The Continuum Hypothesis

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In this paper we give a proof of the Continuum Hypothesis.

Theorem "Continuum Hypothesis": There exist a bijection between the set of real numbers numbers $\mathbb{R}$ and the power set of $\mathbb{N}$.

Proof: Let $\mathbb{R}^+ \subset \mathbb{R}$ be the complete free semialgebra of nonnegative numbers, and let $\mathbb{R}_2^+$ be the complete free semialgebra of dyadic nonnegative numbers, which, by the division algorithm in complete free semialgebras, is isomorphic to $\mathbb{R}^+$. Then, since $\mathbb{R}_2^+$, as a semialgebra over its subsemialgebra $B = \{0,1\}$, is isomorphic to the direct sum $\underset{i\in\mathbb{Z}}{\oplus} B_i$ where $B_i \cong B$ for every i, that is, $\mathbb{R}_2^+ \cong \underset{i\in\mathbb{Z}}{\oplus} B_i$, and since the cardinal number of the indexing set $\mathbb{Z}$ for the direct sum is equal to the cardinal number of $\mathbb{N}$, $\aleph_0$, the cardinal number $|\underset{i\in\mathbb{N}}{\oplus} B_i|$ of the set $\underset{i\in\mathbb{N}}{\oplus} B_i$, indexed by $\mathbb{N}$, is equal to the cardinal number $|\underset{i\in\mathbb{Z}}{\oplus} B_i|$ of $\underset{i\in\mathbb{Z}}{\oplus} B_i$. Thereby, since $\underset{i\in\mathbb{N}}{\oplus} B_i = \bigcap_{k=0}^{\infty}\bigcup_{j=0}^{k}\prod_{i=j}^{\infty} B_i$ and the sets $\mathbb{R}$ and $\mathbb{R}^+$ are equipotent as well as the indexing sets $\{i \in \mathbb{N}\colon i \geq j\}$ for every $\prod_{i=j}^{\infty} B_i$ for all $j \in \mathbb{N}$,

$$|\mathbb{R}| = |\mathbb{R}^+| = \left|\mathbb{R}_2^+\right| = |\underset{i\in\mathbb{Z}}{\oplus} B_i| = |\underset{i\in\mathbb{N}}{\oplus} B_i| = |\bigcap_{k=0}^{\infty}\bigcup_{j=0}^{k}\prod_{i=j}^{\infty} B_i|$$
$$= \inf_{k\in\mathbb{N}}\sup_{j\leq k}|\prod_{i=j}^{\infty} B_i| = \inf_{k\in\mathbb{N}}\sup_{j\leq k} 2^{|\omega|} = \liminf_{k\to\infty} 2^{\aleph_0} = 2^{\aleph_0},$$

therefore, by the Cantor-Bernstein-Schröder theorem, there exist a bijection between $\mathbb{R}$ and the power set of $\mathbb{N}$.

Thus, by the Zermelo-Fraenkel axioms, the Choice axiom, the Peano axioms, the Cantor-Bernstein-Schröder theorem, and the definitions of cardinal number and ordinal number, this theorem in particular proves that there is no cardinal number between the initial transfinite cardinal number $\aleph_0$ and $2^{\aleph_0}$. In general, we have

Corollary "Generalized Continuum Hypothesis": For every cardinal number α, there is no cardinal number between α and 2^α.

A theorem of universal algebra

Thus not only does this theorem prove that the class of cardinal numbers forms a complete strict semiring category, with arrows, the polynomial maps and the exponential maps, which is an e-algebra by the action of the covariant exponential functor semiring e, itself, a functor e-algebra, but also, by definition of limit ordinal number, that the $\mathcal{P}$-algebra of limit ordinal numbers, where $\mathcal{P}$ is the power set functor, is isomorphic to the countable complete strict e-algebra of transfinite cardinals.