ON THE EVALUATION OF CERTAIN ARITHMETICAL FUNCTIONS OF NUMBER THEORY AND THEIR SUMS

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ABSTRACT: In this paper we present a method to get the prime counting function π(x) and other arithmetical functions than can be generated by a Dirichlet series, first we use the general variational method to derive the solution for a Fredholm Integral equation of first kind with symmetric Kernel K(x,y)=K(y,x), after that we find another integral equations with Kernels K(s,t)=K(t,s) for the Prime counting function and other arithmetical functions generated by Dirichlet series, then we could find a solution for π(x) and $\sum_{n\le x} a(n) = A(x)$, solving $\delta J[\phi] = 0$ for a given functional J, so the problem of finding a formula for the density of primes on the interval [2,x], or the calculation of the coefficients for a given arithmetical function a(n), can be viewed as some "Optimization" problems that can be attacked by either iterative or Numerical methods (as an example we introduce Rayleigh-Ritz and Newton methods with a brief description)

Also we have introduced some conjectures about the asymptotic behavior of the series $\Xi_n(x) = \sum_{p\le x} p^n$ =S_n(x) for n>0 , and a new expression for the Prime counting function in terms of the Non-trivial zeros of Riemann Zeta and its connection to Riemman Hypothesis and operator theory.

- *Keywords:* =PNT (prime number theorem), Variational Calculus, Maxima and minima, Integral Transforms.

1. VARIATIONAL METHODS IN NUMBER THEORY:

It was Euler, in the problem of "Brachistochrone" (shortest time) or curve of fastest descent, who introduced the preliminaries of what later would be called, "Calculus of Variations", he solved the problem minimizing the integral below , where "t" is the time employed by the particle to go from (0,0) to another point on the plane (x, y) $x \neq 0$ below (0,0), using Newtonian mechanics he found the expression:

$$\int_a^b dt\sqrt{\frac{1+(\dot{y})^2}{y}}=t$$ (for constant gravity g=9.8 m/s^2 and ignoring friction)

(1.1)

Then he got that the minimum of the integral above was a differential equation describing a cycloid, later Lagrange used this Calculus of Variations to describe the mechanics of a particle introducing the Lagrangian and the action functional S, whose extremum gave precisely the equations of motion for the particles:

$$\frac{\delta S}{\delta q}=0=\frac{d}{dt}\left(\frac{\partial L}{d\dot{q}_j}\right)-\frac{\partial L}{\partial q_j} \qquad S=\int_a^b dtL(q_j,\dot{q}_j,t) \qquad \frac{dq_j}{dt}=\dot{q}_j \quad (1.2)$$

The equations above are the Euler-Lagrange equations of motion for the system defined by the Lagrangian L, this formulation is equivalent to Newton second law, although its use is more extended, as you only need to know the Kinetic part of the particle (usually of the form $T=\frac{1}{2}\sum_{i,j}^{N}Q^{ij}\dot{q}_i\dot{q}_j$) and a potential V related to the force $F=-\nabla(V)$ is the force of the system , Here Q is an Hermitian Matrix.

Now we can ask ourselves if we can generate a similar Variational principle for number theory, for the case of a Fredholm equation of second kind, with K(x,y)=K(y,x) we have:

$$J[f]=a\int_a^b dxf^2(x)+\int_a^b g(x)f(x)dx-\int_a^b dx\int_a^b dyK(x,y)f(x)f(y) \quad (1.2)$$

Taking the functional derivative.

$$\frac{\delta J[f]}{\delta f(x)}=0=af(x)+g(x)-\int_a^b dyK(x,y)f(y) \qquad (1.3)$$

$$a\in ¡ \qquad \frac{\delta F}{\delta\phi(x)}=\lim_{\varepsilon\to 0}\frac{F[\phi+\varepsilon\delta(x-y)]-F[\phi]}{\varepsilon} \qquad (1.4)$$

So we have derived for this especial case the Euler-Lagrange equation for this Integral equation, for the cases of the prime counting function and an arithmetical function that can be generated via Dirichlet series of the form:

$$g(s)=\sum_{n=1}^{\infty}a(n)n^{-s} \qquad A(x)=\sum_{x\geq n>1}a(n) \qquad (1.5)$$

We can give the 2 Functional, so their maximum or minimum are precisely some integral equations defining these arithmetical functions: (a=0)

$$J_0[E]=\int_0^\infty \frac{\ln\zeta(2s)}{2s}E(s)dx-\int_0^\infty ds\int_0^\infty dt(e^{2st}-1)^{-1}E(s)E(t) \qquad E(t)=E_0\pi(e^t) \quad (1.6)$$

$$J_1[\phi]=\int_0^\infty \frac{g(s+c)}{s+c}\phi(s)dx-\int_0^\infty ds\int_0^\infty dte^{-st}\phi(s)\phi(t) \qquad \phi(t)=A_0A(e^t)e^{-ct} \quad (1.7)$$

Minimizing these Functional $\delta J_{0,1}=0$ with respect to E(s) and ϕ(s) and making the change of variable t=ln(x) we get the usual integral equation for π(x) and A(x) namely:

$$g(s+c)=(s+c)\int_0^\infty dx\frac{A(x)}{x^{s+1+c}} \qquad \frac{\ln\zeta(2s)}{2s}=\int_2^\infty dx\frac{\pi(x)}{x(x^{2s}-1)} \quad , \quad \prod_p(1-p^{-2s})=\frac{1}{\zeta(2s)} \quad (1.8)$$

With c>0 and s>1/2 (the election of g(s+c) and 2s inside the second integral is to avoid the singularities inside the integrals due to a pole of the Riemann zeta function at s=1), this allows us to study the prime counting function and other arithmetical functions by using the Optimization techniques, including some iterative methods (gradient-descent, Newton method…) to calculate their "shape", as an example of these iterative methods we can get the Maximum or Minimum for a given Functional in the form:

$$\phi_{n+1}=\phi_n-\frac{\delta J[\phi_n]}{\delta^2 J[\phi_n]} \qquad \|\phi_{n+1}-\phi_n\|<\varepsilon \quad n\to\infty \quad (1) \quad J=J_0\,,\,J_1 \qquad (1.9)$$

So we "stop" the process when the condition on the right (1.9) is fulfilled, of course we need an initial "guess" function, for the exponential prime counting function with J_0 we can use the exponential integral E(t)= $E_i(x)=-\int_{-x}^\infty dt\frac{e^{-t}}{t}$ or simply E(t)=exp(t)/t ,using some upper and lower bounds for π(x), for our arithmetical functions if their sum satisfies that A(x)=O(v(x)), for a known function v(x) we could use our trial function in the form, $v(e^t)e^{-ct}$ and take into account that A(n)-A(n-1)=a(n)

The use of an appropriate trial function to begin with the iterative process is not casual, as the Newton iterative method could fail and diverge for certain trial functions, other inconveniences are that to compute Newton method we should need to know the value of the first and second functional derivative, which is not always available or easy to obtain, other optimization methods although slower in convergence are more secure, generally we will use a Gradient method for infinite-dimensional space to get an initial and convergent set of trial functions and then, to improve the convergence, we will use a Newton-like method.

For the case that our arithmetical function is generated by a power series:

$G(x)=\sum_{n=0}^\infty x^n p(n)$ |x|<R (Radius of convergence) Where $B(t)=\sum_{n\le t}p(n)$

We can use the functional J_1 with $\phi(t)=B(t)$ with c=0 and $G(e^{-s})=g(s)$, to get the values of p(n) and B(t), by obtaining the maximum or minimum of this J.

Another useful method is if we choose our trial function to be of the form $\phi(t) \approx \sum_{n=0}^{N} C_n W_n(t)$ Where the functions $W_n(t)$ are known, so our Functional J becomes

$J_{0,1} = J(C_0, C_1, C_2, ..., C_N)$ And to obtain the C_n we need to solve the system

$\frac{\partial J}{\partial C_n} = 0 \quad n=0,1,2,3,.......,N$ With $N \to \infty$ (Rayleigh-Ritz method)

For the study of Prime counting function $\pi(e^t)$ when $t \to \infty$, we could choose $W_n(t) = E_i(\frac{t}{n})$, n>0 so we can introduce some correction to the PNT $\pi(e^t) \to E_i(t) = Li(e^t)$ for big t, or simply if we use the trial function $aE_i(bt)$ and take $\frac{\partial J}{\partial a} = 0 = \frac{\partial J}{\partial b}$,we calculate the Hessian symmetric Matrix with elements: $\frac{\partial^2 J}{\partial C_i \partial C_j} = H_{i,j}$ We will have a Maximum or a Minimum depending on if H is negative or positive definite.

2 .ON THE ASYMPTOTIC BEHAVIOR OF THE SUMS OVER PRIMES, BEYOND THE PRIME NUMBER THEOREM:

Now that we have given a method to obtain $\pi(x)$ by Variational methods, after that we could study every sum over primes:

$$\pi'(x) = \frac{d\pi(x)}{dx} = \sum_p \delta(x-p) \qquad \sum_{p \le x} f(p) = \int_2^x dt \pi'(t) f(t) \quad \text{p=sum over primes.} \quad (2.1)$$

Where we have introduced the Dirac delta function with definition and properties:

$$\int_a^b dx f(x)\delta(x-a) = f(a) \qquad 2\pi\delta(x-a) = \int_{-\infty}^{\infty} d\omega e^{\pm i\omega(x-a)} \qquad (2.2)$$

Now we would be interested in some asymptotic behavior for the cases $f(t)=t^n$ as a conjecture, we have that the probability of a random integer number being prime is 1/ln(x) then we have the approximate (asymptotic) relationships:

$$\Xi_n(x) = \sum_{p \le x} p^n \to \sum_{k=2}^{x} \frac{k^n}{\ln(k)} \to \int_2^x \frac{t^n dt}{\ln(t)} = Li(x^{n+1}) \qquad \text{n=0, n>0} \qquad (2.3)$$

$$Li(x^{n+1}) = \ln(\ln(x)) + \sum_{k=1}^{\infty}(n+1)^k \frac{\ln^k(x)}{m.m!} + \gamma \quad (2.4)$$

The last identity is found on tables for indefinite integrals, with γ the Euler-Mascheroni constant, for n=0 we get the Prime Number Theorem (PNT) $\pi(x) \to x/\ln(x)$, for n=1 we get the asymptotic relation:

$\Sigma(n) = \sum_{i=1}^{n} p_i = \Xi_1(p_n)$ Setting $p_n \to n\ln(n)$ (3) $Li(x^2) \to \frac{x^2}{2\ln(x)}$ $\quad Li(x) = \int_2^x \frac{dt}{\ln(t)}$

(3) is the asymptotic expression for the n-th prime, we use the European convention so Li(2)=0, expanding the Li(x^2) , keeping only the first term and using the prime number theorem to give an expression for n-th prime, we get : $\Sigma(n) \to \frac{n^2\ln(n)}{2}$

For the case n=-1 the terms inside the sum over k cancel, so we get that the Harmonic prime series (as shown before, by Euler and others) diverges in the form ln(ln(x)) as x→∞ , Or if our conjecture is valid, we can study the growth-rate of the series :

$$\Xi_n(x) = \sum_{p \le x} p^n \to \pi(x^{n+1}) \to Li(x^{n+1}) \qquad \sum_{p \le x} p^n = \int_2^x dt\pi'(t)t^n \to \pi(x^{n+1}) \qquad (2.5)$$

Differentiating both sides: $\frac{\pi'(x)}{\pi'(x^{n+1})} \to (n+1)$ With $\pi'(x) = \frac{d\pi}{dx} \to \frac{1}{\ln(x)}$ (PNT)

The last is how the derivative of the prime counting function behaves for big x, using the relation (2.4) with an function F(x) that is analytic near x=0 and has the limit $\lim_{x\to\infty} F(x) = \infty$,then the asymptotic expression for the sum :

$$\sum_{p \le x} F(p) \to \sum_{n=0}^{\infty} c(n) Li(x^{n+1}) \qquad c(n) = \frac{d^n F}{dx^n}(0)\frac{1}{n!} \qquad (2.6)$$

Where we have used the asymptotic notation $f(x) \to g(x)$ meaning that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$

- *Sum over primes:*

For other convergent prime sums, first we use the identity:

$$\sum_{n=0}^{\infty}(-1)^n[\pi(n) - \pi(n-1) + 1]f(n) = 2f(2) - \sum_{p} f(x) + \sum_{n=0}^{\infty}(-1)^n f(n) \quad (2.7)$$

To obtain a relation between a prime series and an alternating series, valid when both, the alternating series and the sum over all primes for f(x) converge, the main purpose of this is that we can accelerate the convergence of the prime series using only a few values of $\pi(n)$ n<100, as an example putting f(x)=x^{-s} .

$$\sum_{n=0}^{\infty}(-1)^n[\pi(n)-\pi(n-1)+1]n^{-s} = 2^{1-s} - \sum_p p^{-s} + \eta(s) \qquad (2.8)$$

Where we have introduced the Prime Zeta function $P(n) = \sum_p p^{-s}$

$\eta(s) = (1-2^{1-s})\zeta(s)$ Dirichlet Zeta function , The Euler (forward) transform for alternating series is given by the expression:

$$\sum_{n=0}^{\infty}(-1)^n f(n) \approx \sum_{n=0}^{\infty}(-1)^n \frac{\Delta^n f(0)}{2^{n+1}} \qquad \Delta f(n) = f(n+1) - f(n) \qquad (2.9)$$

Euler method allows you to accelerate the convergence of an alternating series like (5) with general terms $(-1)^n[\pi(n)-\pi(n-1)+1]f(n)$ and $(-1)^n f(n)$ providing that f(n)>f(n+1) , if Euler transform is not good or converges worse than the initial series, we could use the backward Euler transform with the Backward difference operator $\nabla f(n) = f(n) - f(n-1)$,the main purpose of using Euler transform is to be able to compute the expression $2f(2) - \sum_p f(x) + \sum_{n=0}^{\infty}(-1)^n f(n)$. For a given f with an small error, knowing only a few values of the Prime counting function, so we can extract the behavior or approximate the sum for the series over all the primes, calculating the sum of the alternating series with general term $(-1)^n f(n)$, which is in general, easier to calculate, this expression to evaluate prime sums can also be applied to products.

Even if the sum is divergent , we could use the functional equation for Riemann function relating $\zeta(s) \quad \rightarrow \quad \zeta(1-s)$ so using the definition of the Prime zeta function in terms of the Möbius formula and Riemann Zeta :

$$\sum_p p^s (-1)^r \ln^r(p) = \sum_{n=1}^{\infty}\mu(n)\frac{d^{r-1}}{ds^{r-1}}\left(\frac{\zeta'(-ks)}{\zeta(-ks)}\right) \qquad ks \neq 2,4,6,... \quad s > 0 \quad (2.10)$$

and r being a positive integer bigger than 1 , we can consider the infinite sum on the right being the regularized value of the divergent prime sum on the left

From equation (1.8) we can find a Lambert series representation for the prime counting function, introducing the notation $E(n) = \pi(e^n)$, and s=-lnx we can find the Lambert series for Prime number counting function.

$$\sum_{n=1}^{\infty}(E(n)-E(n-1))n\frac{x^n}{1-x^n}=-x\frac{\partial}{\partial x}\left(\frac{\zeta(-\ln x)}{1}\right)=\frac{\zeta'(-\ln x)}{\zeta(-\ln x)} \qquad (2.11)$$

In this method we approximate the sum over all the primes $\sum_{p} f(p)$ with the series (Euler transformation of alternating series)

$$2f(2)-\sum_{n=0}^{\infty}(-1)^n f(n)-\sum_{n=0}^{\infty}(-1)^n\frac{\Delta^n}{2^{n+1}}b(0) \qquad b(n)=\left(\pi(n)-\pi(n-1)\right)f(n) \qquad (2.12)$$

So expression (2.12) converges faster thant the sum $\sum_{p} f(p)$

3.SUMS OVER ARITHMETICAL FUNCTIONS:

In many important cases all the arithmetical functions which are useful in number theory has a Dirichlet series in the form $\sum_{n=1}^{\infty}\frac{a(n)}{n^s}=G(s)$, Where G(s) includes powers or quotients of the Riemann zeta function for example

$$\frac{1}{\zeta(s)}=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} \qquad -\frac{\zeta'(s)}{\zeta(s)}=\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s} \qquad \frac{\zeta(2s)}{\zeta(s)}=\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s} \qquad (3.1)$$

$$\frac{\zeta(s)}{\zeta(2s)}=\sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^s} \qquad \frac{\zeta(s-1)}{\zeta(s)}=\sum_{n=1}^{\infty}\frac{\varphi(n)}{n^s} \qquad \frac{\zeta(s)\zeta(s-1)\zeta(s-2)}{\zeta(2s-2)}=\sum_{n=1}^{\infty}\frac{\sigma(n^2)}{n^s} \qquad (3.2)$$

The definition of the functions inside (3.1) and (3.2) is as follows

- The Möbius function, $\mu(n)=1$ if the number 'n' is square-free (not divisible by an square) with an even number of prime factors , $\mu(n)=0$ if n is not squarefree and   if the number 'n' is square-free with an odd number of prime factors.
- The Von Mangoldt function $\Lambda(n)=\log p$, in case 'n' is a prime or a prime power and takes the value 0 otherwise
- The Liouville function $\lambda(n)=(-1)^{\Omega(n)}$ $\Omega(n)$ is the number of prime factors of the number 'n'
- $|\mu(n)|$ is 1 if the number is square-free and 0 otherwise
- $\varphi(n)=n\prod_{p|n}\left(1-\frac{1}{p}\right)$, the meaning of $p\,|\,n$ is that the product is taken only over the primes p that divide 'n'.
- $\sigma(n)$ is the divisor function, it counts the number of divisors of 'n'

To obtain the coefficients of the Dirichlet series we can use the Perron formula

$$\sum_{n=1}^{\infty}\frac{a(n)}{n^s}=G(s)=s\int_{1}^{\infty}\frac{A(x)}{x^{s+1}} \qquad A(x)=\sum_{n\le x}a(n)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{s}G(s)ds \qquad (3.3)$$

If the function G(s) includes powers and quotients of the Riemann zeta function we can use Cauchy's theorem to obtain the explicit formulae for example

$$M(x)=\sum_{n\le x}\mu(n)=-2+2\Re e\left(\sum_{k=1}^{N}\frac{x^{\rho_k}}{\rho_k\zeta'(\rho_k)}\right)+\sum_{n=1}^{\infty}\frac{x^{-2n}}{\zeta'(-2n)(-2n)} \quad (3.4)$$

$$\Psi(x)=\sum_{n\le x}\Lambda(n)=x-\frac{\zeta'(0)}{\zeta(0)}-2\Re e\left(\sum_{k=1}^{N}\frac{x^{\rho_k}}{\rho_k}\right)+\sum_{n=1}^{\infty}\frac{x^{-2n}}{(-2n)} \quad (3.5)$$

$$L(x)=\sum_{n\le x}\lambda(n)=1+\frac{\sqrt{x}}{\zeta(1/2)}+2\Re e\left(\sum_{k=1}^{N}\frac{x^{\rho_k}\zeta(2\rho_k)}{\rho_k\zeta'(\rho_k)}\right) \quad (3.6)$$

$$Q(x)=\sum_{n\le x}|\mu(n)|=1+\frac{6x}{\pi^2}+2\Re e\left(\sum_{k=1}^{N}\frac{x^{\frac{\rho_k}{2}}\zeta\left(\frac{\rho_k}{2}\right)}{\rho_k\zeta'(\rho_k)}\right)+\sum_{n=1}^{\infty}\frac{x^{-n}\zeta(-k)}{(-2n)\zeta'(-2k)} \quad (3.7)$$

$$\Phi(x)=\sum_{n\le x}\varphi(n)=\frac{1}{6}+\frac{3x^2}{\pi^2}+2\Re e\left(\sum_{k=1}^{N}\frac{x^{\rho_k}\zeta(\rho_k-1)}{\rho_k\zeta'(\rho_k)}\right)+\sum_{n=1}^{\infty}\frac{x^{-2n}\zeta(-2k-1)}{(-2n)\zeta'(-2k)} \quad (3.8)$$

$$T(x)=\sum_{n\le x}\sigma(n^2)=\frac{1}{48}-\frac{x}{12}-\frac{x^2}{4}+\frac{5\zeta(3)x^3}{\pi^2}+2\Re e\left(\sum_{k=1}^{N}\frac{x^{\frac{\rho_k}{2}-1}\zeta\left(\frac{\rho_k}{2}+1\right)\zeta\left(\frac{\rho_k}{2}-1\right)}{(\rho_k+2)\zeta'(\rho_k)}\zeta\left(\frac{\rho_k}{2}\right)\right) \quad (3.9)$$

In all cases N must be taken in the limit $N\to\infty$, also we have for the Riemann zeta function

$$\zeta'(-2n)=\frac{(-1)^n\zeta(2n+1)(2n)!}{2^{2n+1}\pi^{2n}} \qquad \zeta'(0)=-\frac{1}{2}\log(2\pi) \qquad \zeta(0)=-\frac{1}{2} \quad (3.10)$$

The sum of an arithmetical function $A(x)=\sum_{n\le x}a(n)$ is an step function, therefore its derivative in distributional sense must satisfy $e^{-bx}\frac{dA(x)}{dx}=\sum_{n=1}^{\infty}\frac{a(n)}{n^b}\delta(x-\log n)$, so if we take the derivative with respect to 'x' and make a change of variable $x=e^t$ inside every formulae (3.4) to (3.9).

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{\sqrt{n}}g(\log n)=2\Re e\left(\sum_{k=1}^{N}\frac{h(\gamma_k)}{\zeta'(\rho_k)}\right)+\sum_{n=1}^{\infty}\frac{1}{\zeta'(-2n)}\int_{-\infty}^{\infty}dxg(x)e^{-x\left(2n+\frac{1}{2}\right)} \quad (3.11)$$

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{\sqrt{n}}g(\log n)=\frac{1}{2\zeta(1/2)}\int_{-\infty}^{\infty}dxg(x)+2\Re e\left(\sum_{k=1}^{N}\frac{\zeta'(2\rho_k)h(\gamma_k)}{\zeta'(\rho_k)}\right) \quad (3.12)$$

$$\sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^{\frac{1}{4}}}g(\log n)=\frac{6}{\pi^2}\int_{-\infty}^{\infty}dxg(x)e^{\frac{3}{4}x}+\Re e\left(\sum_{k=1}^{N}\frac{h\left(\frac{\gamma_k}{2}\right)}{\zeta'\left(\rho_k\right)}\zeta\left(\frac{\rho_k}{2}\right)\right)+\frac{1}{2}\sum_{n=1}^{\infty}\frac{\zeta'(-n)}{\zeta'(-2n)}\int_{-\infty}^{\infty}dxg(x)e^{-x\left(n+\frac{1}{4}\right)} \tag{3.13}$$

$$\sum_{n=1}^{\infty}\frac{\varphi(n)}{\sqrt{n}}g(\log n)=\frac{6}{\pi^2}\int_{-\infty}^{\infty}dxg(x)e^{\frac{3}{2}x}+2\Re e\left(\sum_{k=1}^{N}\frac{h\left(\gamma_k\right)}{\zeta'\left(\rho_k\right)}\zeta\left(\rho_k-1\right)\right)+\sum_{n=1}^{\infty}\frac{\zeta'(-2n-1)}{\zeta'(-2n)}\int_{-\infty}^{\infty}dxg(x)e^{-x\left(2n+\frac{1}{2}\right)} \tag{3.14}$$

$$\sum_{n=1}^{\infty}\frac{\sigma(n^2)}{n^{\frac{5}{4}}}g(\log n)=-\frac{1}{12}\int_{-\infty}^{\infty}g(x)e^{\frac{-x}{4}}-\frac{1}{2}\int_{-\infty}^{\infty}g(x)e^{\frac{3x}{4}}+\frac{15}{\pi^2\zeta(3)}\int_{-\infty}^{\infty}g(x)e^{\frac{7x}{4}}+$$
$$2\Re e\left(\sum_{k=1}^{N}\frac{h\left(\frac{\gamma_k}{2}\right)}{\zeta'\left(\rho_k\right)}\zeta\left(\frac{\rho_k}{2}\right)\zeta\left(\frac{\rho_k}{2}-1\right)\zeta\left(\frac{\rho_k}{2}+1\right)\left(\frac{\frac{\rho_k}{2}+1}{\rho k+2}\right)\right) \tag{3.15}$$

Where $g(x)$ is an smooth test function , so it has a Fourier transform and the integral $H(c)=\int_{-\infty}^{\infty}dxg(x)e^{-cx}$ exists and is finite for every real number (positive or negative) 'c', and $g(\alpha)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dxh(x)e^{i\alpha x}$ or $g(\alpha)=\frac{1}{\pi}\int_{0}^{\infty}dxh(x)\cos(\alpha x)$ depending on if the test function are even or not $h(x)=h(-x)$.

The sum over the Riemann zeros must be done as follows, we take the sum in pairs of zeros $\rho_k(\gamma_k)=\frac{1}{2}+i\gamma_k$ and $\rho_k(\gamma_k)=\frac{1}{2}-i\gamma_k$, also for the imaginary part of the zeros , they must be summed in pairs $i\gamma_k$ and $-i\gamma_k$ to avoid problems of convergence, that is the meaning of the expression $\Re e$ inside each of the formulae (3.12) to (3.15).

These formulae are very similar to the Riemann-Weil explicit formula for the Chebyshev and Von Mangoldt functions $\Psi(x)=\sum_{n\le x}\Lambda(n)$

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}g(\log n)=\frac{1}{4\pi}\int_{-\infty}^{\infty}dr\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{ir}{2}\right)+h\left(\frac{i}{2}\right)-\frac{h(0)}{2}\log\pi-\sum_{k=0}^{\infty}h\left(\gamma_k\right) \tag{3.16}$$

4.EXPRESSION FOR π(x) AS A SUM OVER ZEROS:

If we use the expressions of the Prime Counting function via Mellin transform:

$$P(s) = s\int_0^{\infty} dx\pi(x)x^{-s-1} \quad \to \pi(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\frac{P(s)}{s} \qquad (4.1)$$

Where P(s) is the "Prime zeta function" (sum over the inverse powers of primes) that satisfies:

$$\ln\zeta(s) = \sum_{n=1}^{\infty}\frac{P(ns)}{n} \qquad \to \quad P(s) = \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\ln\zeta(sn) \qquad (4.2)$$

Where we have introduced the Mertens and Möbius functions related by the sum:

$M(x) = \sum_{n\le x}\mu(n)$. Now using the Abel sum-formula with the Möbius function:

$$\sum_{n=1}^{\infty}\mu(n)f(n) = \int_{-\infty}^{\infty} dxM'(x)f(x) \quad \text{and} \quad M_0'(x) = M'(x) = \frac{dM}{dx} \qquad (4.3)$$

$$M_0(x) = \sum_{\rho}\frac{x^{\rho_k}}{\zeta'(\rho_k)\rho_k} - 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n)!\zeta(2n+1)n}\left(\frac{2\pi}{x}\right)^{2n} \begin{cases} M(x) - \frac{1}{2}\mu(x) & \text{iff x} \in Z^{+} \\ M(x) & \text{Otherwise} \end{cases} \qquad (4.4)$$

In case that there are no multiple Non-trivial roots of the Riemann Zeta function so

$$|\zeta'(\rho_k)| \neq 0\,, \quad \zeta(s) = \frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}(-1)^n n^{-s} \;\; \Re\text{e}(\text{ s}) > 0 \;\; , \zeta(1) = \infty \qquad (4.4)$$

If we combine expressions (4.1-2) involving P(s) and perform the Mellin transform, using the next identity obtained by Riemann:

$$\frac{n}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\frac{\ln\zeta(ns)}{ns}x^s = Li(x^{1/n}) - \sum_{\rho}Li(x^{\rho/n}) - \ln 2 + \int_{x^{1/n}}^{\infty} dt\frac{1}{t(t^2-1)\ln t} \qquad (4.5)$$

For n=1 we obtain the "Riemann Prime counting function"

$$J(x) = \sum_{n=1}^{\infty}\frac{\pi(x^{1/n})}{n} = Li(x) - \sum_{\rho}Li(x^{\rho}) - \ln 2 + \int_{x}^{\infty} dt\frac{1}{t(t^2-1)\ln t} \qquad (4.6)$$

Then the Prime counting function in terms of the non-trivial zeroes would read:

$$\pi(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\int_1^{\infty} ds \ln\zeta(ns)\frac{M_0'(n)}{ns} \qquad (4.7)$$

(Mellin inverse transform plus Abel sum formula)

$$\pi(x)=\int_1^{\infty}\frac{du}{u^2}G(u)\left(Li(x^{1/u})-\sum_{\rho}Li(x^{\rho_k/u})-\ln 2+\int_{x^{1/u}}^{\infty}dt\frac{1}{t(t^2-1)\ln t}\right) \quad (4.8)$$

$$\sum_{\rho}\frac{u^{\rho_k}}{\zeta'(\rho_k)}+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)!\zeta(2n+1)n}\left(\frac{2\pi}{u}\right)^{2n}=G(u) \qquad (4.9)$$

And, for the sum over primes of a certain function f(x) using Abel formula and the result above, we have that:

$$\sum_{p\le r}f(p)=\int_2^r dt\pi'(t)f(t)d\pi=\int_2^r dx\int_1^{\infty}\frac{du}{u^2}G(u)\left(\frac{x^{(\frac{1}{u}-1)}}{\ln(x)}-\sum_{\rho}\frac{x^{\frac{\rho}{u}-1}}{\ln(x)}-\frac{1}{x(x^{\frac{2}{u}}-1)\ln(x)}\right)f(x) \qquad (4.10)$$

A faster method to get the Prime counting function is using the definitions for J(x) and M_0 (x), and then using the Möbius inversion formula together with Abel sum formula :

$$J(x)=\sum_{n=1}^{\infty}\frac{\pi(x^{1/n})}{n} \qquad \rightarrow \qquad \pi(\mathrm{x})=\sum_{n=1}^{\infty}\frac{\mu(n)}{n}J(x^{1/n})=\int_1^{\infty}\frac{dM_0(u)}{u}J(x^{1/u}) \qquad (4.11)$$

The sum over the Non-trivial roots of $\zeta(s)$, must be summed in order of increasing $\Im m[\rho]$ imaginary part to avoid some problems with convergence:

$$\sum_{\rho}f(x^{\rho_k})=\sum_{\Im m[\rho]>0}[f(x^{\rho_k})+f(x^{1-\rho_k})] \qquad (4.12)$$

The sum over the Non-trivial zeros , (those different from $\zeta(-2n)=0$ n=1,2,3,4,5,.....) has a deep connection with operator theory, if Riemann Hypothesis is correct then $\rho_{\mathrm{n}}=\frac{1}{2}+iE_n$ Where the E_n are the Eigenvalues of a certain Hermitian operator T , if we could also consider the prime numbers to be some kind of “Eigenvalues” of a certain operator P then we could make the connection:

$$J(x)=\sum_{n=1}^{\infty}\frac{\pi(x^{1/n})}{n}=Tr\left[\hat{1}Li(x)-Li(\sqrt{x}x^{i\hat{T}})-\hat{1}\ln 2+\hat{1}\int_x^{\infty}dt\frac{1}{t(t^2-1)\ln t}\right] \qquad (4.13)$$

$$Tr\left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\frac{e^{sx}}{s}e^{-s\hat{P}}\right]=\pi(x) \qquad \hat{1} = \text{Identity operator}\ ,\ \hat{T}^{\dagger}=\hat{T} \qquad (4.14)$$

Another curious relation involving Riemann Hypothesis, Logarithmic integral and Möbius function comes from combining Riemann-Siegel formula:

$$\frac{\pi(x)-Li(x)}{\frac{\sqrt{x}}{\ln x}} \approx -1-2\sum_{\gamma}\gamma^{-1}\sin(\gamma\ln x) \qquad (4.15)$$

Where the sum on the right is over the imaginary part of Non-trivial zeros of the form $\frac{1}{2}+i\gamma \quad \gamma\in C$, in case RH is true then all the imaginary parts are real,

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