The Continuum Hypothesis

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In this paper we give a proof of the Continuum Hypothesis

Lemma "Continuum Hypothesis": There exist a bijection between the set of real numbers numbers $\mathbb{R}$ and the power set of $\mathbb{N}$.

Proof: Let $\mathbb{R}^+ \subset \mathbb{R}$ be the complete free $\mathbb{N}$-semialgebra of nonnegative numbers, and let $\mathbb{R}_2^+$ be the $\mathbb{N}$-semialgebra of dyadic nonnegative numbers isomorphic to $\mathbb{R}^+$, then, for $\mathbb{R}_2^+$ is a complete free $\mathbb{N}$-semialgebra with its multiplicative cyclic subgroup $\langle 2 \rangle$ as basis, it is isomorphic to the cartesian product $\prod\limits_{i \in \mathbb{N}} B_i$ where $B_i = \{0, 1\}$ for every i, then, since the sets $\mathbb{R}$ and $\mathbb{R}^+$ are equipotent, that is, the cardinal number $|\mathbb{R}|$ of $\mathbb{R}$ and that of $\mathbb{R}^+$, $|\mathbb{R}^+|$, are equal,

$$|\mathbb{R}| = |\mathbb{R}^+| = \left|\mathbb{R}_2^+\right| = |\prod_{i \in \mathbb{N}} B_i| = 2^{\aleph_0},$$

therefore, by the Cantor-Bernstein-Schröder theorem, there exist a bijection between $\mathbb{R}$ and the power set of $\mathbb{N}$.

Thereby, by the ZFC axioms and definition of cardinal number, this lemma proves that there is no cardinal number between the initial transfinite cardinal number $\aleph_0$ and $2^{\aleph_0}$. In general, we have

Corollary "Generalized Continuum Hypothesis": For every cardinal number α, there is no cardinal number between α and 2^{α}.

Thus the set of cardinal numbers is countable and forms a category which is a monoid with arrows the exponential maps.