The Continuum Hypothesis

Daniel Cordero Grau
dcgrau01@yahoo.co.uk

In this paper we give a proof of the Continuum Hypothesis.

Theorem "Continuum Hypothesis": There exist a bijection between the set of real numbers numbers $\mathbb{R}$ and the power set of $\mathbb{N}$.

Proof: Let $\mathbb{R}^+ \subset \mathbb{R}$ be the complete free semialgebra of nonnegative numbers, and let $\mathbb{R}_2^+$ be the complete free semialgebra of dyadic nonnegative numbers, which, by the division algorithm in complete free semialgebras, is isomorphic to $\mathbb{R}^+$. Then, since $\mathbb{R}_2^+$, as a semialgebra over its subsemialgebra $B = \{0, 1\}$, is isomorphic to the direct sum $\underset{i\in\mathbb{Z}}{\oplus} B_i$ where $B_i \cong B$ for every i, that is, $\mathbb{R}_2^+ \cong \underset{i\in\mathbb{Z}}{\oplus} B_i$, and since the cardinal number of the indexing set $\mathbb{Z}$ for the direct sum is equal to the cardinal number of $\mathbb{N}$, $\aleph_0$, the cardinal number $|\underset{i\in\mathbb{N}}{\oplus} B_i|$ of the set $\underset{i\in\mathbb{N}}{\oplus} B_i$, indexed by $\mathbb{N}$, is equal to the cardinal number $|\underset{i\in\mathbb{Z}}{\oplus} B_i|$ of $\underset{i\in\mathbb{Z}}{\oplus} B_i$. Thereby, since $\underset{i\in\mathbb{N}}{\oplus} B_i = \bigcap_{k=0}^{\infty} \bigcup_{j=0}^{k} \prod_{i=j}^{\infty} B_i$ and the sets $\mathbb{R}$ and $\mathbb{R}^+$ are equipotent as well as the indexing sets $\{i \in \mathbb{N} \colon i \geq j\}$ for every $\prod_{i=j}^{\infty} B_i$ for all $j \in \mathbb{N}$,

$$|\mathbb{R}| = |\mathbb{R}^+| = \left|\mathbb{R}_2^+\right| = |\underset{i\in\mathbb{Z}}{\oplus} B_i| = |\underset{i\in\mathbb{N}}{\oplus} B_i| = |\bigcap_{k=0}^{\infty} \bigcup_{j=0}^{k} \prod_{i=j}^{\infty} B_i|$$
$$= \inf_{k\in\mathbb{N}} \sup_{j\leq k} |\prod_{i=j}^{\infty} B_i| = \inf_{k\in\mathbb{N}} \sup_{j\leq k} 2^{|\omega|} = \liminf_{k\to\infty} 2^{\aleph_0} = 2^{\aleph_0},$$

therefore, by the Cantor-Bernstein-Schröder theorem, there exist a bijection between $\mathbb{R}$ and the power set of $\mathbb{N}$.

Thus, by the Zermelo-Fraenkel axioms, the Choice axiom, the Peano axioms, the Cantor-Bernstein-Schröder theorem, and the definitions of cardinal number and ordinal number, this theorem in particular proves that there is no cardinal number between the initial transfinite cardinal number $\aleph_0$ and $2^{\aleph_0}$. In general, we have

Corollary "Generalized Continuum Hypothesis": For every transfinite cardinal number α, there is no cardinal number between α and 2^α.

Proof: Let **Card** be the class of transfinite cardinal numbers and let **Ord** be class of infinite initial ordinal numbers. By definition of transfinite cardinal number, **Card** is **Ord**, that is, there exist an order-preserving isomorphism between these two well-ordered classes, **Card** $\cong$ **Ord**. By definition of infinite initial ordinal number and the Cantor-Bernstein-Schröder theorem, the well-ordered class **Ord** is the nondiscrete idempotent algebra with just one object, ω, upon which the exponential functor e acts, thereby, by definition of functor monoidal category, since e is a monoid, **Ord** is an infinite countable well-ordered algebra isomorphic to ω, that is, **Ord** $\cong \boldsymbol{\omega}$. As a consequence, **Card**, the well-ordered class of transfinite cardinal numbers, is isomorphic to ω.

Now applying the transfinite induction principle for well-ordered classes isomorphic to ω on the well-ordered subclass $\mathbf{Card}_\omega$ of transfinite cardinal numbers α for which there exist no cardinal number β such that

$$\alpha < \beta < 2^\alpha,$$

we have, by the theorem, that the first element of the well-ordered class of the transfinite cardinal numbers **Card**, $\aleph_0$, is in this subclass. And for every transfinite cardinal number α, if its immediate predecessor β is such that there exist no transfinite cardinal number between β and 2^β, that is, if 2^β is the immediate successor of β, then, since the immediate successor of the immediate predecessor of an element is the same element, $\alpha = 2^\beta$, thereby, since there exist no cardinal number between 2^β and 2^{2^β}, by the induction hypothesis and because the exponentiation is a transfinite cardinal order-preserving function, there exist no cardinal number between α and 2^α, so that α also belongs to this subclass. Therefore this subclass, $\mathbf{Card}_\omega$, is the class of transfinite cardinal numbers, **Card**, that is, for every transfinite cardinal α, there exist no cardinal number between α and 2^α.

As a consequence, there exist no inaccessible cardinals. In fact, the class of transfinite cardinal numbers is countable.

A theorem of universal algebra

Therefore not only does the theorem prove that the class of transfinite cardinal numbers is a large category which forms an infinite countable complete semiring, with arrows, the polynomial maps and the exponential maps, which is an algebra by the action of the covariant exponential functor semiring e, itself, a functor algebra, but also, by definition of initial limit ordinal number, that the algebra of initial limit ordinal numbers, on which the power set functor $\mathcal{P}$ acts, is isomorphic to the complete algebra of transfinite cardinals.

The theorem in categorical logic

In categorical logic, not only does the theorem prove that all first order theories are infinite linear orders that have transfinite cardinal number $\alpha \leq 2^{\aleph_0}$, but also therefore, that all higuer order theories are continuums for they are partial orders which are countably infinite products of first order theories.

The theorem in topos theory

In topos theory, this theorem proves that the category **Card** of (small) cardinal numbers is a topos, and that its topos of sheaves is the category $\mathbf{Sets}^{\mathbf{Card}^*}$, denoting the dual category of the category **Card** by $\mathbf{Card}^*$, of the set-valued contravariant functors on **Card** to **Sets** which assign to every cardinal number β its set of cardinal functions on β all of which turn out to be the cardinal continuous functions on the topology of the cardinal numbers.

Bibliography

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