The Continuum Hypothesis

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In this paper we give a proof of the Continuum Hypothesis.

Theorem "Continuum Hypothesis": There exist a bijection between the set of real numbers numbers $\mathbb{R}$ and the power set of $\mathbb{N}$.

Proof: Let $\mathbb{R}^+ \subset \mathbb{R}$ be the complete free semialgebra of nonnegative numbers, and let $\mathbb{R}_2^+$ be the complete free semialgebra of dyadic nonnegative numbers, which, by the division algorithm in complete free semialgebras, is isomorphic to $\mathbb{R}^+$. Then, since $\mathbb{R}_2^+$, as a semialgebra over its subsemialgebra $B = \{0, 1\}$, is isomorphic to the direct sum $\underset{i\in\mathbb{Z}}{\oplus} B_i$ where $B_i \cong B$ for every i, that is, $\mathbb{R}_2^+ \cong \underset{i\in\mathbb{Z}}{\oplus} B_i$, and since the cardinal number of the indexing set $\mathbb{Z}$ for the direct sum is equal to the cardinal number of $\mathbb{N}$, $\aleph_0$, the cardinal number $|\underset{i\in\mathbb{N}}{\oplus} B_i|$ of the set $\underset{i\in\mathbb{N}}{\oplus} B_i$, indexed by $\mathbb{N}$, is equal to the cardinal number $|\underset{i\in\mathbb{Z}}{\oplus} B_i|$ of $\underset{i\in\mathbb{Z}}{\oplus} B_i$. Thereby, since $\underset{i\in\mathbb{N}}{\oplus} B_i = \bigcap_{k=0}^{\infty} \bigcup_{j=0}^{k} \prod_{i=j}^{\infty} B_i$ and the sets $\mathbb{R}$ and $\mathbb{R}^+$ are equipotent as well as the indexing sets $\{i \in \mathbb{N} \colon i \geq j\}$ for every $\prod_{i=j}^{\infty} B_i$ for all $j \in \mathbb{N}$,

$$|\mathbb{R}| = |\mathbb{R}^+| = \left|\mathbb{R}_2^+\right| = |\underset{i\in\mathbb{Z}}{\oplus} B_i| = |\underset{i\in\mathbb{N}}{\oplus} B_i| = |\bigcap_{k=0}^{\infty} \bigcup_{j=0}^{k} \prod_{i=j}^{\infty} B_i|$$
$$= \inf_{k\in\mathbb{N}} \sup_{j\leq k} |\prod_{i=j}^{\infty} B_i| = \inf_{k\in\mathbb{N}} \sup_{j\leq k} 2^{|\omega|} = \lim_{k\to\infty} \inf 2^{\aleph_0} = 2^{\aleph_0},$$

therefore, by the Cantor-Bernstein-Schröder theorem, there exist a bijection between $\mathbb{R}$ and the power set of $\mathbb{N}$.

Thus, by the Zermelo-Fraenkel axioms, the Choice axiom, the Peano axioms, the Cantor-Bernstein-Schröder theorem, and the definitions of cardinal number and ordinal number, this theorem in particular proves that there is no cardinal number between the initial transfinite cardinal number $\aleph_0$ and $2^{\aleph_0}$. In general, we have

Corollary "Generalized Continuum Hypothesis": For every cardinal number α, there is no cardinal number between α and 2^α.

A theorem of universal algebra

Thus not only does this theorem prove that the class of cardinal numbers is a large category which forms a countable complete strict semiring, with arrows, the polynomial maps and the exponential maps, which is an e-algebra by the action of the covariant exponential functor semiring e, itself, a functor e-algebra, but also, by definition of limit ordinal number, that the $\mathcal{P}$-algebra of limit ordinal numbers, where $\mathcal{P}$ is the power set functor, is isomorphic to the complete strict e-algebra of transfinite cardinals.

The theorem in categorical logic

In categorical logic, as all first order theories are linear orders, this theorem proves that all higuer order theories are continuums since they are preorders which are countable products of linear orders having transfinite cardinal number $\alpha \leq 2^{\aleph_0}$.

The theorem in topos theory

In topos theory, not only does this theorem prove that the category **Card** of (small) cardinal numbers is a topos, but also that the category of set-valued contravariant functors which assign to every function $f\colon \beta \to \alpha$ for every cardinal numbers α and β its set of cardinal-valued functions on β is its topos of sheaves $\mathbf{Sets}^{\mathbf{Card}^*}$ where $\mathbf{Card}^*$ is the dual category of **Card**.