

## The Continuum Hypothesis

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In this paper we give a proof of the Continuum Hypothesis.

Theorem "Generalized Continuum Hypothesis": For every transfinite cardinal number  $\alpha$ , there is no cardinal number between  $\alpha$  and  $2^\alpha$ .

Proof: Let **Card** be the class of transfinite cardinal numbers and let **Ord** be class of infinite initial ordinal numbers. We prove the equivalence of these two large categories showing that there exist a full and faithful functor  $T: \mathbf{Card} \rightarrow \mathbf{Ord}$  such that each  $\beta \in \mathbf{Ord}$  is isomorphic to  $T\alpha$  for some  $\alpha \in \mathbf{Card}$ .

Let  $T: \mathbf{Card} \rightarrow \mathbf{Ord}$  be the well-ordering functor which, by the well-ordering principle, assigns to every transfinite cardinal number  $\alpha$  its infinite initial ordinal number,  $T\alpha$ . The functor  $T$  is full because to every pair of transfinite cardinal numbers  $\alpha$  and  $\alpha'$  and to every arrow  $g: T\alpha \rightarrow T\alpha'$  in **Ord** there is an arrow  $f: \alpha \rightarrow \alpha'$  in **Card** such that  $Tf = g$  because, by definition of transfinite cardinal number, every infinite initial ordinal number is its transfinite cardinal number, and so, both well orders are isomorphic, that is, there exist an order-preserving isomorphism between **Ord** and **Card**,  $\mathbf{Card} \cong_{\mathcal{C}} \mathbf{Ord}$ . The functor  $T$  is faithful because to every pair of transfinite cardinal numbers  $\alpha$  and  $\alpha'$  and to every pair of arrows  $f_1, f_2: \alpha \rightarrow \alpha'$  the equality  $Tf_1 = Tf_2$  implies  $f_1 = f_2$  because every  $f_i$  is unique since **Card** is also a preorder. And to each infinite ordinal number  $\beta$  there is an order-preserving bijection on it to its well-ordered transfinite cardinal number  $T|\beta|$  because, by definition of transfinite cardinal number and infinite initial ordinal number, it is equal to its infinite initial ordinal number up to isomorphism, that is,  $\beta \cong T|\beta|$ . Therefore  $\mathbf{Card} \cong \mathbf{Ord}$ .

Thus, because equivalence of categories is equivalence of theories, the equivalence between the categories **Card** and **Ord** proves, by the Gödel-Bernays-von Neumann axioms and by the fundamental theorem of cardinal arithmetic, since there is no initial ordinal number between  $\omega$  and  $\omega^\omega$ , that there is no cardinal number between the transfinite cardinal numbers  $\aleph_0$  and  $2^{\aleph_0}$ , and that, in general, there is no cardinal number between the transfinite cardinal numbers  $\alpha$  and  $2^\alpha$ . As a consequence, there exist no inaccessible cardinals. In fact, the class of transfinite cardinal numbers is countable.

### A theorem of universal algebra

Thus, not only does the theorem prove that the class of transfinite cardinal numbers is an infinite countable nondiscrete large category which is a closed complete and cocomplete semiring, with arrows, the polynomial maps and the exponential maps, that is an algebra by the action of the covariant exponential functor semiring  $e$ , itself, a functor algebra, but also, by the well-ordering principle, that the closed complete and cocomplete algebra of infinite initial ordinal numbers, **Ord**, is isomorphic to the closed complete and cocomplete algebra of transfinite cardinals, **Card**.

### The theorem in categorical logic

In categorical logic, not only does the theorem prove that all first order theories are infinite linear orders that have transfinite cardinal number  $\alpha \leq 2^{\aleph_0}$ , but also therefore, that all higher order theories are continuums for they are partial orders which are countably infinite products of first order theories.

### The theorem in topos theory

Finally, in topos theory, this theorem not only proves that the category **Card** of (small) cardinal numbers is a topos, but also that its topos of sheaves is the category **Sets**<sup>**Card**\*</sup>, denoting the dual category of the category **Card** by **Card**<sup>\*</sup>, of the set-valued contravariant functors on **Card** to **Sets** which assign to every cardinal number  $\beta$  its set of cardinal functions on  $\beta$  all of which turn out to be the cardinal continuous functions on the topology of the cardinal numbers.

### Bibliography

- Gödel, *The consistency of the continuum hypothesis*, Princeton: Princeton University Press 1940  
Mac Lane, *Categories for the working mathematician*, Berlin: Springer 1998