The Continuum Hypothesis

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In this paper we give a proof of the Continuum Hypothesis.

Theorem "Continuum Hypothesis": There exist a bijection between the set of real numbers numbers $\mathbb{R}$ and the power set of $\mathbb{N}$.

Proof: Let $\mathbb{R}^+ \subset \mathbb{R}$ be the complete free semialgebra of nonnegative numbers, and let $\mathbb{R}_2^+$ be the complete free semialgebra of dyadic nonnegative numbers, which, by the division algorithm in complete free semialgebras, is isomorphic to $\mathbb{R}^+$. Then, since $\mathbb{R}_2^+$, as a semialgebra over its subsemialgebra $B = \{0, 1\}$, is isomorphic to the direct sum $\bigoplus\limits_{i \in \mathbb{Z}} B_i$ where $B_i \cong B$ for every i, that is, $\mathbb{R}_2^+ \cong \bigoplus\limits_{i \in \mathbb{Z}} B_i$, and since the cardinal number of the indexing set $\mathbb{Z}$ for the direct sum is equal to the cardinal number of $\mathbb{N}$, $\aleph_0$, the cardinal number $|\bigoplus\limits_{i \in \mathbb{N}} B_i|$ of the set $\bigoplus\limits_{i \in \mathbb{N}} B_i$, indexed by $\mathbb{N}$, is equal to the cardinal number $|\bigoplus\limits_{i \in \mathbb{Z}} B_i|$ of $\bigoplus\limits_{i \in \mathbb{Z}} B_i$. Thereby, since $\bigoplus\limits_{i \in \mathbb{N}} B_i = \bigcap\limits_{k=0}^{\infty} \bigcup\limits_{j=0}^{k} \prod\limits_{i=j}^{\infty} B_i$ and the sets $\mathbb{R}$ and $\mathbb{R}^+$ are equipotent as well as the indexing sets $\{i \in \mathbb{N} \colon i \geq j\}$ for every $\prod\limits_{i=j}^{\infty} B_i$ for all $j \in \mathbb{N}$,

$$\begin{aligned} |\mathbb{R}| = |\mathbb{R}^+| = \left|\mathbb{R}_2^+\right| = |\bigoplus_{i \in \mathbb{Z}} B_i| = |\bigoplus_{i \in \mathbb{N}} B_i| = |\bigcap_{k=0}^{\infty} \bigcup_{j=0}^{k} \prod_{i=j}^{\infty} B_i| \\ = \inf_{k \in \mathbb{N}} \sup_{j \leq k} |\prod_{i=j}^{\infty} B_i| = \inf_{k \in \mathbb{N}} \sup_{j \leq k} 2^{|\omega|} = \lim_{k \to \infty} \inf 2^{\aleph_0} = 2^{\aleph_0}, \end{aligned}$$

therefore, by the Cantor-Bernstein-Schröder theorem, there exist a bijection between $\mathbb{R}$ and the power set of $\mathbb{N}$.

Thus, by the Zermelo-Fraenkel axioms, the Choice axiom, the Peano axioms, the Cantor-Bernstein-Schröder theorem, and the definitions of cardinal number and ordinal number, this theorem in particular proves that there is no cardinal number between the initial transfinite cardinal number $\aleph_0$ and $2^{\aleph_0}$. In general, we have

Corollary "Generalized Continuum Hypothesis": For every transfinite cardinal number α, there is no cardinal number between α and 2^{α}.

Proof: The class of transfinite cardinal numbers is the class of infinite initial ordinal numbers because, by definition of transfinite cardinal number, every transfinite cardinal number is an infinite initial ordinal number, and every infinite initial ordinal number is a transfinite cardinal number. As the class of infinite initial ordinal numbers, the class of transfinite cardinal numbers is well-ordered, has $\aleph_0$ as its first element and each of its elements, by definition of initial ordinal number, has an immediate predecessor, that is, the well-ordered class of transfinite cardinal numbers is isomorphic to ω. Now applying the transfinite induction principle for well-ordered classes isomorphic to ω on the well-ordered subclass of transfinite cardinal numbers α for which there exist no cardinal number β such that

$$\alpha < \beta < 2^{\alpha},$$

we have, by the theorem, that the first element of the well-ordered class of the transfinite cardinal numbers, $\aleph_0$, is in this subclass. And for every transfinite cardinal number α, if its immediate predecessor β is such that there exist no transfinite cardinal number between β and 2^{β}, that is, if 2^{β} is the immediate successor of β, then, since the immediate successor of the immediate predecessor of an element of a well-ordered class is the same element, $\alpha = 2^{\beta}$, thereby, since there exist no cardinal number between 2^{β} and $2^{2^{\beta}}$, by the induction hypothesis and because the exponentiation is a transfinite cardinal order-preserving function, there exist no cardinal number between α and 2^{α}, so that α also belongs to this subclass. Therefore this subclass is the class of transfinite cardinal numbers, that is, for every transfinite cardinal α, there exist no cardinal number between α and 2^{α}.

As a consequence, the inaccessible cardinals distinct than $\aleph_0$ are only 2^{α} for any transfinite cardinal number α.

A theorem of universal algebra

Thus not only does this theorem prove that the class of transfinite cardinal numbers is a large category which forms a countable complete strict semiring, with arrows, the polynomial maps and the exponential maps, which is an e-algebra by the action of the covariant exponential functor semiring e, itself, a functor e-algebra, but also, by definition of infinite initial ordinal number, that the $\mathcal{P}$-algebra of infinite initial ordinal numbers, where $\mathcal{P}$ is the power set functor, is isomorphic to the complete strict e-algebra of transfinite cardinals.

The theorem in categorical logic

In categorical logic, the theorem proves that all first order theories are infinite linear orders that have transfinite cardinal number $\alpha \leq 2^{\aleph_0}$, and also therefore, that all higuer order theories are continuums for they are partial orders which are countably infinite products of first order theories.

The theorem in topos theory

In topos theory, this theorem proves that the category **Card** of (small) cardinal numbers is a topos, and that its topos of sheaves is the category $\mathbf{Sets}^{\mathbf{Card}^*}$, denoting the dual category of the category **Card** by $\mathbf{Card}^*$, of the set-valued contravariant functors on **Card** to **Sets** which assign to every cardinal number β its set of cardinal functions on β all of which turn out to be the cardinal continuous functions on the topology of the cardinal numbers.

Bibliography

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