $Tg\colon T\alpha' \to T\alpha''$ and $T(f \circ g)\colon T\alpha \to T\alpha''$ are arrows in **Ord** by definition of T, so, Tf and Tg is a composable pair of arrows in **Ord**, every category C has all its composable pairs of arrows, the arrows in a preorder C are unique, and **Ord** is a preorder.

The functor T is full because to every pair of cardinals α and α' and to every arrow $g\colon T\alpha \to T\alpha'$ in **Ord** there is an arrow $f\colon \alpha \to \alpha'$ in **Card** such that $Tf = g$, for **Ord** is a preorder and T satisfies the condition above: the arrow $f\colon \alpha \to \alpha'$ is in **Card** if, and only if, the arrow $Tf\colon T\alpha \to T\alpha'$ is in **Ord**. The functor T is faithful because to every pair of cardinals α and α' and to every pair of arrows $f_1,\ f_2\colon \alpha \to \alpha'$ in **Card** the equality $Tf_1 = Tf_2$ implies $f_1 = f_2$, since **Card** is a preorder and by definiton of T. Finally, since every cardinal $|\beta|$ is isomorphic to the initial ordinal $T|\beta|$ of its equipotence class, and every initial ordinal β is isomorphic to the cardinal $|\beta|$ of its equipotence class by definition of initial ordinal, cardinal and T, to each initial ordinal β there is an isomorphism between β and the initial ordinal $T|\beta|$ of the equipotence class of β, that is, β is isomorphic to $T|\beta|$, $\beta \cong T|\beta|$, therefore, **Card** $\cong$ **Ord**. ■

Thus, for isomorphic categories are isomorphic theories, as isomorphic categories are isomorphic structures and isomorphic structures are isomorphic theories by the fundamental theorem of mathematical logic, by definition of isomorphism of structures, by definition of isomorphism of categories, by the axioms of mathematical logic and by the axioms of the theory of categories, which include the Gödel-Bernays-von Neumann axioms, and since there is no initial ordinal between ω and ω^ω by the theorem on the comparability of ordinals and $\aleph_0^{\aleph_0} = 2^{\aleph_0}$ by the fundamental theorem of cardinal arithmetic, the isomorphism between categories **Card** and **Ord** proves that there is no cardinal number between the transfinite cardinal numbers $\aleph_0$ and $2^{\aleph_0}$, and that, in general, there is no cardinal number between any transfinite cardinal number α and 2^α. As a consequence, there exist no inaccessible cardinals. In fact, the class of transfinite cardinals is isomorphic to ω, because the order-preserving function f of ω to it, which assigns to each finite ordinal α the α-th transfinite cardinal is an isomorphism, which is unique, by transfinite construction.

The theorem in universal algebra

Thus, not only does the theorem prove that the class of cardinals **Card** is an infinite countable nondiscrete large category which is a closed complete and cocomplete semiring, with arrows, the polynomial maps and the exponential maps, that is an algebra by the action of the covariant exponential functor semiring e, itself, a functor algebra, but also that the closed complete and cocomplete algebra of initial ordinals **Ord** is isomorphic to the closed complete and cocomplete algebra of cardinals **Card**.

The theorem in categorical logic

In categorical logic, as all first order theories are infinite well orders isomorphic to ω and have thereby transfinite cardinal smaller than the cardinal of the continuum, not only does the theorem proves that the theories **Card** and **Ord** are isomorphic, but also that all higuer order theories are continuums or greater, for they are, at least, partial orders isomorphic to infinite countable products of first order theories.

The theorem in topos theory

In topos theory, not only does the theorem prove that the category **Card** of cardinals is a topos isomorphic to the topos of initial ordinals **Ord**, nor that is its topos of sheaves the category $\mathbf{Sets}^{\mathbf{Card}^*}$ of the set-valued contravariant functors on **Card** to **Sets** which assign to every cardinal number β its set of cardinal functions on β, denoting the dual category of the category **Card** by $\mathbf{Card}^*$, all of which turn out to be the continuous cardinal functions on the topology of cardinals, but also, that the topos of sheaves of cardinals $\mathbf{Sets}^{\mathbf{Card}^*}$ is isomorphic to the topos of sheaves of initial ordinals $\mathbf{Sets}^{\mathbf{Ord}^*}$.

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