The Continuum Hypothesis

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In this paper we give a proof of the Continuum Hypothesis.

Theorem "Continuum Hypothesis": There exist a bijection between the set of real numbers numbers $\mathbb{R}$ and the power set of $\mathbb{N}$.

Proof: Let $\mathbb{R}^+ \subset \mathbb{R}$ be the complete free semialgebra of nonnegative numbers, and let $\mathbb{R}_2^+$ be the complete free semialgebra of dyadic nonnegative numbers, which, by the division algorithm in complete free semialgebras, is isomorphic to $\mathbb{R}^+$. Then, for $\mathbb{R}_2^+$, as a semialgebra over its subsemialgebra $B = \{0, 1\}$, is isomorphic to a countable direct sum of copies of B, that is, $\mathbb{R}_2^+ \cong \underset{i\in\mathbb{N}}{\oplus} B_i$ where $B_i \cong B$ for every i; since $\underset{i\in\mathbb{N}}{\oplus} B_i = \bigcap_{k=0}^{\infty} \bigcup_{j=0}^{k} \prod_{i=0}^{j} B_i$ and the sets $\mathbb{R}$ and $\mathbb{R}^+$ are equipotent, that is, the cardinal number $|\mathbb{R}|$ of $\mathbb{R}$ and that of $\mathbb{R}^+$, $|\mathbb{R}^+|$, are equal,

$$
\begin{aligned}
|\mathbb{R}| = |\mathbb{R}^+| = \left|\mathbb{R}_2^+\right| = |\underset{i\in\mathbb{N}}{\oplus} B_i| = |\lim_{n\to\infty} \bigcap_{k=0}^{n} \bigcup_{j=0}^{k} \prod_{i=0}^{j} B_i| \\
= \inf_{k\in\mathbb{N}} \sup_{j<k} |\prod_{i=0}^{j} B_i| = \inf_{k\in\mathbb{N}} \sup_{j<k} 2^{|j|} = \lim_{k\to\infty} \inf 2^{|k|} = 2^{\aleph_0},
\end{aligned}
$$

therefore, by the Cantor-Bernstein-Schröder theorem, there exist a bijection between $\mathbb{R}$ and the power set of $\mathbb{N}$.

Thereby, by the ZFC axioms and definition of cardinal number, this theorem proves that there is no cardinal number between the initial transfinite cardinal number $\aleph_0$ and $2^{\aleph_0}$. In general, we have

Corollary "Generalized Continuum Hypothesis": For every cardinal number α, there is no cardinal number between α and 2^{α}.

A theorem of universal algebra

Thus this theorem not only proves that the class of cardinal numbers forms a complete strict semiring category, with arrows, the polynomial maps and the exponential maps, which is an e-algebra by the action of the covariant exponential functor semiring e, itself, a functor e-algebra, but also, by definition of ordinal number, that the $\mathcal{P}$-algebra of limit ordinal numbers, where $\mathcal{P}$ is the power set functor, is isomorphic to the countable complete strict e-algebra of transfinite cardinals.