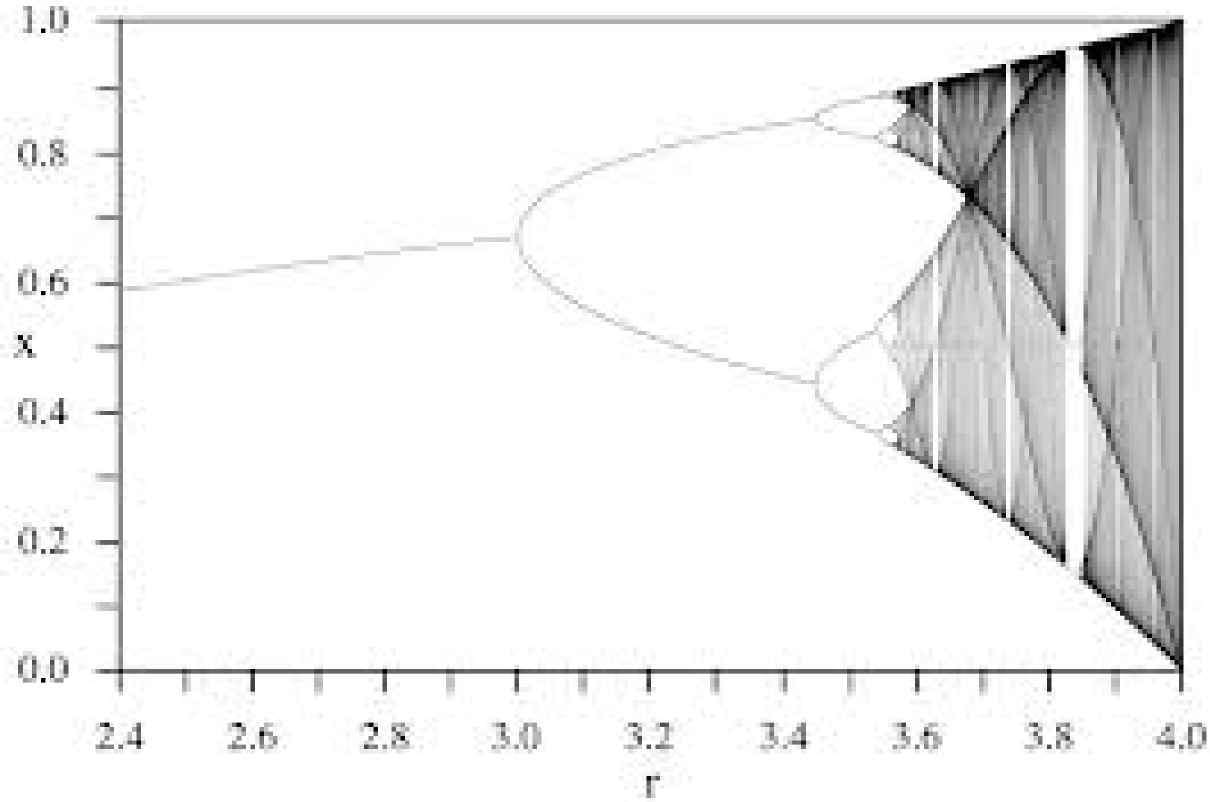
THEORETICAL ECOLOGY

A SUPPORTING ARTICLE TO THE RESEARCH

FAMOUS MODELS USED IN THIS RESEARCH

BIFURCATION THEORY - A famous model in the field of mathematics.

Models are often used to describe real ecological reproduction processes of single or multiple species. These can be modelled using stochastic branching processes. Examples are the dynamics of interacting populations (predation competition and mutualism), which, depending on the species of interest, may best be modeled over either continuous or discrete time. Other examples of such models may be found in the field of mathematical epidemiology where the dynamic relationships that are to be modeled are host-pathogen interactions.



Bifurcation diagram of the logistic map

Bifurcation theory is used to illustrate how small changes in parameter values can give rise to dramatically different long run outcomes, a mathematical fact that may be used to explain drastic ecological differences that come about in qualitatively very similar systems. The difference equation is intended to capture the two effects of reproduction and starvation.

Because ecological systems are typically nonlinear, they often cannot be solved analytically and in order to obtain sensible results, nonlinear, stochastic and computational techniques must be used.

$$X_{n+2} \simeq X_n + 2\varepsilon X_n - 2X_n^3.$$

For $\varepsilon < 0$ the only fixed point solution of the second iteration $X_{n+2} = X_n$ is $X = 0$; for $\varepsilon > 0$ a new (stable) fixed point of the second iteration develops with $X = \varepsilon^{1/2}$ corresponding to the period doubled solution $X_n = (-1)^n \varepsilon^{1/2}$.

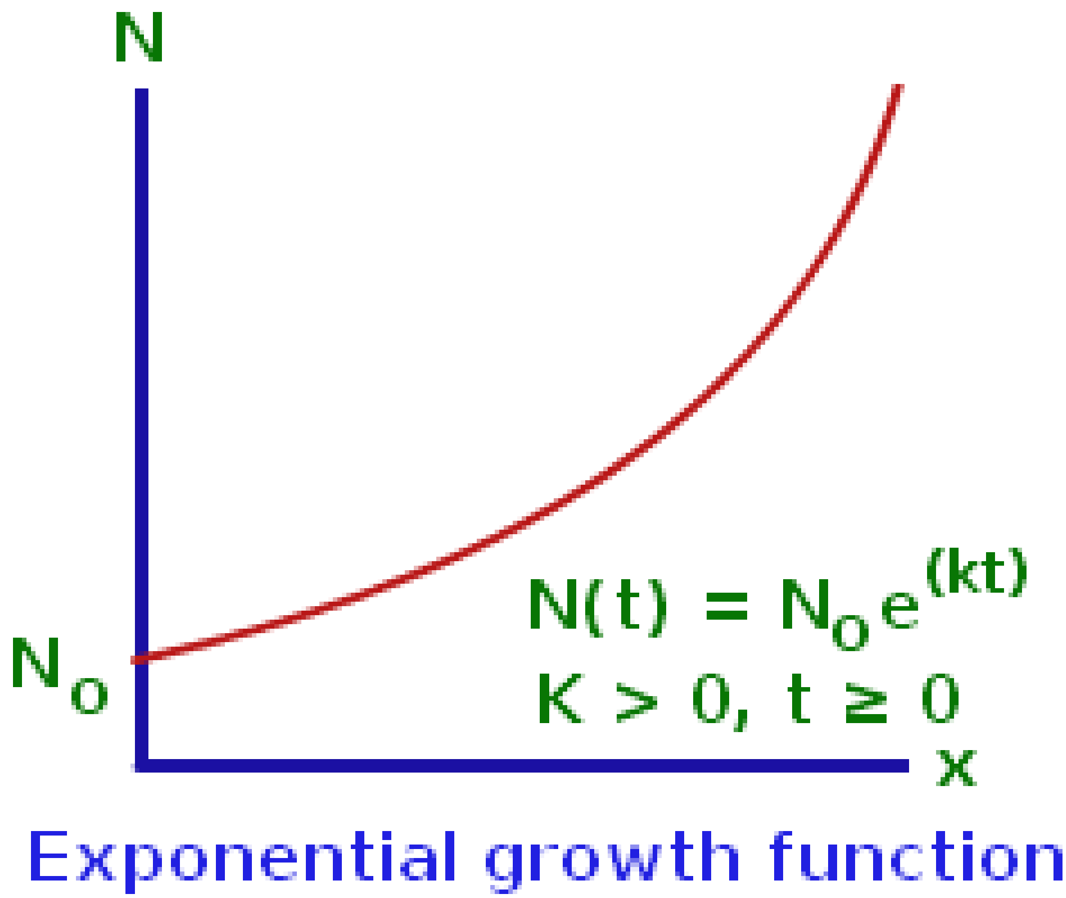
Applied theoretical ecology yields results which are used in the real world. For example, optimal harvesting theory draws on optimization techniques developed in economics, computer science and operations research, and is widely used in fisheries.

EXPONENTIAL GROWTH

The most basic way of modeling population dynamics is to assume that the rate of growth of a population depends only upon the population size at that time and the per capita growth rate of the organism. In other words, if the number of individuals in a population at a time t, is N(t), then the rate of population growth is given by:

$$N(t) = N(0)\, e^{rt}$$

where r is the per capita growth rate, or the intrinsic growth rate of the organism. It can also be described as r = b-d, where b and d are the per capita time-invariant birth and death rates, respectively.



The population grows when $r > 0$, and declines when $r < 0$. The model is most applicable in cases where a few organisms have begun a colony and are rapidly growing without any limitations or restrictions impeding their growth (e.g. bacteria inoculated in rich media).

A simple modification of the exponential growth is to assume that the intrinsic growth rate varies with population size. This is reasonable: the larger the population size, the fewer resources available, which can result in a lower birth rate and higher death rate.

Hence, we can replace the time-invariant r with r'(t) = (b –a*N(t)) – (d + c*N(t)), where a and c are constants that modulate birth and death rates in a population dependent manner (e.g. intraspecific competition). Both a and c will depend on other environmental factors which, we can for now, assume to be constant in this approximated model. The differential equation is now.

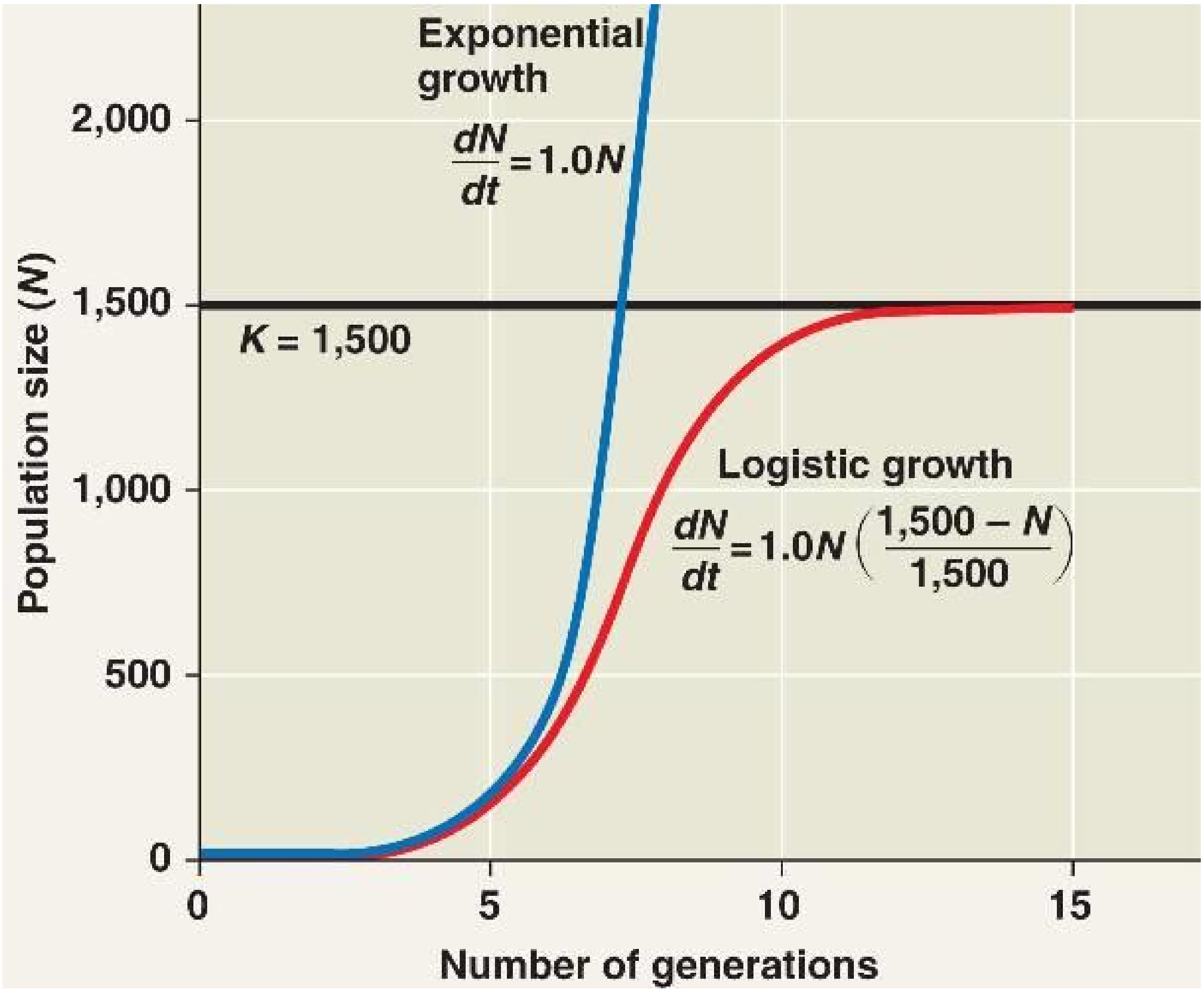
$$\frac{dN(t)}{dt} = ((b - aN(t)) - (d - cN(t)))N(t)$$

This can be rewritten as:

$$\frac{dN(t)}{dt} = rN(t)\left(1 - \frac{N}{K}\right)$$

where r = b-d and K = (b-d)/(a+c).

The biological significance of K becomes apparent when stabilities of the equilibria of the system are considered. It is the carrying capacity of the population. The equilibria of the system are N = 0 and N = K. If the system is linearized, it can be seen that N = 0 is an unstable equilibrium while K is a stable equilibrium.



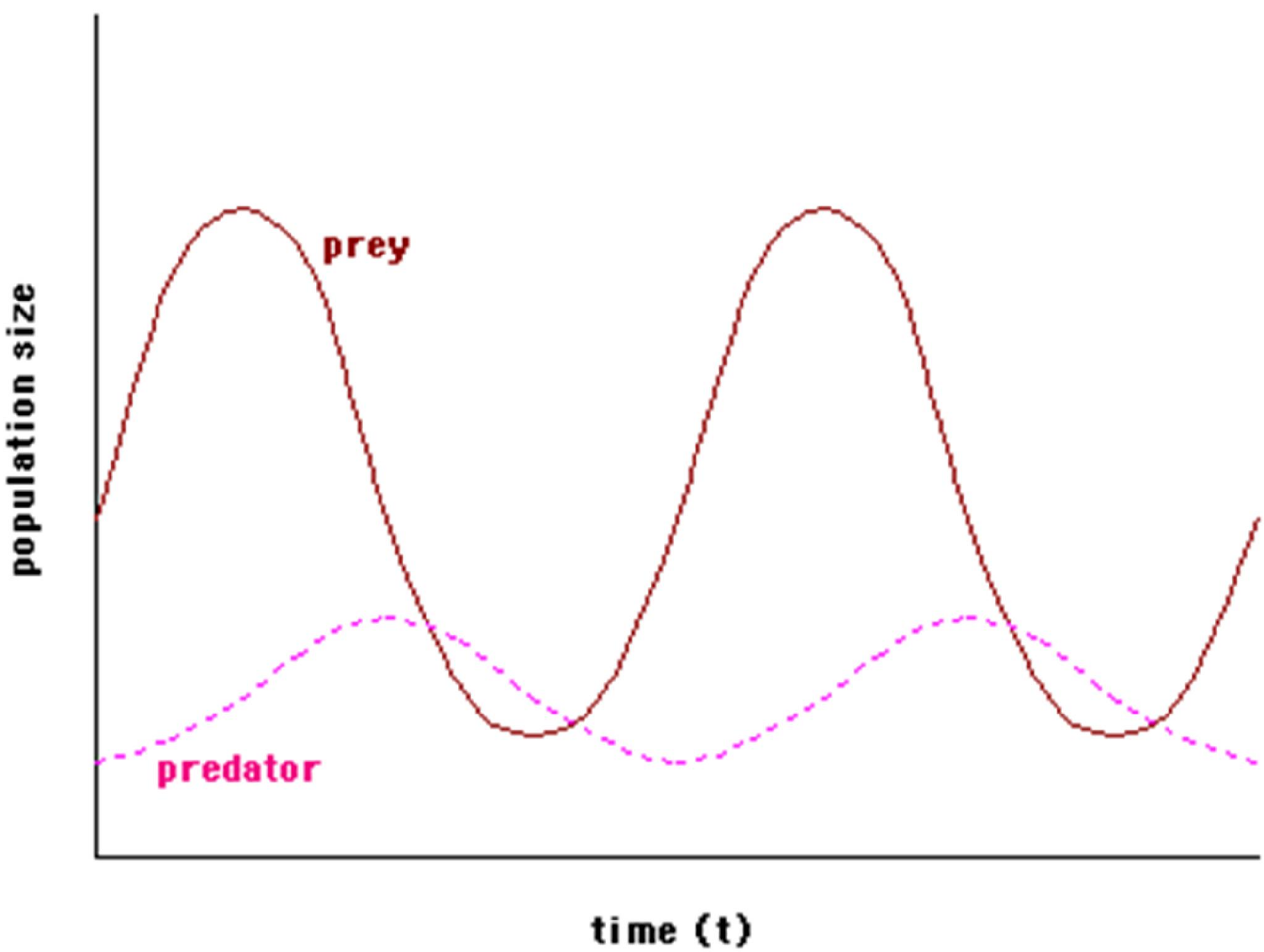
PREDATOR-PREY

Predator-prey interactions exhibit natural oscillations in the populations of both predator and the prey. It is one of the earliest and most recognised ecological models, known as the Lotka-Volterra model:

$$\frac{dN(t)}{dt} = N(t)(r - \alpha P(t))$$
$$\frac{dP(t)}{dt} = P(t)(c\alpha N(t) - d)$$

where N is the prey and P is the predator population sizes, r is the rate for prey growth, taken to be exponential in the absence of any predators, α is the prey mortality rate for per-capita predation (also called ‘attack rate’), c is the efficiency of conversion from prey to predator, and d is the exponential death rate for predators in the absence of any prey.

Volterra originally used the model to explain fluctuations in fish and shark populations after fishing was curtailed during the First World War. However, the equations have subsequently been applied more generally.



HOST- PATHOGEN

The second interaction, that of host and pathogen, differs from predator-prey interactions in that pathogens are much smaller, have much faster generation times, and require a host to reproduce. Therefore, only the host population is tracked in host-pathogen models. Compartmental models that categorize host population into groups such as susceptible, infected, and recovered (SIR) are commonly used.

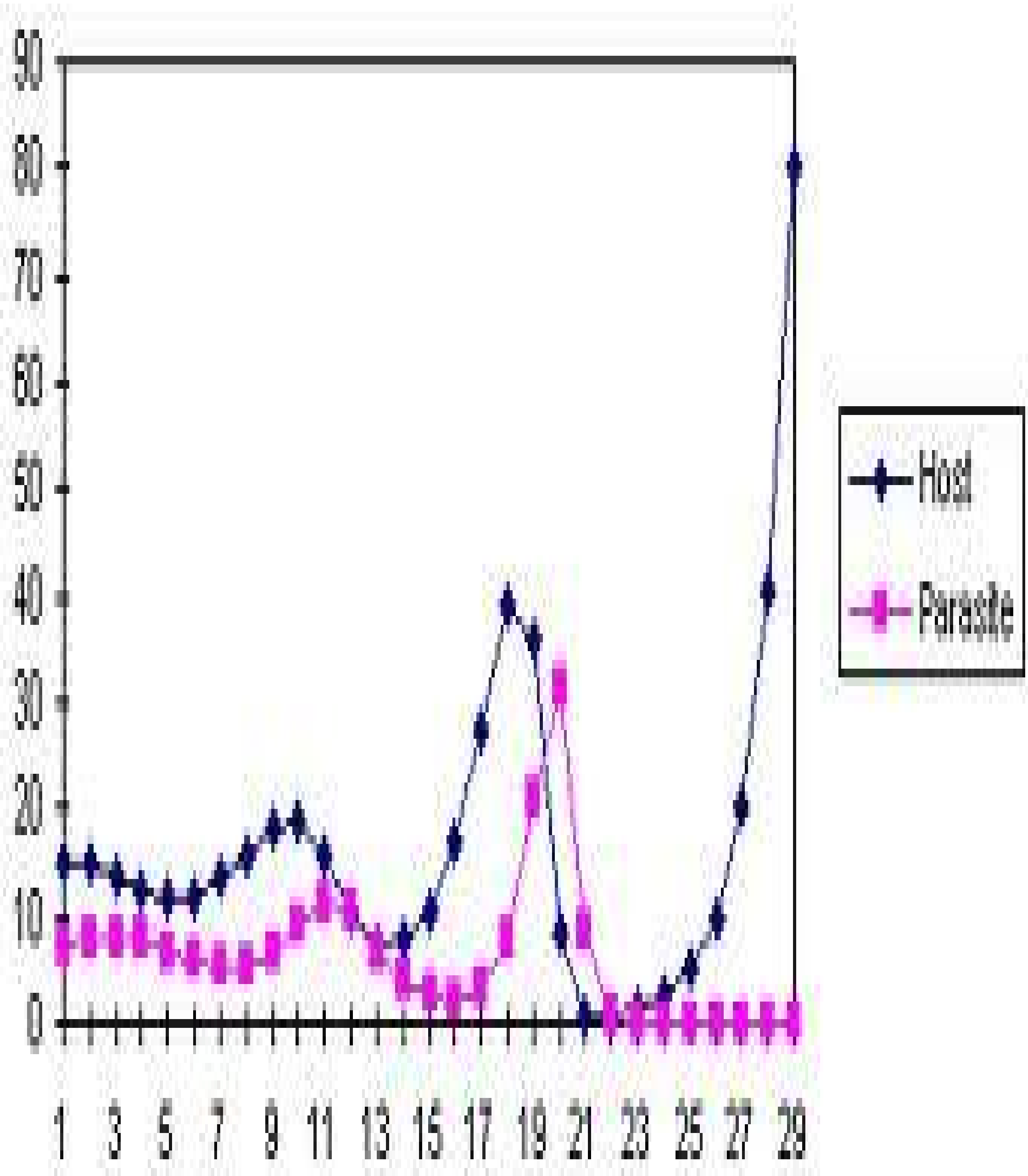
HOST- PARASITOID

The third interaction, that of host and parasitoid, can be analyzed by the Nicholson-Bailey model, which differs from Lotka-Volterra and SIR models in that it is discrete in time. This model, like that of Lotka-Volterra, tracks both populations explicitly. Typically, in its general form, it states:

$$N_{t+1} = \lambda\ N_t\ [1 - f(N_t, P_t)]$$

$$P_{t+1} = c\ N_t\ f(N_t, p_t)$$

Where $f(N_t, P_t)$ describes the probability of infection (typically, Poisson distribution), λ is the per-capita growth rate of hosts in the absence of parasitoids, and c is the conversion efficiency, as in the Lotka-Volterra model.



MODEL FOR GROWTH

RESULT

For us, mathematics is the study of *patterns and relationships.* Patterns are ideas; idealized observations about Nature that help our minds grasp her wonder. Patterns in counting end up becoming the mathematics of numbers and algebra. Using these patterns in studying ecology, environment, biodiversity or evolution is extremely beneficial.
For example, while studying populations we discover that the numbers of any species tend to grow in certain ways. The mathematics does not predict the future, it just tells you what you can likely expect to happen. What we learn about population growth has important implications for your life and mine.

The longer we study Nature and patterns, the more patterns we see; the more mathematics we create. Contrary to some popular opinions, mathematics is alive and growing; and there is beauty and excitement to be experienced. **Mathematics provides a special way of looking at the world.**