The Continuum Hypothesis

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In this paper we give a proof of the Continuum Hypothesis.

Theorem "Generalized Continuum Hypothesis": For every transfinite cardinal number α, there is no cardinal number between α and 2^{α}.

Proof: Let **Card** be the class of transfinite cardinal numbers and let **Ord** be class of infinite initial ordinal numbers. We prove the equivalence of these two large categories showing that there exist a full and faithful functor $T\colon$ **Card**$\to$**Ord** such that each $\beta \in$ **Ord** is isomorphic to $T\alpha$ for some $\alpha \in$ **Card**.

Let $T\colon$ **Card**$\to$**Ord** be the well-ordering functor which, by the well-ordering principle, assigns to every transfinite cardinal number α its infinite initial ordinal number, $T\alpha$. The functor T is full because to every pair of transfinite cardinal numbers α and α' and to every arrow $g\colon T\alpha \to T\alpha'$ in **Ord** there is an arrow $f\colon \alpha \to \alpha'$ in **Card** such that $Tf = g$ because, by definition of transfinite cardinal number, every infinite initial ordinal number is its transfinite cardinal number, and so, both well orders are isomorphic, that is, there exist an order-preserving isomorphism between **Ord** and **Card**, **Card** $\cong_{\subset}$**Ord**. The functor T is faithful because to every pair of transfinite cardinal numbers α and α' and to every pair of arrows $f_1,\ f_2\colon \alpha \to \alpha'$ the equality $Tf_1 = Tf_2$ implies $f_1 = f_2$ because every f_i is unique since **Card** is also a preorder. And to each infinite ordinal number β there is an order-preserving bijection on it to its well-ordered transfinite cardinal number $T|\beta|$ because, by definition of transfinite cardinal number and infinite initial ordinal number, it is equal to its infinite initial ordinal number up to isomorphism, that is, $\beta \cong T|\beta|$. Therefore **Card** $\cong$ **Ord**.

Thus, by the Gödel-Bernays-von Neumann axioms, by the fundamental theorem of cardinal arithmetic, because there is no initial ordinal number between ω and ω^{ω}, and because equivalent categories are equivalent theories, the equivalence between the categories **Card** and **Ord** proves that there is no cardinal number between the transfinite cardinal numbers $\aleph_0$ and $2^{\aleph_0}$, and that, in general, there is no cardinal number between any transfinite cardinal number α and 2^{α}. As a consequence, there exist no inaccessible cardinals. In fact, the class of transfinite cardinal numbers is countable.

A theorem of universal algebra

Thus, not only does the theorem prove that the class of transfinite cardinal numbers is an infinite countable nondiscrete large category which is a closed complete and cocomplete semiring, with arrows, the polynomial maps and the exponential maps, that is an algebra by the action of the covariant exponential functor semiring e, itself, a functor algebra, but also, by the well-ordering principle, that the closed complete and cocomplete algebra of infinite initial ordinal numbers, **Ord**, is isomorphic to the closed complete and cocomplete algebra of transfinite cardinals, **Card**.

The theorem in categorical logic

In categorical logic, as all first order theories are infinite linear orders that have transfinite cardinal number $\alpha < c$, where c is the cardinal of the continuum, the theorem proves that all higuer order theories are continuums, for they are partial orders which are countably infinite products of first order theories.

The theorem in topos theory

Finally, in topos theory, this theorem not only proves that the category **Card** of (small) cardinal numbers is a topos, but also that its topos of sheaves is the category $\mathbf{Sets}^{\mathbf{Card}^*}$, denoting the dual category of the category **Card** by $\mathbf{Card}^*$, of the set-valued contravariant functors on **Card** to **Sets** which assign to every cardinal number β its set of cardinal functions on β all of which turn out to be the cardinal continuous functions on the topology of the cardinal numbers.

Bibliography

Gödel, *The consistency of the continuum hypothesis,* Princeton: Princeton University Press 1940

Mac Lane, *Categories for the working mathematician*, Berlin: Springer 1998